

2019-03

# Optimization of dividend payouts and reinsurance policies under a set ruin probability target

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NM-AIST

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<https://doi.org/10.58694/20.500.12479/308>

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**OPTIMIZATION OF DIVIDEND PAYOUTS AND REINSURANCE POLICIES  
UNDER A SET RUIN PROBABILITY TARGET**

**Christian Kasumo**

**A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of  
Doctor of Philosophy in Applied Mathematics and Computational Science of the  
Nelson Mandela African Institution of Science and Technology**

**Arusha, Tanzania**

**March, 2019**

## ABSTRACT

This dissertation is devoted to the mathematical investigation of the topic: *Optimization of Dividend Payouts and Reinsurance Policies under a Set Ruin Probability Target*. Its purposes are, first, to determine the optimal reinsurance and dividend policies for an insurance company whose surplus is modelled by a diffusion-perturbed classical risk process and, second, to determine the reinsurance and dividend strategies under a set ruin probability target. The dissertation concerns itself with three aspects of risk theory: (a) minimization of infinite ruin probability, which resulted in one published journal paper; (b) maximization of dividend payments, resulting in a second published paper; and (c) computing optimal dividend barriers based on set ruin probability targets, whose research paper is still in draft form. All three papers are based on a diffusion-perturbed classical risk process compounded by quota-share and excess-of-loss reinsurance. By means of the dynamic programming approach and the application of Itô's formula, the Hamilton-Jacobi-Bellman (HJB) equations for the optimization problems were derived. Additionally, the corresponding second-order Volterra integrodifferential equations (VIDEs) were obtained. These VIDEs were then transformed into Volterra integral equations (VIEs) of the second kind which were subsequently solved using the fourth-order block-by-block method based on Simpson's Rule to determine the optimal value functions. The results of the problem of minimizing the ruin probability show that the optimal reinsurance policy is  $(k^*, a^*) = (0, \infty)$ , where  $k^*$  and  $a^*$  are, respectively, the optimal retention levels for quota-share and excess-of-loss reinsurance. This applies to both the Cramér-Lundberg (CLM) and diffusion-perturbed models (DPM). For the dividend maximization problem, results indicate that for the CLM the optimal reinsurance policy is  $(k^*, a^*) = (1, \infty)$  for small claims and  $(k^*, a^*) = (1, 10)$  for large claims. The optimal dividend barrier levels for small and large claims in the CLM, respectively, are  $b^* = 10.27$  and  $b^* = 9.35$ . For the DPM, the optimal reinsurance policy is the same as for the CLM, with optimal dividend barriers  $b^* = 12.35$  for small claims and  $b^* = 11.50$  for large claims. This means higher optimal dividend barriers should be used for small claims than for large ones. With regard to ruin probability targets, results show that the optimal dividend barrier increases as the ruin probability reduces.

## DECLARATION

I, **Christian Kasumo**, do hereby declare to the Senate of the Nelson Mandela African Institution of Science and Technology, that this dissertation is my own original work and that it has neither been submitted nor being concurrently submitted for degree award in any other institution.

CHRISTIAN KASUMO



23/03/2019

Name and Signature of Candidate

Date

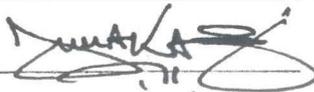
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## CERTIFICATION

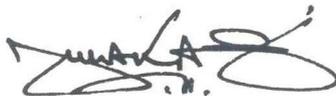
The undersigned certify that they have read and hereby recommend for examination a dissertation entitled *Optimization of Dividend Payouts and Reinsurance Policies under a Set Ruin Probability Target*, in fulfilment of the requirements for the degree of Doctor of Philosophy of the Nelson Mandela African Institution of Science and Technology.



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## ACKNOWLEDGEMENTS

It has been exciting to study for my PhD in the Department of Applied Mathematics and Computational Science at the Nelson Mandela African Institution of Science and Technology (NM-AIST), Arusha, Tanzania, for the past three years. I wish to pay tribute to several persons who have played a part, directly or indirectly, in facilitating my completion of this dissertation. Let me begin by giving all the glory to God without whom none of my efforts would have counted for anything. He is my Rock and Fortress, my Light and my Salvation, the Lifter of my head. I am particularly grateful to God for giving me life and good health, as well as for His guidance, protection and love.

I extend my special thanks to my supervisors Prof. Juma Kasozi of Makerere University, Kampala, Uganda, and Prof. Dmitry Kuznetsov of NM-AIST, Arusha, Tanzania, for their invaluable guidance and support throughout my research. They ensured that I made adequate progress in my research and offered useful corrective suggestions throughout. I am grateful for their vast mathematical knowledge and understanding which has made my PhD research a wonderful learning experience. In particular, I thank them for their patience with me and for putting up with my many shortcomings during this period.

It was, in fact, Prof. Kasozi who recommended that I consider studying at NM-AIST. If it were not for him, I might probably never have come to NM-AIST. Prof. Kuznetsov took me under his wings and taught me many things not only as my lecturer in some of the courses I took but also as Head of Department for Applied Mathematics and Computational Science.

My thanks are also due, albeit posthumously, to the late Dr. Yaw Nkansah-Gyekye who was a mentor and inspiration to me for the better part of two years of my three-year stay at NM-AIST. I will never forget his constant encouragement to complete my studies on time.

I extend special thanks to my very good friend Dr. Geoffrey Omarch with whom I have shared many memorable experiences throughout our studies at NM-AIST and who has been a great encouragement to me. His friendship means a lot to me and is something I will treasure for the rest of my life. Indeed, Dr. Omarch has become more of a brother to me than a friend. I do not forget to thank my colleagues in the PhD programme in Applied Mathematics and Computa-

tional Science, Furaha Chuma and Slaa Qanne, for the mutual encouragement we shared.

My family deserve very special mention as they had to put up with my absence for much of the past three years. When the going got tough and I was tempted to give up, my wife Baptista and my daughters, Chabota, Choolwe, Chimuka and Christabel, were my pillar of strength and urged me to soldier on. When I was home and busy working on my research, they were very understanding and gave me the time and space I needed to complete my research. Without their constant encouragement I would not have completed my studies at NM-AIST.

All this work would not have been possible without the financial support of the African Development Bank Group through the Support to Science and Technology Education Project (SSTEP) administered by the Zambian Ministry of Higher Education. This scholarship could not have come at a better time than it did, and I will never cease to be appreciative for it. I wish to also thank my employers Mulungushi University for granting me paid study leave to enable me to pursue my studies in Tanzania. In particular, the Dean of the School of Science, Engineering and Technology at Mulungushi University, Prof. Douglas Kunda, deserves special mention as the one who believed in me and played an important role in securing funding for my studies under SSTEP.

## **DEDICATION**

To my family:

Baptista, Chabota, Choolwe, Chimuka and Christabel

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## LIST OF ABBREVIATIONS AND NOTATION

### Abbreviations

|               |   |
|---------------|---|
| <b>HJB</b>    | Hamilton-Jacobi-Bellman                                       |
| <b>VIDE</b>   | Volterra integrodifferential equation                         |
| <b>VIE</b>    | Volterra integral equation                                    |
| <b>SDE</b>    | Stochastic differential equation                              |
| <b>CLM</b>    | Cramér-Lundberg model   |
| <b>DPM</b>    | Diffusion-perturbed model                                     |
| <b>ECOMOR</b> | Excédent du coût moyen relatif ('excess of the average cost') |
| <b>QS</b>     | Quota-share   |
| <b>XL</b>     | Excess-of-loss  |
| <b>i.i.d.</b> | independent and identically distributed                       |
| <b>iff</b>    | if and only if  |
| <b>w.r.t.</b> | with respect to   |
| <b>s.t.</b>   | such that   |
| <b>a.s.</b>   | almost surely   |
| <b>viz.</b>   | namely  |

### Notation

|                                     |  |
|-------------------------------------|--|
| $\mathbb{R}$                        | the real numbers   |
| $\mathbb{R}^+$                      | the non-negative real numbers  |
| $\mathbb{Q}$                        | the rational numbers   |
| $\mathbb{Z}$                        | the integers   |
| $\mathbb{P}$                        | probability measure  |
| $\Omega$                            | sample (or event) space on which probabilities are defined                     |
| $\mathcal{F}$                       | $\sigma$ -algebra of the probability space                                     |
| $\{\mathcal{F}_t\}$                 | filtration   |
| $(\Omega, \mathcal{F}, \mathbb{P})$ | probability space  |
| $A^T$                               | transpose of the matrix $A$  |
| $\det A$                            | determinant of the $n \times n$ matrix $A$                                     |
| $C^q[c, d]$                         | space of functions that are $q$ -times continuously differentiable on $[c, d]$ |
| $\mathcal{A}$                       | infinitesimal generator of an Itô process                                      |
| $W_t$                               | standard 1-dimensional Brownian motion   |
| $\xi_t$                             | white noise process  |
| $S_t$                               | aggregate claims process   |
| $N_t$                               | claim arrival process  |
| $\lambda$                           | claim arrival intensity or rate  |

|                   |   |
|-------------------|---|
| $X_i$             | $i$ -th claim amount                                      |
| $U_t$             | surplus process of insurance company                      |
| $u$               | initial surplus   |
| $b$               | dividend barrier level                                    |
| $D_t^b$           | cumulative dividend payments                              |
| $\sigma$          | volatility or diffusion coefficient                       |
| $\theta$          | reinsurer's safety loading                                |
| $\eta$            | insurer's safety loading                                  |
| $\delta$          | discount rate for dividends                               |
| $c$               | premium rate  |
| $k$               | retention level for quota-share reinsurance               |
| $a$               | retention level for excess-of-loss reinsurance            |
| $\psi(u)$         | ruin probability  |
| $\phi(u)$         | survival probability                                      |
| $\overline{D}$    | dividend strategy   |
| $\overline{R}$    | reinsurance strategy                                      |
| $\Pi_u^{D,R}$     | set of all admissible dividend and reinsurance strategies |
| $V_b(u)$          | dividend value function under strategy $b$                |
| $\rho$            | adjustment coefficient                                    |
| $k^\rho$          | asymptotically optimal QS retention                       |
| $k^*$             | optimal QS retention level                                |
| $a^*$             | optimal XL retention level                                |
| $b^*$             | optimal dividend barrier level                            |
| $\tau$            | ruin time   |
| $M_X(r)$          | moment-generating function of $X$                         |
| $f(x)$            | claim-size density function                               |
| $F(x)$            | claim-size distribution function                          |
| $\overline{F}(x)$ | tail distribution of $F$ ; $\overline{F} = 1 - F$         |
| $G(x)$            | integral of $F$   |
| $\forall$         | for all   |
| $\exists$         | there exists  |
| $\square$         | end of proof  |
| $s \wedge t$      | the minimum of $s$ and $t$ (or $\min(s, t)$ )             |
| $x^+ = x \vee 0$  | positive part of $x$ (or the maximum of $x$ and 0)        |
| $\sim$            | distributed as  |
| $\cong$           | approximately equal to                                    |

# CHAPTER ONE

## INTRODUCTION

### 1.1 Background

This study focuses on optimization of dividend payouts and reinsurance policies under a set ruin probability target. The purpose of the study is to determine the optimal reinsurance and dividend policies for an insurance company whose surplus is modelled by a diffusion-perturbed classical risk process and determine the reinsurance and dividend strategies under a set ruin probability target. Chapter 1 outlines the background of the study, defines some important concepts used in the study and presents the problem statement. It also indicates the objectives of the study as well as the research questions which guide the research process. In addition, the chapter states the significance of the study and outlines the methodology used in undertaking the study. It concludes with the format of the study.

Insurance companies exist to provide cover against risks whose occurrence results in financial loss. These risks include accidents, fire, floods, sickness, disability or even death. Some of these risks threaten all persons and others are restricted to the owners of property, while still others are typical for some individuals or for special occupations. Insurance companies provide cover against such risks in exchange for a payment by the insured (called a *premium*). Insurance cover or protection is accomplished through a pooling mechanism whereby many individuals who are vulnerable to a common risk are joined together into a risk pool. By pooling both the financial contributions and the risks of a large number of policyholders, the insurer is typically able to absorb losses incurred over any given period much more easily than the uninsured individual or firm would do.

However, in providing insurance cover, insurers are themselves exposed to risk and they address their risk by employing several measures. These measures are called control variables and include sharing some of their risk through reinsurance, investing part of the surplus, coming up with attractive and competitive insurance products (portfolio selection), or setting reasonable

premiums which are not only affordable but also profitable (Avanzi, 2009). But insurance companies also have to enhance shareholder value through payment of dividends.

To describe the ideas about insurance in general and before introducing further terminology, it is necessary to define the following terms and concepts:

### 1.1.1 Insurance, reinsurance and dividends

#### Definition 1.1.1

**Risk theory:** This refers to a body of techniques used for modelling and measuring the risk associated with a portfolio of insurance contracts (Dutang *et al.*, 2008).

#### Definition 1.1.2

**Portfolio:** This refers to an aggregate or collection of insurance contracts or policies covering similar risks (Mikosch, 2004).

#### Definition 1.1.3

**Claim:** A *claim* is the amount that the insurance company has to pay out to its policyholders as provided for by the insurance policy.

#### Definition 1.1.4

**Surplus:** A *surplus* is the excess of total income collected over total claims and other expenses paid out in a given period of time. The first time the surplus of an insurance company becomes negative is called the *ruin time* and the associated probability is the *ruin probability*.

#### Definition 1.1.5

**Dividend:** A *dividend* refers to a payment of the portion of the surplus reserves given back to the shareholders of a company (Basu, 2016). Dividends are taxable payments declared by the insurer's board of directors and given to the shareholders out of the company's current or retained earnings (Kasozi and Paulsen, 2005a).

#### Definition 1.1.6

**Barrier strategy:** A *barrier strategy*  $b$  pays out any surplus above  $b$  as dividends immediately so that the surplus is brought back to  $b$ . Below  $b$ , nothing is paid out (Kasozi and Paulsen, 2005a).

### **Definition 1.1.7**

**Reinsurance:** *Reinsurance* is the transfer, whether in whole or in part, of a risk assumed by an insurer (the *cedent*), to another insurer (the *reinsurer*) (Centeno and Simões, 2009).

The study investigates the feasibility of covering its risk through a combined proportional (quota-share) and non-proportional (excess-of-loss) reinsurance arrangement. This is done under the assumption that the company pays out dividends to its shareholders. The classical compound Poisson risk model (or Cramér-Lundberg model) has been widely used for modelling the surplus of insurance companies (Mikosch, 2004). This model says that the surplus of an insurance company is the sum of the initial capital and the total premium income received less the total amount of claims paid out to policyholders. In this model, the premium income is received at a constant rate and is deterministic, while the claims are of random sizes and are paid out at random times (Schmidli, 2008).

Important financial applications of the classical risk process include the modelling of future dividend payments to shareholders as well as the possibility of allowing the company to reinsure by ceding some of its claims and premium income to a reinsurer. In connection with both of these applications, ruin probabilities can be computed based on the controlled classical risk model (i.e., a model incorporating both reinsurance and dividends). Ruin refers to the event that occurs when, given a positive initial surplus, the company's surplus or wealth becomes negative or enters  $(-\infty, 0)$ . When this happens, it does not necessarily mean that the company is bankrupt. But it means that in that instant the company is not operating profitably and therefore must take action in order to make the business profitable once again (Charpentier, 2010). Ruin may occur due to a claim, due to oscillations from the Brownian motion or because of fluctuations in the risky investment. The probability of ruin is a natural measure of risk in the insurance business. This probability needs to be controlled for the continued survival of the insurance company as a going concern.

Modern collective risk theory dates back to the acclaimed doctoral thesis of the Swedish actuary Filip Lundberg in 1903 who introduced a simple model capable of describing the dynamics of a homogeneous insurance portfolio. The ruin probability minimization problem based on Lundberg's model was subjected to a rigorous mathematical treatment in 1930 by the Swedish

actuary and statistician Harald Cramér (Mikosch, 2004). Since then, the focus has traditionally been on ruin probability in an infinite time horizon. In these models an insurance company can avoid ruin by allowing its risk reserve to grow without limit. However, according to the Italian statistician Bruno de Finetti (1957), this assumption was not realistic. This is because it was based on the idea that an older company, purely by virtue of its age, could hold more capital than a younger company bearing similar risks. Therefore, as an alternative, de Finetti proposed the dividend maximization problem as more realistic (Avanzi, 2009). The classical problem in the Cramér-Lundberg model, therefore, has been not only the payment of dividends to shareholders but also the maximization of these dividends in the best possible way, that is, finding the optimal dividend strategies.

But dividend maximization alone would result in certain ruin, as Gerber and Shiu (2006) have pointed out. Hence the need for companies to consider other measures (such as the control variables already alluded to) by way of reducing the probability of ruin. This study seeks to find the optimal dividend and reinsurance strategies for the diffusion-perturbed classical risk process. There are two reasons for using a diffusion-perturbed model as a starting point. The first reason is to account for fluctuations in the total claim amount process. In this way, the diffusion model represents reality better since uncertainty is a normal aspect of everyday life. The second reason is to achieve some generalization from which the classical risk process can be considered as a special case.

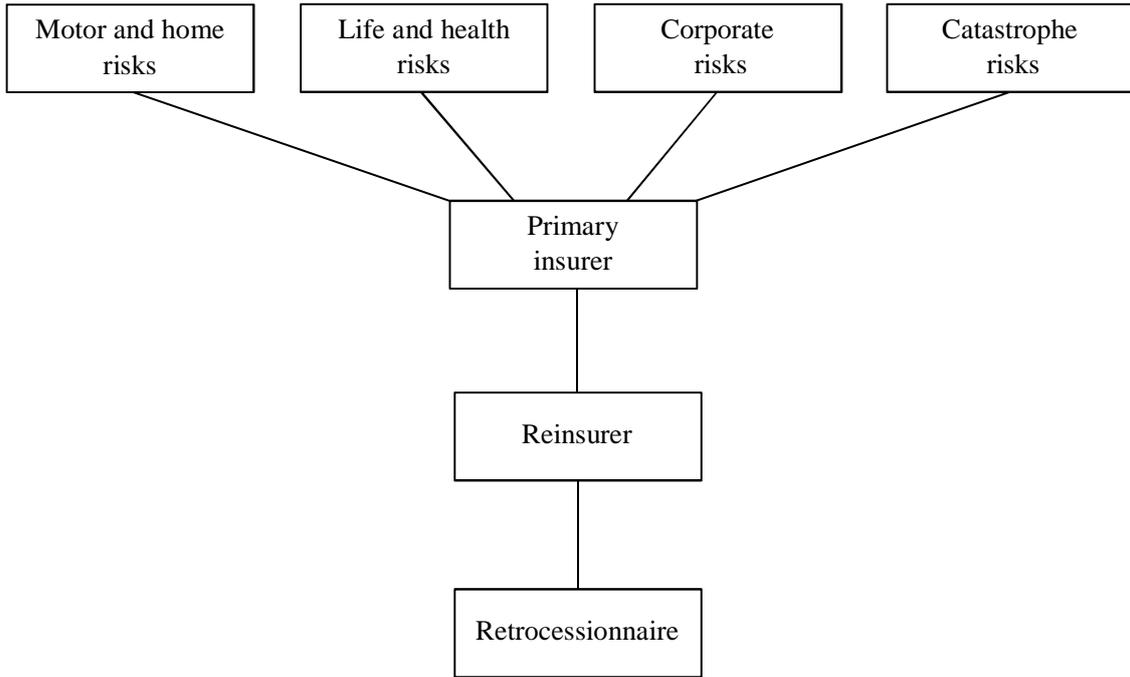
Reinsurance is an important strategy for risk management in insurance and consists in ceding part of the insurer's risk to a reinsurer, in exchange for a reinsurance premium. It is an intelligent mechanism for reducing the insurer's risk retention at a sustainable reinsurance premium. Reinsurance creates value for the insurer by bolstering the insurer's underwriting capacity and, in turn, empowering the insurer to assume losses from its policyholders which it would otherwise be incapable of underwriting (Lo, 2016). Apart from helping the cedent to manage financial risk, increase capacity and achieve marketing goals, reinsurance also benefits policyholders by ensuring availability and affordability of necessary coverage (Li, 2008).

Li (2008) has further pointed out that reinsurance enables insurers to provide the amount of coverage requested by policyholders, even if that amount is beyond a single insurer's reten-

tion limit. He further argues that reinsurance also benefits the insurance industry as a whole in that it provides a greater spread of risk. This is because if each company has less exposure to catastrophic loss, the industry as a whole is better protected. Thus, reinsurance strengthens the insurance industry financially and provides a more stable and reliable marketplace for insurance companies, policyholders and investors (Li, 2008). This study employs a combination of proportional (quota-share) and excess-of-loss reinsurance to minimize the ultimate ruin probability of an insurance company, as well as to maximize the dividend payouts to the shareholders. In quota-share reinsurance, premiums and losses are shared proportionately between the cedent and the reinsurer, while in excess-of-loss reinsurance the reinsurer covers all of the cedent's claims or losses exceeding a prespecified retention limit or deductible.

Reinsurance allows insurers to transfer some of their risks to other insurers at the expense of making less potential profit, since they have to divert a portion of all their premium incomes to the reinsurer. In other words, with reinsurance, the first-line insurer passes on some of its premium income to a reinsurer who, in turn, covers a certain proportion of the claims that occur (Albrecher and Thonhauser, 2009). Figure 1 shows how risks are transferred from policyholders (individuals and companies) to reinsurers via primary insurers. The transfer of risk, for a suitable premium, from reinsurers to other reinsurers is called *retrocession*. This study seeks to investigate the hypotheses that by providing a first-line insurer with additional underwriting capacity, reinsurance reduces the company's probability of ruin. But if, in addition, the company pays dividends to the shareholders from its surplus, it must decide on a dividend strategy that maximizes these dividend payouts. Thus, the study seeks to determine a reinsurance and dividend strategy that minimizes the probability of ultimate ruin as well as maximizes the dividend payouts for an insurance company.

While insurance is a mechanism for coping with risk, the object of *risk theory* is to give a mathematical analysis of random fluctuations in the insurance business and to discuss the various means of protection against the inconvenient effects of those fluctuations. *Classical risk theory* focuses its attention on the *outflow* process, looking first at *claim numbers*, then at the distribution of *claim sizes* and finally combining these two into an *aggregate claim amount process* (Malinovskii, 2000). It should be noted that claims arrive at an insurance company in a random manner due to the fact that in most cases financial losses occur abruptly. Because of this, the



**Figure 1:** Transfer of risk from policyholders to reinsurers

mathematical analysis of an insurance business calls for the use of *probability theory*. In this regard, the definitions of terms and concepts from probability theory lead to the theory of a filtered probability space (see Definition 1.1.14).

**Definition 1.1.8**

**Sample space:** A *sample space*  $\Omega$  is the set of all possible outcomes of some random experiment (Kijima, 2003).

**Definition 1.1.9**

**$\sigma$ -algebra:** A  $\sigma$ -algebra (or  $\sigma$ -field) is a collection  $\mathcal{F}$  of subsets of  $\Omega$  with the following properties:

- (i)  $\emptyset, \Omega \in \mathcal{F}$
- (ii) If  $A \subset \Omega \in \mathcal{F}$ , then  $A^c = \Omega \setminus A \in \mathcal{F}$ .
- (iii) If  $A_1, A_2, \dots \in \mathcal{F}$ , then  $\bigcup_{k=1}^{\infty} A_k, \bigcap_{k=1}^{\infty} A_k \in \mathcal{F}$

(Kijima, 2003)

**Example:** Let  $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ . Then we can define  $\sigma$ -algebras:

- (i)  $\mathcal{F}_0 = \{\emptyset, \Omega\}$  (trivial)
- (ii)  $\mathcal{F}_1 = \{\emptyset, \Omega, \{\omega_3\}, \{\omega_4\}, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_1, \omega_2, \omega_3\}, \{\omega_1, \omega_2, \omega_4\}\}$
- (iii)  $\mathcal{F}_2 = 2^\Omega$  (the class of *all* subsets of  $\Omega$ )

where

- (i)  $\mathcal{F}_0$ : No information except  $\Omega$ .
- (ii)  $\mathcal{F}_1$ : Partial information, since it is not possible to distinguish between occurrence of either  $\omega_1$  or  $\omega_2$ .
- (iii)  $\mathcal{F}_2$ : Full information, since  $\{\omega_1\}$ ,  $\{\omega_2\}$ ,  $\{\omega_3\}$  and  $\{\omega_4\}$  can be observed.

It should be noted that  $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$ , meaning that the information accumulated over time only increases or grows, so nothing can be forgotten. Appropriate probability measures can be defined on each  $\sigma$ -algebra and should satisfy the properties in Definition 1.1.10. The initial information at time zero is usually given by the trivial  $\sigma$ -algebra  $\mathcal{F}_0 = \{\emptyset, \Omega\}$ .

**Definition 1.1.10**

**Probability measure:** Let  $\mathcal{F}$  be a  $\sigma$ -algebra of subsets of  $\Omega$ . Then the *probability measure* is a set function  $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$  defined on  $\mathcal{F}$  and satisfying the following properties:

- (i)  $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1$
- (ii) For any event  $A \in \mathcal{F}$ ,  $\mathbb{P}(A) \in [0, 1]$
- (iii) If  $A_1, A_2, \dots \in \mathcal{F}$ , then  $\mathbb{P}(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mathbb{P}(A_k)$
- (iv) If  $A_1, A_2, \dots$  is a sequence of *disjoint* sets in  $\mathcal{F}$ , then  $\mathbb{P}(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mathbb{P}(A_k)$ . It follows that if  $A, B \in \mathcal{F}$ , then  $A \subseteq B$  implies  $\mathbb{P}(A) \leq \mathbb{P}(B)$

(Grimmett and Stirzaker, 2001; Kijima, 2003)

**Definition 1.1.11**

**Random variable:** Let  $\Omega$  be a non-empty finite set and let  $\mathcal{F}$  be the  $\sigma$ -algebra of all subsets of  $\Omega$ . A *random variable* is an  $\mathcal{F}$ -measurable function  $X : \Omega \rightarrow \mathbb{R}$  if the probability of each outcome of  $X$  can be calculated based on the probability measure  $\mathbb{P}$  (Øksendal, 2003; Kijima, 2003). That is, a random variable  $X$  is a rule that assigns a number  $X(\omega)$  to each outcome  $\omega \in \Omega$  of a random experiment.

**Definition 1.1.12**

**Probability space:** A *probability space* is the triple  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\Omega$  is a non-empty set,  $\mathcal{F}$  is a  $\sigma$ -algebra consisting of subsets of  $\Omega$  and  $\mathbb{P}$  the probability measure on  $\Omega$  (Øksendal, 2003; Geiss, 2010).

**Definition 1.1.13**

**Filtration:** Let  $\Omega$  be a non-empty finite set. A *filtration*  $\{\mathcal{F}_t\}_{t \in \mathbb{R}^+}$  is a sequence of  $\sigma$ -algebras  $\mathcal{F}_0, \mathcal{F}_2, \dots, \mathcal{F}_n$  s.t. each  $\sigma$ -algebra in the sequence contains all the sets contained by the previous  $\sigma$ -algebra (Grimmett and Stirzaker, 2001). A filtration is said to *satisfy the usual conditions* if it is right-continuous and  $\mathbb{P}$ -complete (that is, if for  $0 \leq s \leq t$ ,  $\mathcal{F}_0 \subseteq \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ , and  $\mathcal{F}_0$  contains all the  $\mathbb{P}$ -null sets of  $\mathcal{F}$ ).

**Note:** In other words, if information that has evolved up to time  $t$  is readily available, nothing more can be learned by peeking infinitesimally far into the future.

**Definition 1.1.14**

**Filtered probability space:** A probability space endowed with a filtration is called a *filtered probability space*, denoted by  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}^+}, \mathbb{P})$  (Irgens and Paulsen, 2004).

**Remark 1.1.1**

This filtered probability space constitutes the stochastic basis for the study. It will be assumed that all stochastic quantities and random variables (see Definitions 1.1.11 and 1.1.16) are defined on a large enough filtered probability space. We assume throughout this dissertation that the filtrations always satisfy the usual conditions.

**Note:** The assumption of right-continuity implies that the *ruin time* (see Definition 1.1.4 and Section 3.2(f)), which is one of the most important random times considered in finance and eco-

nomics, is a stopping time (Schmidli, 2000). The following definition recalls the mathematical concept of *stopping time*.

**Definition 1.1.15**

**Stopping time:** For a given filtered probability space, a *stopping time* (also called *Markovian time*) is any non-negative random variable  $\tau$  with values in  $\mathbb{R}^+$ , including possibly the value  $\infty$ , s.t.  $\{\tau \leq t\} \in \mathcal{F}_t$  for any  $t \geq 0$  (Constantinescu, 2006).

**Definition 1.1.16**

**Stochastic process:** A *stochastic process* is a parameterized collection of random variables  $\{X_t\}_{t \in \mathcal{T}}$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and assuming values in  $\mathbb{R}^n$ . In other words, it is a family of random variables  $\{X_t : t \in \mathcal{T}\}$  parameterized by time  $t \in \mathcal{T}$ , where  $\mathcal{T}$  is the parameter set of the stochastic process. (Øksendal, 2003; Kijima, 2003)

**Note:** Since insurance companies can make policy decisions on the basis of information that has evolved up to the present time, the policy factors should be in some way measurable w.r.t. the filtration. The following definition recalls the concept of an *adapted process*.

**Definition 1.1.17**

**Adapted process:** A stochastic process  $X = \{X_t\}_{t \in \mathbb{R}^+}$  defined on a filtered probability space is said to be *adapted to the filtration*  $\{\mathcal{F}_t\}_{t \in \mathbb{R}^+}$  if for each  $t \geq 0$   $X_t$  is  $\mathcal{F}_t$ -measurable (Calin, 2012).

**Remark 1.1.2**

Stochastic processes are mathematical tools for modelling systems that vary randomly in time. Uncertainty is an integral part of the insurance business. Therefore, stochastic processes are particularly used to describe financial and other forms of uncertainty that arise in the insurance business. There are several examples of stochastic processes which are used in real life to model uncertainty depending on the situation at hand. These include, among others, the Poisson process, compound Poisson process, Lévy process, Wiener process or Brownian motion and geometric Brownian motion. These processes are used to model claims and/or to account for uncertainty in this study.

**Definition 1.1.18**

**Poisson process:** An integer-valued stochastic process  $N = \{N_t\}_{t \in \mathbb{R}^+}$  is a *Poisson process* if

it starts at zero, if it has independent and stationary increments  $N(t+s) - N(s)$  for  $0 \leq s < t$ , if the increments are Poisson distributed with intensity  $\lambda$  and if it has càdlàg sample paths (Mikosch, 2004).

**Definition 1.1.19**

**Claim number process:** The *claim number process* is a counting process  $N = \{N_t\}_{t \in \mathbb{R}^+}$  which models the number of claims that occurred up to time  $t \geq 0$ , that is,  $N_t = \#\{i \geq 1 : T_i \leq t\}, t \geq 0$  (Mikosch, 2004).

**Remark 1.1.3**

The most common claim number process is the Poisson process because of its good theoretical properties, as given in Definition 1.1.18. In his 1903 thesis, Lundberg exploited it as a model for the claim number process  $N$  (Mikosch, 2004).

**Definition 1.1.20**

**Homogeneous Poisson Process:** A Poisson process with a mean value function  $\mu$  defined as  $\mu(t) = \lambda t, t \geq 0$  for some  $\lambda > 0$ , is said to be *homogeneous*, inhomogeneous otherwise (Mikosch, 2004).

**Remark 1.1.4**

(i) The quantity  $\lambda$  is the *intensity* or *rate* of the homogeneous Poisson process. If  $\lambda = 1$ , then  $N$  is called a *standard homogeneous Poisson process*. However, because the homogeneous Poisson process does not always describe claim arrivals in an adequate way, other processes for modelling the number of claims have been developed, for example, mixed Poisson and renewal processes. Mikosch (2004) further asserts that a homogeneous Poisson process  $\{N_t\}_{t \in \mathbb{R}^+}$  with intensity  $\lambda$ :

- (a) has càdlàg sample paths,
- (b) starts at zero,
- (c) has *independent* and *stationary* increments, that is, for any  $0 \leq s \leq t$  and  $h > 0$ ,  $N_{s,t}$  has the same distribution as  $N_{s+h,t+h}$ . That is, the random variables  $N_t - N_s$  and  $N_{t+h} - N_{s+h}$  have the same distribution or probability law. This means that the

probability law of the number of claim arrivals in any interval of time depends on the length of the interval (Ramasubramanian, 2005).

(d) is  $Pois(\lambda t)$  distributed for every  $t > 0$ .

(e) is a non-decreasing process. That is, if  $0 \leq s < t$ , then  $N_s \leq N_t$ . Note that  $N_t - N_s$  denotes the number of claims in the interval  $(s, t]$  (Ramasubramanian, 2005).

(ii) A process on  $\mathbb{R}^+$  with properties (a)-(c) above is called a *Lévy process* (Mikosch, 2004).

(iii) The homogeneous Poisson process is one of the prime examples of Lévy processes with vast applications in real life (Mikosch, 2004). In particular, it is used in the models considered in this dissertation for modelling the number of claims received by the insurance company from policyholders.

(iv) Ramasubramanian (2005) adds two additional properties of the claim number process  $N$ :

(a) The probability of two or more claim arrivals in a very short time span is negligible, that is,  $\mathbb{P}(N_h \geq 2) = o(h)$ , as  $h \downarrow 0$ .

(b) In a very short time interval, the probability of exactly one claim arrival is roughly proportional to the length of the interval, that is,  $\exists \lambda > 0$  s.t.  $\mathbb{P}(N_h = 1) = \lambda h + o(h)$ , as  $h \downarrow 0$ .

**Note:** Since the models discussed in Chapter 3 incorporate a diffusion term (that is, a diffusion coefficient  $\sigma$  and a standard one-dimensional Brownian motion  $W_t$ ), it is important to now define a Wiener process or Brownian motion.

**Definition 1.1.21**

**Wiener process:** Let  $\{W_t\}_{t \in \mathbb{R}^+}$  be a stochastic process defined on some filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}^+}, \mathbb{P})$ . The process  $\{W_t\}_{t \in \mathbb{R}^+}$  is a one-dimensional *Brownian motion* or *Wiener process* if:

(i) It starts at the origin, that is,  $W_0 = 0$ .

(ii) It has independent and stationary increments.

- (iii) The increments  $W_t - W_s$  are Gaussian with mean zero and variance  $t - s$ , independently of time  $t$ , that is,  $W_t - W_s \sim N(0, t - s)$ .
- (iv) The sample paths or trajectories  $t \in [0, \infty) \rightarrow W_{t,\omega}$  of the process  $\{W_t\}$  are continuous functions of time.

(Kijima, 2003; Calin, 2012)

**Remark 1.1.5**

- (i) An  $n$ -dimensional Brownian motion is an  $\mathbb{R}^n$ -valued stochastic process

$$W = \{(W_t^1, W_t^2, \dots, W_t^n)\}_{t \in \mathbb{R}^+}$$

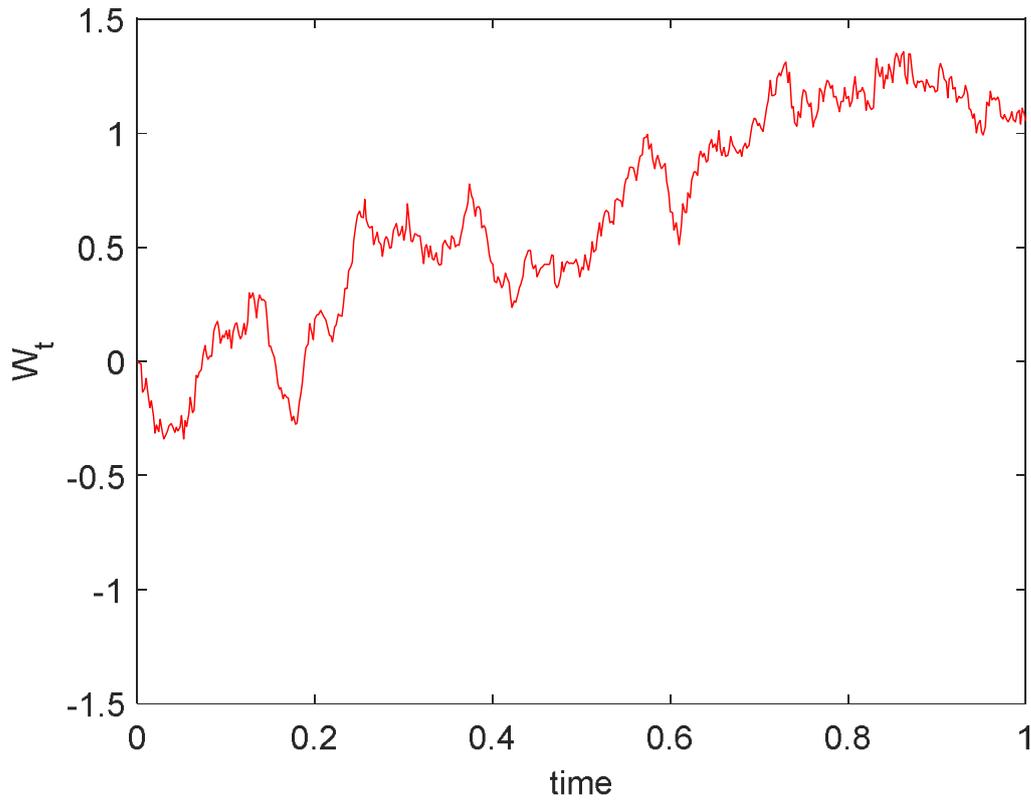
whose components  $W_t^i$  ( $i = 1, 2, \dots, n$ ) are independent one-dimensional Brownian motions.

- (ii) The derivative of Brownian motion is also a stochastic process and referred to as a *white noise* process, denoted by  $\xi_t$ . Thus,  $dW_t = \xi_t dt$ . White noise is introduced to model uncertainties in the underlying deterministic differential equation.

**Definition 1.1.22**

**Sample paths of Brownian motion:** An occurrence of a Brownian motion  $\{W_t\}_{t \in \mathbb{R}^+}$  observed from time 0 to time  $T$  is called a *realisation* or *sample path* of the process on the time interval  $[0, T]$ . (Kijima, 2003)

Fig. 2 illustrates such a path, generated using the MATLAB code *samppath.m* in *Appendix 5*.



**Figure 2:** Sample path of standard Brownian motion

According to Kijima (2003), paths as functions of  $t$ ,  $W_{t,\omega}$  for  $\omega \in \Omega$ , have the following properties: Almost every sample path

- (i) is a continuous function of time  $t$ ,
- (ii) is nowhere differentiable, that is, not differentiable at any point of time,
- (iii) has infinite variation on any interval, no matter how small the interval is, and
- (iv) has quadratic variation on  $[0, t]$  equal to  $t$ , for any  $t \in [0, T]$ .

In the following definition and the associated corollary we introduce the concept of *martingales* which is essential for proving the verification theorem (Theorem 3.5.2) in Chapter 3.

**Definition 1.1.23**

**Martingale:** Consider a real-valued  $\mathcal{F}_t$ -adapted stochastic process  $X = \{X_t\}_{t \in \mathbb{R}^+}$  satisfying

$\mathbb{E}[|X_t|] < \infty$  for all  $t \geq 0$ . Then, for  $0 \leq s < t$ ,

(i)  $X$  is a *supermartingale* if  $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$ .

(ii)  $X$  is a *submartingale* if  $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$ .  $X$  is a submartingale if  $-X$  is a supermartingale.

(iii)  $X$  is a *martingale* if  $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ .

(Calin, 2012)

### Corollary 1.1.6

A process  $X = \{X_t\}_{t \in \mathbb{R}^+}$  is a martingale iff it is both a supermartingale and a submartingale (Calin, 2012).

### Remark 1.1.7

The martingale property basically asserts that the best prediction of the value or expected value of a stochastic process  $X$  at some future time  $t$  given the information available up to time  $s \leq t$  is precisely the value of the process at time  $s$ .

### Definition 1.1.24

**Semimartingale:** A stochastic process  $X$  is called a *continuous semimartingale* if  $\exists$  processes  $A \in \mathcal{V}_0^c$  (the family of all continuous and adapted processes with paths of finite variation which vanish at  $t = 0$ ) and  $M \in \mathcal{M}_0^{loc,c}$  (the family of continuous local martingales with  $M_0 = 0$ ) s.t.

$$X_t = X_0 + A_t + M_t \quad \forall t \geq 0 \text{ a.s.} \quad (1.1.1)$$

(Kountzakis, 2015)

The following theorem has been widely proved (see, for example, Calin, 2012):

### Theorem 1.1.8

A one-dimensional Brownian motion  $W = \{W_t\}_{t \in \mathbb{R}^+}$  is a martingale.

**Remark 1.1.9**

- (i) Brownian motion, and later Brownian motion with drift, was proposed by Bachelier in 1900 as a model for stock prices. However, this model had one drawback, namely, that the stock prices could assume negative values (see Fig. 2). For this reason, mathematicians and financial analysts have since resorted to the geometric Brownian motion as a better model for stock prices. However, this is outside the scope of this dissertation and is therefore not considered here.
- (ii) A standard Brownian motion is actually a special case of a Brownian motion with drift coefficient  $\mu = 0$  and diffusion or volatility coefficient  $\sigma = 1$  (Constantinescu, 2006).

**Definition 1.1.25**

**Itô process:** A continuous semimartingale  $\{X_t\}_{t \in \mathbb{R}^+}$  (see Definition 1.1.24) is said to be an *Itô process* if  $\exists$  progressively measurable processes  $\{\mu_t\}_{t \in \mathbb{R}^+}$  and  $\{\sigma_t\}_{t \in \mathbb{R}^+}$  s.t.  $\int_0^t (|\mu_s| + \sigma_s^2) ds < \infty$ , a.s., and

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s. \quad (1.1.2)$$

where  $X_0$  is  $\mathcal{F}_0$ -measurable,  $(\mu_t)_{0 < t < T}$  and  $(\sigma_t)_{0 < t < T}$  are  $\mathcal{F}_t$ -adapted processes,  $\int_0^T |\mu_s| ds < \infty$   $\mathbb{P} - a.s.$  and  $\int_0^T |\sigma_s|^2 ds < \infty$   $\mathbb{P} - a.s.$  (Haugh, 2010; Zitkovic, 2015). In differential notation,  $dX_t = \mu_t dt + \sigma_t dW_t$ .

**Theorem 1.1.10**

**1-dimensional Itô's formula:** Let  $X_t$  be an Itô process as given in Definition 1.1.25. Let  $g(t, x) \in C^2([0, \infty) \times \mathbb{R})$ , i.e.,  $g$  is twice continuously differentiable on  $[0, \infty) \times \mathbb{R}$ . Then

$$Y_t = g(t, X_t)$$

is also an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t) dt + \frac{\partial g}{\partial x}(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2 \quad (1.1.3)$$

where  $(dX_t)^2 = (dX_t) \cdot (dX_t)$  is computed according to the rules

$$dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0, \quad dW_t \cdot dW_t = dt$$

(Øksendal, 2003)

The following example, which is given in Øksendal (2003), shows the application of Itô's 'multiplication rules'. Consider the Itô integral  $I = \int_0^t W_s dW_s$ . Choose  $X_t = W_t$  and  $g(t, x) = \frac{1}{2}x^2$ . Then  $Y_t = g(t, W_t) = \frac{1}{2}W_t^2$ , so that by Itô's formula

$$dY_t = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dW_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} (dW_t)^2 = W_t dW_t + \frac{1}{2} dt$$

Hence,  $d(\frac{1}{2}W_t^2) = W_t dW_t + \frac{1}{2} dt$ , i.e., by direct integration,  $\frac{1}{2}W_t^2 = \int_0^t W_s dW_s + \frac{1}{2}t$ .

### Definition 1.1.26

**Compound Poisson process:** A stochastic process  $\{S_t\}_{t \in \mathbb{R}^+}$  is called a *compound Poisson process* if it can be represented by  $S_t = \sum_{i=1}^{N_t} X_i, t \geq 0$ , where  $\{N_t\}_{t \in \mathbb{R}^+}$  is a (homogeneous) Poisson process and  $X_1, X_2, \dots$  are i.i.d. random variables that are also independent of  $\{N_t\}_{t \in \mathbb{R}^+}$ . (Kijima, 2003)

### Remark 1.1.11

- (i) Depending on the choice of the counting process  $\{N_t\}$ , there are different models for the total claim amount process  $S_t$  (see Definition 1.1.27). For example, in the Cramér-Lundberg model, where  $\{N_t\}$  is a homogeneous Poisson process,  $S_t$  is modelled as a compound Poisson process. Another prominent model for  $S_t$  is called the renewal or Sparre-Andersen model, where  $\{N_t\}$  is a renewal process Mikosch, 2004).
- (ii) The information about the asymptotic growth of the total claim amount enables one to give advice as to how much premium should be charged in a given time period in order to avoid bankruptcy (ruin) in the portfolio. Common classical premium calculation principles include the net or equivalence principle, the expected value principle, the variance

principle and the standard deviation principle, all of which are discussed in Section 1.1.3 below.

**Definition 1.1.27**

**Total claim amount process:** The *total (or aggregate) claim amount process*  $S_t$  is defined as  $S_t = \sum_{i=1}^{N_t} X_i, t \geq 0$ , where  $\{N_t\}$  is the claim number process defined in Definition 1.1.19 above.

**1.1.2 Claim size distributions**

Collective risk theory disregards individual claims in favour of the *total* gain or loss of the insurance company. Therefore, this section will discuss the issues that are addressed by collective risk theory. A claim size distribution is a distribution that represents the sizes of claims that the insurance company has to pay. In the literature, many different types of distributions have been suggested. They can be grouped into two:

- (i) **Light-tailed distributions.** These are distributions of small claims. A distribution  $F$  is called *light-tailed* iff

$$\int_{\mathbb{R}} e^{\beta x} F(dx) < \infty \text{ for some } \beta > 0. \tag{1.1.4}$$

(1.1.4) implies that light-tailed distributions have finite exponential moments. For any light-tailed distribution  $F$  on the positive half-real line  $\mathbb{R}^+ = [0, \infty)$ , all moments are finite, that is,  $\int_0^\infty x^k F(dx) < \infty \forall k > 0$ . Examples of light-tailed distributions include the exponential (common for its desirable properties), gamma, Weibull (for scale parameter greater than 1) and truncated normal.

- (ii) **Heavy-tailed distributions.** Heavy-tailed distributions are an important tool for actuaries working in insurance where many insurable events have low likelihoods and high severities and the associated insurance policies require adequate pricing and reserving. These are distributions of large claims. Such claim size distributions typically occur in a reinsurance portfolio, where the largest claims are insured. Also, most claims data in general insurance are skewed to the right and therefore call for use of heavy-tailed distributions which are highly right-skewed to model these claims. A distribution  $F$  is said to

be (right-) *heavy-tailed* iff

$$\int_{\mathbb{R}} e^{\beta x} F(dx) = \infty \quad \forall \beta > 0, \quad (1.1.5)$$

that is, iff  $F$  does not possess any positive exponential moment. Examples of heavy-tailed distributions include, among others, Pareto, Burr, Cauchy, Weibull (for scale parameter less than 1) and log-normal distributions (Foss *et al.*, 2013).

This study uses the exponential and Pareto distributions for modelling small and large claims, respectively. This is because the exponential distribution has the advantage of mathematical tractability, while the Pareto distribution function has a computationally simple form, requiring only algebraic calculations and no limit processes, and has a typically heavy long tail. These properties have made the exponential and Pareto families of distributions the distributional families of choice for modelling claim sizes in a variety of actuarial applications (Bahnemann, 2015).

### 1.1.3 Premium calculation principles

One of the control variables of insurance is the setting of insurance premiums that are not only profitable but also affordable. A *premium* is the price of the insurance coverage, that is, a series of regular payments made to the insurer by the policyholder in exchange for protection against random events (Dickson, 2017). Thus, while the insurance company will set the premium in such a way as to create an adequate insurance fund to cover its liabilities at time  $t > 0$ , it must recognize that very high premiums may cause the company to lose customers. This is because other insurance companies in the market might cover the same risks at lower and more attractive premiums.

A basic insurance premium for a policyholder is the expected value of a loss  $X$ , hence the *net premium principle*. But the insurance company normally has to charge premiums that are enough to cover the portion of claims exceeding the expected loss. This necessitates the use by the insurer of a positive amount  $\theta = c - \mathbb{E}[X] > 0$  referred to as the *safety loading*. A *premium calculation principle* is a general rule that assigns a premium  $c$  to any given risk  $X$

(Young, 2004; Heras *et al.*, 2012), that is, it is of the form  $c = \varphi(X)$ , where  $\varphi$  is some function (Dickson, 2017). The following are some of the well-known premium calculation principles used in insurance:

(i) *Net premium principle (NPP)*

$$c = \mathbb{E}[S], \quad (1.1.6)$$

where  $S$  is the aggregate claim amount process and  $\mathbb{E}[S]$  the expected risk (or the mean aggregate claims in the portfolio). This premium calculation principle is also called the *pure risk* premium principle (Dickson, 2017).

(ii) *Expected value premium principle (EVPP)*: This is the most widely studied premium principle and is given by

$$c = \mathbb{E}[S] + \theta\mathbb{E}[S] = (1 + \theta)\mathbb{E}[S], \quad (1.1.7)$$

provided  $\mathbb{E}[S] < \infty$ , where  $\theta > 0$  is the safety loading (or pure premium loading factor) on the premium  $\mathbb{E}[S]$ . When  $\theta = 0$ , we obtain the *NPP* in (a) (Laeven, 2011; Montserrat, 2014).

(iii) *Variance premium principle (VPP)*: Let  $V[S]$  denote the variance of the loss amount. Then

$$c = \mathbb{E}[S] + \theta V[S]. \quad (1.1.8)$$

(Laeven, 2011; Montserrat, 2014)

(iv) *Modified variance premium principle (MVPP)*: Let  $V[S]$  denote the variance of the loss amount. Then

$$c = \mathbb{E}[S] + \theta \frac{V[S]}{\mathbb{E}[S]}. \quad (1.1.9)$$

(Furman and Zitikis, 2007; Kaluszka, 2001)

(v) *Standard deviation premium principle (SDPP)*

$$c = \mathbb{E}[S] + \theta \sqrt{V[S]}. \quad (1.1.10)$$

(Laeven, 2011; Montserrat, 2014)

- (vi) *Zero-utility premium principle (ZUPP)*: This states that premium  $c$  is the solution of the equation

$$u(0) = \mathbb{E}[u(c - S)], \quad (1.1.11)$$

where  $u(\cdot)$  is a utility function satisfying  $u'(x) \geq 0$ ,  $u''(x) \leq 0$  for  $x > 0$  (that is,  $u$  is concave) and  $u$  is not an exponential function (Young, 2004).

- (vii) *Swiss premium principle (SPP)*: If  $z \in [0, 1]$  and  $u$  is a strictly monotonic, continuous function on  $\mathbb{R}$ , the premium  $c$  for an aggregate claim process  $S$  is the solution of

$$u((1 - z)c) = \mathbb{E}[u(S - zc)]. \quad (1.1.12)$$

The case  $z = 1$  and substitution  $u(x) \rightarrow -u(-x)$  leads to the *ZUPP* (Young, 2004).

- (viii) *Exponential premium principle (EPP)*:

$$c = \frac{1}{\theta} \ln \mathbb{E}[e^{\theta S}]. \quad (1.1.13)$$

This principle can be derived from the *ZUPP* by taking the utility function to be exponential. Here,  $\theta = -\frac{d}{dx} \ln \frac{d}{dx} u$ . An extensive treatment of utility theory can be found, for example, in Bühlmann (1970), Bowers *et al.* (1997), Centeno and Guerra (2008) and Li and Zhao (2014).

## 1.2 Problem Statement

Insurance companies are always exposed to the risk of ruin due to the possibility of many small claims or few large ones. They have to strike a balance between satisfying policyholders by maintaining their ability to meet claims as and when they are made and enhancing shareholder value through the periodic redistribution of their wealth in the form of dividends. This study seeks to find the best combination of reinsurance and dividend payment policy for the survival of an insurance company. This is done by modifying the diffusion-perturbed classical com-

pound Poisson risk process to include reinsurance and dividend payments. The problem is to find the optimal dividend strategy, that is, the dividend barrier that maximizes the total expected discounted dividend payments made up to the time of ruin, as well as the optimal reinsurance strategy so that the ruin probability of the firm is minimized. Ruin probability targets will also be imposed to determine the optimal dividend and reinsurance strategies.

### **1.3 Rationale**

If an insurance company reinsures its portfolio using proportional (quota-share) and a non-proportional form of reinsurance (such as excess-of-loss (XL)), what would be the optimal reinsurance policy that minimizes the probability of ultimate ruin? What would be the result if the insurance company also pays out some of its surplus in the form of dividends to its shareholders? Furthermore, what would be the effect of imposing a ruin probability constraint on the optimal reinsurance and dividends? Though many studies have combined reinsurance and dividends in a risk process, few have applied dividend payouts and reinsurance to the diffusion-perturbed classical risk process. This is the motivation for the current study, particularly the use of two reinsurance programmes for one and the same insurance portfolio.

### **1.4 Research Objectives**

#### **1.4.1 General objective**

To develop and analyse a mathematical model for optimization of dividend payouts and reinsurance policies for a diffusion-perturbed classical risk process under a set ruin probability target.

#### **1.4.2 Specific objectives**

- (i) To formulate a dividend-and-reinsurance model based on the diffusion-perturbed classical risk process;

- (ii) To analyse the developed model and solve it analytically and/or numerically;
- (iii) To find the optimal reinsurance policy that maximizes dividend payouts;
- (iv) To determine reinsurance and dividend strategies under a set ruin probability target.

### **1.5 Research Questions**

- (i) How can a dividend-and-reinsurance model based on the diffusion-perturbed classical risk process be formulated?
- (ii) How can the developed model be solved by analytical and/or numerical methods?
- (iii) What is the optimal combinational reinsurance policy for maximization of dividend payouts?
- (iv) What are the reinsurance and dividend strategies under a set ruin probability target?

### **1.6 Significance of the study**

- (i) It will help decision-makers (such as managers of insurance companies) to make meaningful choices of measures for risk reduction and enhancement of shareholder value.
- (ii) It will serve to broaden society's awareness of and appreciation for the role of insurance companies in maintaining a delicate balance between stability and profitability through risk reduction and dividend maximization.
- (iii) It will add to the existing body of knowledge on mathematical applications in the insurance industry and provide a basis for future research on reinsurance and dividend payout policies in the context of general insurance.

## 1.7 Outline of the Study

In this dissertation, we assume that the insurance company has the goals of minimizing its ruin probability and maximizing the dividend payouts to its shareholders under a set ruin probability target. To accomplish these objectives, the company chooses to enter into proportional (quota-share) and non-proportional (excess-of-loss) reinsurance arrangements. The dissertation comprises five chapters. Chapter 1 gives a general background to the study, defines some key terms used in the study and gives the problem statement. The study objectives and research questions are also given in Chapter 1, which also indicates the study's significance and justification, as well as the methodology which was used in undertaking the study. Chapter 2 reviews relevant literature on reinsurance, ruin probability minimization and dividend maximization to provide a theoretical framework for the study. Great care has been taken to define terms and concepts before using them.

The mathematical model used in this dissertation is the subject of Chapter 3 which first describes the model assumptions, variables and parameters. The chapter then introduces the diffusion-perturbed classical risk process as the basic model and describes the model's most important components. We then enrich this model by incorporating quota-share and excess-of-loss reinsurance whereby the retained loss function is  $R(x, k, a) = kx \wedge a$  and the retained (or insurance) premium is  $c^{\bar{R}} = c - (1 + \theta)\lambda\mathbb{E}[(kx - a)^+]$ , where  $\theta$  is the safety loading. The optimization problems are defined and the respective value functions presented in Chapter 3. Finally, stochastic control theory and stochastic calculus are used to derive the HJB equations, the integrodifferential equations and the integral equations which are solved in order to determine the optimal strategies.

Chapter 4 presents the numerical method for solving the integral equations for the survival probability and dividend value functions, as well as numerical results based on selected light- and heavy-tailed claim distributions. The chapter also discusses these results by way of determining the optimal policies to be followed. In the fifth chapter we summarise the dissertation, conclude the study, offer some recommendations for insurance companies and regulators and propose possible extensions to this work. In summary, the dissertation is structured as shown in Table 1 below:

**Table 1:** Outline of the study

| <b>Chapter</b> | <b>Contents</b>  |
|----------------|--|
| 1              | Provides an introduction and some background information on insurance, reinsurance and dividends, the objectives, significance and rationale of the study, as well as the problem statement and the research questions |
| 2              | Reviews relevant existing literature and provides a theoretical framework for the study  |
| 3              | Outlines the methods and materials used in conducting the study  |
| 4              | Presents, analyzes, discusses and interprets the research findings   |
| 5              | Draws appropriate conclusions and offers recommendations based on the research findings from the primary research and the literature review  |

## **1.8 Conclusion**

This chapter has introduced the study by giving a comprehensive background and stating the problem to be solved through this research. The objectives of the study have also been presented, along with the associated research questions. Chapter 2 reviews literature relevant to the application of stochastic optimal control theory to dividend maximization and ruin probability minimization problems for a diffusion-perturbed classical risk model compounded by proportional and excess-of-loss reinsurance.

## CHAPTER TWO

### LITERATURE REVIEW

#### 2.1 Introduction

The purpose of this chapter is to review the relevant literature on optimization problems involving ruin probabilities and dividend payouts in the presence of reinsurance. The study of an insurer's optimal reinsurancedividend strategy has received much attention in the actuarial literature. This review focuses on those studies that deal with the objectives of minimizing the ultimate ruin probability and maximizing dividends paid out to the shareholders. The review will be subdivided into six segments: (a) reinsurance, (b) dividends, (c) reinsurance and ruin probability, (d) reinsurance and dividends, (e) dividends and ruin probability targets and (f) block-by-block method.

#### 2.2 Reinsurance

Reinsurance is one of the many ways of risk-sharing (see Definition 1.1.7). All the different forms of reinsurance serve the same purpose, viz., reducing an excessive number of claims and/or the impact of large claims. Since the risk is shared between the cedent and the reinsurer, the volatility in the insurer's portfolio is significantly reduced. Munich Re (2010) distinguishes between two types of reinsurance arrangements: *facultative reinsurance* which covers individual risks in the insurance portfolio and *treaty reinsurance* which is an obligatory agreement providing reinsurance cover for an entire portfolio of risks or encompassing a block of the cedent's book of business. Table 2 indicates the characteristics of these two types of reinsurance arrangements.

Mikosch (2004) has pointed out that reinsurance treaties are of two types: *Random walk* type reinsurance which includes proportional, excess-of-loss and stop-loss reinsurance, and *extreme value* type reinsurance which includes largest claims and ECOMOR reinsurance (excédent du

coût moyen relatif or ‘excess of the average cost’). *Proportional*, or pro rata, reinsurance is a common form of reinsurance for claims of ‘moderate’ size, and requires the reinsurer to cover a fraction of each claim equal to the fraction of total premiums ceded to the reinsurer. Proportional reinsurance treaties are traditionally subdivided into two forms: quota-share and surplus reinsurance. Quota-share reinsurance is a common type of proportional reinsurance in which the cedent and the reinsurer agree to share claims and premiums in the same proportion which remains constant throughout the portfolio (Dam and Chung, 2017). With surplus reinsurance the reinsurer agrees to accept an individual risk with sum insured in excess of the direct retention limit set by the cedent (Ladoucette and Teugels, 2004).

**Table 2:** Characteristics of facultative and treaty reinsurance

| Facultative (Individual risk)   | Treaty (Book of business)  |
|---|--|
| - Individual risk review  | - No individual risk acceptance by the reinsurer over an extended period of time                         |
| - Right to accept or reject each risk on its own merit  | - Obligatory acceptance by the reinsurer of covered business   |
| - A profit is expected by the reinsurer in the short and long term, and depends primarily on the reinsurer’s risk selection process | - A long-term relationship in which the reinsurer’s profitability is expected, but measured and adjusted |
| - Adapts to short-term ceding philosophy of the insurer   | - Less costly than “per risk” reinsurance  |
| - A facultative certificate is written to confirm each transaction  | - One treaty contract encompasses all subject risks  |
| - Can reinsure a risk that is otherwise excluded from a treaty  |  |
| - Can protect a treaty from adverse underwriting results  |  |

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Munich Re (2010)

Within the framework of non-proportional reinsurance treaties, four forms may be cited. In *excess-of-loss* reinsurance, the reinsurer covers all individual losses exceeding a retention or deductible  $a \in [0, \infty)$ . In a *stop-loss* reinsurance contract, the reinsurer covers losses in the portfolio that exceed a well-defined limit or retention  $K$ . It should be noted that excess-of-loss and stop-loss reinsurance treaties are equivalent when a single risk comes into play (Ladoucette and Teugels, 2004). In *largest claims* reinsurance, at time  $t = 0$  the reinsured amount combines the values of the  $r$  largest claims in the portfolio occurring in the time period  $[0, t]$ . *ECOMOR* reinsurance, which is slightly more popular, may be defined as an excess-of-loss treaty with a random retention or deductible determined by the  $(r + 1)$ -th largest claim in the portfolio.

Hu and Zhang (2015) introduced a general risk model involving dependence structure with common Poisson shocks. Under a combined quota-share and excess-of-loss reinsurance arrangement, they studied the optimal reinsurance strategy for maximizing the insurer's adjustment coefficient and established that excess-of-loss reinsurance was optimal from the insurer's point of view. Zhang and Liang (2016) studied the optimal retentions for an insurance company that intends to transfer risk by means of a layer reinsurance treaty. Under the criterion of maximizing the adjustment coefficient, they obtained the closed-form expressions of the optimal results for the Brownian motion as well as the compound Poisson risk models and concluded that under the expected value principle excess-of-loss reinsurance is better than any other layer reinsurance strategies while under the variance premium principle pure excess-of-loss reinsurance is no longer the optimal layer reinsurance strategy. Both of these studies, however, used the criterion of maximizing the adjustment coefficient rather than minimizing the insurer's ruin probability.

Cani and Thonhauser (2017) studied the surplus process of an insurance company within the Cramér-Lundberg framework with the aim of controlling its performance by means of dynamic reinsurance. With the objective of finding a general dynamic reinsurance strategy that maximizes the expected discounted surplus level integrated over time, they obtained an integrodifferential equation for the problem and showed that for proportional reinsurance it is optimal not to reinsure (i.e., the optimal retention is 1), while for XL reinsurance the optimal strategy is buying no reinsurance (i.e., the optimal retention is  $\infty$ ), followed by taking exactly the initial surplus as the optimal retention and finally taking some constant retention level as the maxi-

mizing criterion for large initial surplus. However, this study considered the optimal combined quota-share and excess-of-loss reinsurance problem under the criterion of maximizing the expected discounted surplus rather than minimizing the insurer's ruin probability.

Zhang *et al.* (2018) investigated the optimal proportional reinsurance strategies that have the potential to result in mutual benefit to both insurer and reinsurer. They obtained explicit expressions for the optimal quota-share retention and the corresponding optimal value function, with results highlighting the key role played by the reinsurer's safety loading in determining the optimal quota-share retention level. Zhang *et al.* (2018) also showed that the mutual benefit for the insurer and reinsurer is guaranteed by utility improvement constraints in that reinsurance arrangements enable both of them to achieve higher expected utility of wealth.

Hu *et al.* (2018) investigated an optimal reinsurance-investment problem involving the interests of both the insurer and reinsurer on the assumption that the insurer's surplus follows a jump-diffusion surplus process and that the insurer purchases proportional reinsurance from the reinsurer, both of them being allowed to invest in a risky and risk-free asset. By applying stochastic control methods, they derived the corresponding HJB equation and obtained the optimal reinsurance-investment strategies for maximizing the exponential utility. Hu *et al.* (2018) established that the insurer's reinsurance strategy is different from that of the reinsurer, and that the optimal reinsurance strategies are dependent on the reinsurer's safety loading and the claim-size distribution used. With regard to investment, the study found that if the price of the risky asset is dependent on the risk model, then the optimal investment strategy depends on both the financial and insurance markets. Otherwise, the optimal investment policy is only influenced by the parameters in the financial market as well as the investor's risk preference. This study, however, only considered proportional reinsurance without addressing dividend payments.

Similar to Hu *et al.* (2018), Wang *et al.* (2018) studied the optimal investment problem for both an insurer and a reinsurer. The insurer's surplus grows according to a jump-diffusion risk model, the insurer can purchase proportional reinsurance from the reinsurer and both of them are allowed to invest in a risky and risk-free asset. However, Wang *et al.* (2018) assumed that the price of the risky asset follows a constant elasticity of variance (CEV) model. With the objective of maximizing the exponential utility, they used stochastic control techniques to

derive the corresponding HJB equations and obtained the respective closed-form expressions for the reinsurance-investment strategies for the insurer and reinsurer. The study established that the insurer's reinsurance strategy is different from that of the reinsurer. In comparison with the geometric Brownian motion model, the optimal investment strategies under the CEV model were found to contain an additional modification factor, reflecting decisions on the part of the insurer and reinsurer to hedge the volatility risk. However, like Hu *et al.* (2018) this study did not address the problem of dividend maximization.

### 2.3 Dividends

Insurance companies have to strike a balance between risk control (through reinsurance) and payment of a portion of their surplus to the shareholders. Such payments are referred to as *dividends* (see also Definition 1.1.5). However, in distributing dividends to shareholders, two fundamental questions have to be addressed, viz., *when* dividends should be distributed, and *how much* of the surplus should be distributed. The answer to these questions is referred to as a dividend strategy (Avanzi, 2009), the most common dividend strategies being the band, barrier, threshold and impulse strategies.

A band strategy is characterized by three sets which partition the state space of the reserve process. Each set is associated with a certain dividend payment action for the current reserve (Albrecher and Thonhauser, 2009), meaning that under this strategy dividends are paid according to the band in which the surplus is located. Further discussions on band strategies can be found in Azcue and Muler (2014). A barrier strategy  $b$ , first proposed by de Finetti (1957), is a special type of band strategy comprising only two bands, viz.,  $A = (0, b]$  and  $B = (b, \infty)$  (Avanzi, 2009; see also Definition 1.1.6). In this strategy, all the reserve above  $b$  is paid out immediately and subsequently all incoming premiums that lead to a surplus above  $b$  are immediately distributed as dividends. This is the optimal strategy in the absence of fixed transaction costs, that is, there is a level  $b^*$  so that whenever the surplus exceeds  $b^*$ , the excess is paid out as dividends (Paulsen, 2007; Zou *et al.*, 2009).

A threshold strategy involves payment of dividends continuously at a rate  $c$  whenever the current reserve is above level  $b$ . This strategy may be extended by fixing multiple thresholds  $b_i$  and associated intensities or rates  $c_i$ . A simple type of impulse strategy involves fixing two levels  $b_1$  and  $b_2$  with  $b_2 > b_1 \geq 0$ . If the surplus is above or equal to  $b_2$ , then pay out the amount  $b_2 - b_1$  immediately, otherwise do nothing until the reserve reaches the level  $b_2$  again (Avanzi, 2009; Albrecher and Thonhauser, 2009). The optimal dividend and reinsurance problem has generated much interest among researchers in recent years. Most of the references considered here typically deal with the dividend flow as a controlled process with the objective of maximizing the expected total discounted dividends paid out to shareholders up to the company's ruin. The ruin time is a stopping time (see Definitions 1.1.4 and 1.1.15), representing the moment at which the reserve process hits zero.

Liang and Yao (2010) studied the optimization of dividend and reinsurance strategies under a ruin probability constraint, applying only proportional reinsurance. Løkka and Zervos (2008) considered the optimal dividend and equity issuance strategies in the diffusion model without reinsurance. They concluded that whenever there is a deficit it is optimal to inject capitals to guarantee no bankruptcy when the costs of collecting capitals are relatively low, otherwise the company should not inject capitals and should allow bankruptcy to take place. But it needs to be noted that this study did not consider any reinsurance; the only risk control measure included was the issuance of equity.

Belhaj (2010) studied the optimal dividend problem in a jump-diffusion model for a firm faced with two types of liquidity risks, viz., a Brownian risk and a Poisson risk, and showed that the optimal dividend policy was of barrier type. A similar and more recent study is that of Wenyuan (2016) who obtained a linear barrier type optimal dividend strategy. Kasozi *et al.* (2011) used homotopy analysis method (HAM) to maximize dividend payments in the Cramér-Lundberg model under a barrier strategy but found that the HAM was not convergent when applied to a model with stochastic return on investments.

Zhang and Cheung (2018) studied the optimal dividend problem for an insurance company whose surplus evolves according to a spectrally-negative Lévy process with a barrier-type dividend strategy whereby dividend payments are made discretely rather than continuously. They

derived expressions for the Gerber-Shiu expected discounted penalty function and the moments of total discounted dividends payable until ruin. They determined the optimal dividend barriers and found that for exponential claims the optimal barrier for the Brownian motion model was lower than that for the compound Poisson model. For each model, the expected discounted dividends prior to ruin first increase and then decrease in the barrier level  $b$ , and the optimal barriers are independent of the initial surplus. However, apart from dealing with discrete rather than continuous dividend payments, the study by Zhang and Cheung (2018) differs from the current study in that it does not include reinsurance and is not based on a jump-diffusion risk process.

Along the lines of Zhang and Cheung (2018), Noba *et al.* (2018) considered the optimal periodic dividend problem for Lévy risk processes and showed that for exponentially distributed jumps the completely monotone assumption on the Lévy density is a sufficient condition for the optimality of a periodic barrier strategy. Noba *et al.* (2018) also expressed the optimal strategies and value functions in terms of scale functions. Junca *et al.* (2018) considered the de Finetti problem for spectrally one-sided Lévy processes with control strategies that are absolutely continuous with respect to the Lebesgue measure. They solved the optimal dividend problem with terminal cost and the absolutely continuous assumption for the case of a spectrally-negative Lévy process with a completely monotone Lévy density. Junca *et al.* (2018) extended their results to the solution of constrained dividend problems and spectrally-positive processes. Zhao *et al.* (2017) investigated an optimal dividend and capital injection problem for a spectrally-positive Lévy process for the restricted dividends case and obtained the optimal strategy and explicit optimal value function. Numerical results for hyper-exponential and Gamma distributed compound Poisson positive jumps were also given.

Bai *et al.* (2017) studied a class of optimal dividend and investment problems for an insurance company whose underlying reserve process evolves according to a Sparre Andersen model in which the claim frequency is modelled by a renewal process rather than a standard compound Poisson process. By means of the dynamic programming principle they derived the associated HJB equation and showed that the value function is the unique constrained viscosity solution to the HJB equation on a cylindrical domain on which the problem is defined. Bai *et al.* (2017) also assumed that the financial market in which the company invests is described by the stan-

standard Black-Scholes model. However, this study focused on the Cramér-Lundberg model only and did not take into account reinsurance of any kind. Pérez, Yamazaki and Yu (2018) investigated the dividend maximization problem with capital injection under the constraint that the cumulative dividend strategy is absolutely continuous and based on a general spectrally-negative Lévy process. They obtained the optimal strategy and value function which they expressed in terms of the scale functions. They showed that the solution is a refraction-reflection strategy that reflects the surplus from below at zero and decreases the drift at a suitable threshold.

## 2.4 Reinsurance and ruin probability

Schmidli (2002) studied the problem of minimizing the probability of ruin by investment and reinsurance. He considered a classical risk model and allowed investment into a risky asset modelled as a Black-Scholes model as well as proportional reinsurance. Using the HJB approach, he found optimal levels of investment and reinsurance which minimize the ruin probability. Among other things, he found that, for Pareto distributed claim sizes (for the case where the whole insurance risk is reinsured), investment and reinsurance decrease the ruin probability considerably for larger initial capital. But in addition to assuming investment by the insurer, this study used only proportional reinsurance as a control variable.

Under the criterion of maximizing the adjustment coefficient and the expected utility of terminal wealth, Centeno (2005) considered the optimal excess-of-loss retention limits for two dependent classes of insurance risks. In the dynamic setting, Bai *et al.* (2013) also sought to find the optimal excess of loss reinsurance that minimizes the ruin probability for the diffusion approximation risk model. Based on the aggregate claims process  $S_t = \sum_{l=1}^2 \left( \sum_{i=1}^{N_l(t)+N(t)} X_i^{(l)} \right)$  and using the variance premium principle, Liang and Yuen (2014) studied the optimal proportional reinsurance problem for both the compound Poisson and diffusion approximation risk models. They found that the optimal reinsurance strategies of the two classes of insurance business are the same, and that the claim size distributions and the counting processes have no effect on the optimal reinsurance strategy.

Dickson and Waters (2006) focused on a dynamic reinsurance strategy to minimize the ruin probability. They derived an expression for the finite time ruin probability for discrete and continuous time by using the Bellman optimality principle. Moreover, they showed how the optimal strategies are determined, respectively, by approximating the compound Poisson aggregate claims distributions by translated gamma distributions and by approximating the compound Poisson process by a translated gamma process. Liu and Yang (2004) reconsidered the model in Hipp and Plum (2000) and incorporated a risk-free interest rate. Since closed-form solutions could not be obtained in this case, they provided numerical results for optimal strategies for maximizing the survival probability under different claim-size distribution assumptions.

More recently, taking ruin probability as a risk measure for the insurer, Li *et al.* (2015) investigated a dynamic optimal reinsurance problem with both fixed and proportional transaction costs for an insurer whose surplus process is modelled by a Brownian motion with positive drift. Under the assumption that the insurer takes non-cheap proportional reinsurance, they formulated the problem as a mixed regular control and optimal stopping problem and established that the optimal reinsurance strategy was to never take reinsurance if proportional costs were high and to wait to take the reinsurance when the surplus hits a level. Additionally, they obtained the explicit expression for the survival probability under the optimal reinsurance strategy and found it to be larger than that with the aforementioned strategies.

In their study on finite-time ruin probabilities in a risk model under quota-share reinsurance, Dam and Chung (2017) investigate the effect of quota-share reinsurance on the insurer's risk model  $U_n = u + \alpha \sum_{i=1}^n Y_i - \alpha \sum_{i=1}^n X_i$ , where  $u$  is the insurer's initial surplus,  $Y_i$  represents the premium income in the  $i$ -th period,  $X_i$  is the claim amount in the  $i$ -th period and  $\alpha \in [0, 1]$  is the cedent's retention level for quota-share reinsurance. Assuming that the premium incomes and claim amounts take values in finite sets of non-negative numbers, Dam and Chung (2017) derived the explicit formula for the joint ruin probability of the cedent and the reinsurer and obtained upper bounds of ruin probabilities of both companies by martingale and inductive methods. In particular, they showed that the QS retention level that minimizes the joint ruin probability and the upper bound of ruin probabilities for both companies is  $\alpha^* = \frac{u}{u+v}$ , where  $u$  and  $v$  are, respectively, the initial surpluses of the cedent and reinsurer.

Liang and Young (2018) considered the problem of minimizing the ruin probability for an insurance company that invests in two risk-free assets (a bank account and a bond), purchases proportional reinsurance and pays proportional transaction costs on the sale and purchase of bonds. On the assumption that the company's surplus is modelled by a Brownian motion with drift compounded by proportional reinsurance and investments in a money market and bond and incorporating transaction costs, they obtained the optimal investment and reinsurance strategies.

## **2.5 Reinsurance and dividends**

Nansubuga *et al.* (2016) considered maximization of dividend payouts under infinite ruin probability constraints. Using a diffusion-perturbed classical risk process as their basic model, they studied the dividend maximization problem in the presence of investments of Black-Scholes type and with a ruin probability constraint. They obtained the expected present value of the dividends and derived the corresponding Volterra integral equations (VIEs). They solved these VIEs using the block-by-block method, which is fully developed in Paulsen *et al.* (2005), and established the optimal barrier to use to pay dividends provided the ruin probability is no larger than the predetermined tolerance. This study incorporated investment as a risk control measure but did not consider any reinsurance.

Chen and Yuen (2016) studied the optimal dividend and reinsurance problem in the limit diffusion setting, under the assumptions that the dividend payments were subject to transaction costs and that the company could take out two reinsurance policies with two different reinsurers. But due to the complexity of the problem, they considered proportional reinsurance only. Chikodza and Esunge (2014) investigated a combined dividend and proportional reinsurance regular-singular control problem in the context of jump diffusions, but based on a model different from the one considered in this dissertation.

Yao *et al.* (2016) investigated a combined optimal financing, reinsurance and dividend distribution problem for a large insurance portfolio in a company that can control its surplus by buying proportional reinsurance, paying dividends and raising money dynamically. Similarly to Cheng

and Zhao (2016), they assumed non-cheap reinsurance and their work also included transaction costs and liquidation or terminal values at bankruptcy. In contrast to Cheng and Zhao (2016) who assumed that the reinsurance premium is calculated using the variance principle, Yao *et al.* (2016) assumed the expected value principle. Similar to Wu (2013) and motivated by Centeno (1985), Zhang *et al.* (2007) considered optimal combinational quota-share and excess-of-loss reinsurance policies for a Cramér-Lundberg model but without dividend payments to shareholders. Then the ruin probability is  $\psi(x) = f(x)$  and the optimal strategy is  $(1, m^*)$ , that is, a pure excess-of-loss. But, as has already been noted, their model does not include dividend payouts.

Højgaard and Taksar (1999) studied the problem of risk control and dividend distribution policies for a diffusion model compounded by proportional reinsurance. They sought to find a policy that maximizes the expected discounted dividends paid until ruin. They assumed that the insurance company chooses to invest part of its surplus in a risky asset and to control its risk exposure through the choice of business activity. They showed that the optimal value function is concave and found that a barrier strategy is optimal. Asmussen *et al.* (2000) considered the issue of optimal risk control and dividend distribution policies under excess-of-loss reinsurance which is the most common in the reinsurance industry. They used a diffusion approximation for the reserve process and reparametrized the problem by considering the drift term as the basic control parameter, which leads to a mixed regular/singular stochastic control problem. They then derived a Hamilton-Jacobi-Bellman variational inequality (HJBVI) in the case of unbounded rate of dividends and proved that the value function is a classical solution of the associated HJBVI. They constructed the solution in the case of unbounded and bounded support of the distribution of the claims. Their conclusion was that the optimal excess of loss reinsurance is always better than the optimal proportional reinsurance and this conclusion coincides with Bai *et al.* (2010). However, similar to the present study, Asmussen *et al.* (2000) did not assume transaction costs and taxes when dividends are paid out.

Although Bai *et al.* (2010) also studied the optimal excess-of-loss reinsurance and dividend problem, by contrast they used the classical risk process as their basic model and allowed for transaction costs as well as taxes in the payment of dividends. They formulated the problem as a stochastic impulse control problem. By solving the corresponding quasi-variational inequal-

ity, they obtained analytical solutions for the optimal return function and the optimal strategy. Also using a diffusion approximation to the classical risk process, Wu (2013) considered a combined quota-share and excess-of-loss reinsurance and dividends in the presence of capital injections. He showed that the optimal combinational reinsurance strategy is pure excess-of-loss reinsurance (that is,  $k_t^* = 1$ ).

In their study, Liang and Palmowski (2018) considered the problem of maximizing the expected discounted utility of dividend payments for an insurance company that controls its risk exposure through the purchase of proportional reinsurance and whose reserve process evolves in time according to a diffusion process. They obtained the optimal surplus thresholds and found that the optimal threshold for non-cheap reinsurance is lower than that for cheap reinsurance. They further established that in the non-cheap reinsurance case it is optimal for the insurer not to take reinsurance when the surplus is larger than the optimal threshold. In the cheap reinsurance case the insurer is more willing to divert its claim risks to the reinsurer even when its own reserves are relatively large. Furthermore, Liang and Palmowski (2018) also established that the optimal dividend rate and reinsurance retention level are lower for non-cheap than for cheap reinsurance.

In the literature, studies on the dividend-reinsurance problem may be categorised into those dealing with the Cramér-Lundberg model or classical risk process and those dealing with the diffusion-perturbed model. For the classical risk process, the optimal dividend-proportional reinsurance problem has been considered by Azcue and Muler (2005) and Liang and Yao (2010), while the dividend-XL reinsurance problem has been studied by Asmussen *et al.* (2000). For the diffusion risk model, the optimal dividend-proportional reinsurance problem has been studied by Højgaard and Taksar (1999), while the dividend-XL reinsurance problem has been studied by Mnif and Sulem (2005). Wu (2013) considered a diffusion approximation to the diffusion-perturbed classical risk process. Other extensions of the optimal dividend problem can be found in the survey studies of Albrecher and Thonhauser (2009) and Avanzi (2009) and references therein.

This study therefore seeks to make a contribution by optimizing dividend payouts and a combination of proportional (quota-share) reinsurance and excess-of-loss reinsurance under a set

ruin probability target. The study thus provides a framework for sustainable combinations of proportional and non-proportional reinsurance programmes and dividend payout policies for insurance companies.

## **2.6 Dividends and ruin probability targets**

This section reviews studies that have emerged in the actuarial literature focusing on dividend maximization under solvency constraints. This is necessitated by the fact that there has to be a delicate trade-off between stability and profitability. Maximizing dividend payments leads to certain ruin (which is unacceptable for the policyholders), while maximizing survival probability results in a reduction in solvency capital, thus making dividend distribution impossible (which is unacceptable for the shareholders). To strike a balance, we seek to maximize dividend payments under a ruin probability constraint or target (Hipp, 2003).

Paulsen (2003) solved the dividend optimization problem of a firm under solvency constraints and showed that the optimal policy is of barrier type. Dickson and Drekcic (2006) considered dividend optimization under a ruin probability constraint but for models different from ours. He *et al.* (2008) studied the optimal control problem for an insurance company that adopts a proportional reinsurance policy under solvency constraints. They gave a rigorous probability proof on the bankrupt probability decreasing with respect to some dividend barrier.

Using the fundamental tool of scale functions and fluctuation theory, Hernández *et al.* (2018) extended the results of Hernández and Junca (2015) for spectrally one-sided Lévy risk processes by introducing a longevity feature in the classical dividend problem through addition of a constraint on the time of ruin of the firm. Hipp (2016) studied control for minimizing ruin probability as well as maximizing dividend payments. In particular, he considered an optimal control problem concerned with maximizing the total expected discounted dividend payments with a ruin constraint and found that a ruin constraint is cheaper when an appropriate reinsurance cover is available. A review of recent studies on optimal reinsurance under ruin probability constraints is given by Karageyik and Şahin (2016).

## 2.7 Block-by-block method

The models studied in this dissertation result in Volterra integral equations (VIEs) of the second kind. As Press *et al.* (1992) have pointed out, there is general consensus that the block-by-block method, first proposed by Young (1954), is the best of the higher order methods for solving VIEs of the second kind. The block-by-block methods are essentially extrapolation procedures which produce a block of values at a time. They are advantageous over linear multistep and step-by-step methods in that they can be of higher order and still be self-starting. Apart from not requiring special starting procedures or values, block-by-block methods have a simple structure, allow for easy switching of step-size and have the ability to compute several values of the unknown function at the same time (Linz, 1985; Katani and Shahmorad, 2012).

In addition, the block-by-block method is chosen in this study over such methods as saddle-point approximation, importance sampling simulation, upper and lower bounds, Fast-Fourier Transform (FFT) and diagonally implicit multistep block (Gatto and Mosimann, 2012; Gatto and Baumgartner, 2016; Baharum *et al.*, 2018) because it is a fourth-order method while most of the other methods are of order less than four. In fact, some of the methods mentioned above are used for directly computing ruin probabilities and not for solving integrodifferential or integral equations.

Other methods have been used to solve integrodifferential equations arising in engineering such as the local Galerkin integral equation and thin plate spline collocation methods for solving second-order Volterra integrodifferential equations (VIDEs) with time-periodic coefficients (Assari and Dehghan, 2018; Assari, 2018). Both of these methods are meshless and therefore do not require any background interpolation. As for the collocation method proposed by Cardone *et al.* (2018), although it has the advantages of variable step-size implementation, high order of convergence, strong stability and a high degree of flexibility, it suffers from the order-reduction phenomenon when applied to stiff problems since it does not have a uniform order of convergence.

In the literature, two-, three- and four-block block-by-block methods have been used to solve Volterra integral equations of the second kind (e.g., Linz (1969) for non-linear VIEs; Saify (2005) for a system of linear VIEs). More recently, Kasozi and Paulsen (2005a) used the two-block block-by-block method to study the flow of dividends under a constant interest force. They derived a linear VIE and applied a fourth-order block by-block method of Paulsen *et al.* (2005) in conjunction with Simpson's rule to solve the Volterra integral equation for the optimal dividend barrier. In another study, Kasozi and Paulsen (2005b) applied a fourth-order block-by-block method to the numerical solution of the VIE for ultimate ruin in the Cramér–Lundberg model compounded by a constant force of interest. More pertinent literature on the block-by-block method is available, for example, in (Paulsen, 2003; Paulsen and Gjessing, 1997).

## **2.8 Conclusion**

Much more literature exists on reinsurance, dividends and ruin probability targets but for this dissertation the literature reviewed in this chapter has been selected as a good representative sample of the literature in this work. This chapter has therefore reviewed the relevant literature on proportional and excess-of-loss reinsurance, ruin probability and dividend payouts. The next chapter outlines the model formulation process and identifies the model to be studied.

## CHAPTER THREE

### MATERIALS AND METHODS

#### 3.1 Introduction

This chapter presents the model assumptions, outlines the model formulation process and derives the model for the study. The chapter also presents the derivation of integrodifferential and integral equations which will later be solved and closes with an outline of the methods and materials to be used in solving the optimization problems.

#### 3.2 Model assumptions

- (i) The company has a *fixed premium rate*, only depending on the safety loadings of the insurer and reinsurer;
- (ii) The insurer takes *cheap reinsurance* (in which the insurer's safety loading is the same as that of the reinsurer);
- (iii) The insurer takes a *combination of quota-share and excess-of-loss reinsurance* on the same insurance portfolio;
- (iv) *No transaction costs* are charged when dividends are paid out (though in practice these cannot be ignored);
- (v) There are *no capital injections* (that is, the insurer's surplus only consists of the initial capital and the premium income up to time  $t$ , less any claim amounts paid out);
- (vi) Dividends are distributed *until ruin* (that is, the ruin time is a stopping time); and
- (vii) Dividend payouts are made under a *set ruin probability target*.

Traditional collective risk theory is concerned with the fluctuations in the surplus of an insurance company. In general, the company's *surplus* process can be expressed quite simply as:

$$Surplus = Initial\ capital + Income - Outflow$$

Income is primarily made up of premium payments received from policyholders but could also include returns arising from investment of the surplus or capital injections by the shareholders. The outflow of payments may be due to settlement of claims to policyholders, dividend payouts to shareholders, corporate debt repayments to financial institutions or transaction costs. This dissertation does not assume investment, capital injection, debt repayment or transaction costs. In other words, the income process of the insurance company comprises the initial capital and the premium income and grows linearly in time at a constant rate. Thus, bearing this and the foregoing assumptions in mind, and allowing for volatility in the surplus, the model can be formulated.

### 3.3 Model Formulation

#### 3.3.1 Basic model without reinsurance and dividends

To lay ground for the study, we assume that all random variables and stochastic quantities are defined on a filtered probability space satisfying the usual conditions (see Definition 1.1.13). In the absence of dividend payouts and reinsurance, the surplus of an insurance company is governed by the diffusion-perturbed classical risk process:

$$U_t = u + ct + \sigma W_t - \sum_{i=1}^{N_t} X_i, \quad t \geq 0 \quad (3.3.1)$$

where  $u = U_0 \geq 0$  is the initial reserve,  $c = (1 + \theta)\lambda\mu > 0$  is the premium rate calculated according to the expected value premium principle (see Section 1.1.3(b)),  $\theta$  is a safety loading,  $\{N_t\}$  is a homogeneous Poisson process with intensity  $\lambda > 0$  as defined in Definition 1.1.20 and  $\{X_i\}$  is an i.i.d. sequence of strictly positive random variables with distribution function

$F$ . The claim arrival process  $\{N_t\}$  and claim sizes  $\{X_i\}$  are assumed to be independent. Here  $\{W_t\}$  is a standard Brownian motion (see Definition 1.1.21) independent of the compound Poisson process, with  $W_t \sim N(0, t)$  and  $\sigma W_t \sim N(0, \sigma^2 t)$ . We assume that  $\mathbb{E}[X_i] = \mu < \infty$  and  $F(0) = 0$ . The Brownian term  $\sigma W_t$  represents random fluctuations in the surplus process; without volatility in the surplus process, (3.3.1) is the well-known Cramér-Lundberg model (CLM) or classical risk process.

### 3.3.2 Model with proportional and excess-of-loss reinsurance

We proceed like in Centeno (1985) where the insurer took a combination of quota-share and XL reinsurance arrangements. Most of the actuarial literature dealing with reinsurance as a risk control mechanism only considers pure quota-share or excess-of-loss reinsurance. However, in reality the insurer has the choice of a combination of the two, hence the use of a combination of quota-share and XL reinsurance in this study. In quota-share reinsurance the reinsurance company covers a fixed ratio of each claim and therefore the retained loss function is  $R(x, k) = kx$  for some retention level  $k \in [0, 1]$ . In excess-of-loss reinsurance a retention level  $a \in [0, \infty)$  is fixed in such a way that, in exchange for a reinsurance premium, the reinsurance company covers the amount of the claim exceeding  $a$ , so that the retained loss function is  $R(x, a) = x \wedge a$ . But if these two reinsurance forms are combined, then the retained loss function takes the form  $R(x, k, a) = kx \wedge a$ . Thus, the diffusion-perturbed risk model compounded by QS and XL reinsurance is given by

$$U_t^{\bar{R}} = u + c^{\bar{R}}t + \sigma W_t - \sum_{i=1}^{N_t} kX_i \wedge a, \quad t \geq 0 \quad (3.3.2)$$

where  $c^{\bar{R}} = c - c^R = c - (1 + \theta)\lambda\mathbb{E}[(kX_i - a)^+]$  represents the insurance premium (see also (3.4.6)). When the retention limit  $a$  of the XL reinsurance is infinite, the treaty becomes a *pure quota-share* reinsurance, while a quota-share level  $k = 1$  makes it a *pure excess-of-loss* reinsurance treaty. Though these two scenarios are somewhat extreme, they are still real possibilities for an insurance company. We assume that the reinsurance is *cheap*, meaning that the reinsurer uses the same safety loading as the insurer.

### 3.3.3 Model with reinsurance and dividends

If we also let the cumulative amount of dividend payouts distributed to shareholders up to time  $t$  be  $D_t^b$ , where  $b$  is a dividend barrier level, then the dividend and reinsurance strategy is a pair of  $\mathcal{F}_t$ -adapted measurable processes  $(\bar{D}, \bar{R})$ . Here,  $\bar{D} = (D_t^b)_{t \in \mathbb{R}^+}$  is a dividend strategy and  $\bar{R} = (R_t)_{t \in \mathbb{R}^+}$  is a reinsurance strategy combining quota-share and XL reinsurance. Thus, given a dividend and reinsurance strategy  $(\bar{D}, \bar{R})$ , the insurer's controlled surplus process now becomes

$$U_t^{\bar{D}, \bar{R}} = U_t^{\bar{R}} - D_t^b \quad (3.3.3)$$

where the process  $U_t^{\bar{R}}$ , defined in (3.3.2), is the insurer's surplus in the presence of reinsurance. The controlled surplus process (3.3.3) has dynamics

$$dU_t^{\bar{D}, \bar{R}} = c^{\bar{R}} dt + \sigma dW_t - d \left( \sum_{i=1}^{N_t} k X_i \wedge a \right) - dD_t^b. \quad (3.3.4)$$

The time of ruin is defined as  $\tau^{\bar{D}, \bar{R}} = \inf\{t \geq 0 | U_t^{\bar{D}, \bar{R}} < 0\}$  and the probability of ultimate ruin is defined as  $\psi^{\bar{D}, \bar{R}} = \mathbb{P}(U_t^{\bar{D}, \bar{R}} < 0 \text{ for some } t > 0)$ . It is necessary at this point to define an admissible strategy.

#### Definition 3.3.1

A dividend strategy  $\bar{D}$  is said to be *admissible* if

- (i)  $0 \leq \tau_1^{\bar{D}}$  and for  $n \geq 1$ ,  $\tau_{n+1}^{\bar{D}} > \tau_n^{\bar{D}}$  on  $\{\tau_n^{\bar{D}} < \infty\}$ .
- (ii)  $\tau_n^{\bar{D}}$  is a stopping time w.r.t.  $\{\mathcal{F}_t\}_{t \in \mathbb{R}^+}$ ,  $n = 1, 2, \dots$  (see Definition 1.1.15).
- (iii)  $\mathbb{P}(\lim_{n \rightarrow \infty} \tau_n^{\bar{D}} \leq T) = 0 \forall T \geq 0$ .

#### Definition 3.3.2

A reinsurance strategy  $\bar{R}$  is said to be *admissible* if the quota-share and excess-of-loss reinsurance retentions  $k$  and  $a$ , respectively, belong to certain sets, that is,  $k \in [0, 1]$ ,  $a \in [0, \infty)$ .

A dividend and reinsurance strategy  $(\bar{D}, \bar{R})$  is therefore said to be *admissible* if it simultane-

ously satisfies Definitions 3.3.1 and 3.3.2 and, additionally, if at any time prior to ruin a lump sum dividend payment is smaller than the size of the available liquidity reserves in the time interval  $[t, t+]$ , i.e.,

$$D_{t+}^b - D_t^b \leq \max\{0, U_t^{\bar{D}, \bar{R}}\} \text{ for } t \leq \tau^{\bar{D}, \bar{R}} \quad (3.3.5)$$

This condition simply means that ruin does not occur through a dividend payout. Furthermore,  $D_t^b$  must be non-negative, non-decreasing, right-continuous with left limits (or càdlàg). Given an admissible dividend and reinsurance strategy  $(\bar{D}, \bar{R}) \in \Pi_u^{D, R}$ , where  $\Pi_u^{D, R}$  is the set of all admissible dividend and reinsurance strategies, the total expected discounted dividends paid out to the shareholders until ruin when the initial capital is  $u \geq 0$  are given by

$$V_u^{\bar{D}, \bar{R}} = \mathbb{E}_u \left[ \int_0^{\tau^{\bar{D}, \bar{R}}} e^{-\delta t} dD_t^b \right] \quad (3.3.6)$$

where  $\delta > 0$  is the constant rate at which dividends are discounted and  $\mathbb{E}_u$  denotes expectation w.r.t.  $\mathbb{P}_u$ , the probability measure conditioned on the initial capital  $U_0 = u$ . Define the optimal value function of this problem as

$$V_u^* = \sup_{(\bar{D}, \bar{R}) \in \Pi_u^{D, R}} V_u^{\bar{D}, \bar{R}} \quad (3.3.7)$$

The problem is formulated as an optimal control problem in which the controls are the retention levels for quota-share and XL reinsurance (that is,  $k$  and  $a$ , respectively) as well as the dividend barrier level  $b > 0$ . Solution of the problem will lead to finding the optimal dividend and reinsurance strategy  $(\bar{D}, \bar{R})^* = (D_t^*, R_t^*)$ , where  $D_t^* = (b^*)$  and  $R_t^* = (k^*, a^*)$ , with  $b^*$  denoting the optimal dividend barrier level and  $k^*$  and  $a^*$  are, respectively, the optimal retention levels for quota-share and XL reinsurance, so that  $V_u^{(\bar{D}, \bar{R})^*} = V_u^*$ .

When  $D_t^b = 0$  then no dividends are paid out, and the problem reduces to a ruin probability minimization problem. For this problem, we define the value function as

$$\psi^{\bar{R}}(u) = \mathbb{P}(U_t \leq 0 \text{ for some } t \geq 0 | U_0^{\bar{R}} = u) = \mathbb{P}(\tau^{\bar{R}} < \infty | U_0^{\bar{R}} = u) \quad (3.3.8)$$

where  $\psi(u)$  is the probability of ultimate ruin under the policy  $\bar{R}$  when the initial surplus is  $u$ .

Then the objective is to find the optimal value function, that is, the minimal ruin probability

$$\psi(u) = \inf_{\bar{R} \in \mathcal{R}} \psi^{\bar{R}}(u) \quad (3.3.9)$$

and optimal policy  $(\bar{R})^* = (k^*, a^*)$  s.t.  $\psi^{\bar{R}^*}(u) = \psi(u)$ . Alternatively, we can find the values of  $k^*$  and  $a^*$  which maximize the probability of ultimate survival  $\phi(u) = 1 - \psi(u)$ , so that the optimal value function becomes

$$\phi(u) = \sup_{\bar{R} \in \mathcal{R}} \phi^{\bar{R}}(u) \quad (3.3.10)$$

where  $\mathcal{R}$  is the set of all reinsurance policies.

### 3.4 Dividend models

In this section, the Hamilton-Jacobi-Bellman (HJB) equation is derived, followed by integrodifferential and integral equations for the value function  $V$ . The HJB equation is a partial differential equation which is central to optimal control theory. The solution to the HJB equation is the value function, which gives the optimal cost-to-go for a given dynamical system with an associated cost function. The equation is a result of the theory of *dynamic programming* pioneered in the 1950s by Richard Bellman and his coworkers (see Øksendal (2003), Chapters 10 and 11 for details). For purposes of simplifying notation, let  $V = V^{\bar{D}, \bar{R}}$  and  $v(u) = V_u^{(\bar{D}, \bar{R})^*} = V_u^*$ . If  $v$  is twice continuously differentiable, then applying standard arguments of stochastic control theory (see, for example, Fleming and Soner, 2006; Pham, 2009), we can show that for any  $u \geq 0$ , the value function fulfills the *dynamic programming principle* (DPP), hence the following lemma whose proof proceeds as in Azcue and Muler (2014:12-14) and is given in *Appendix I*.

#### 3.4.1 Dynamic programming principle

##### Lemma 3.4.1

For any initial surplus  $u > 0$  and any stopping time  $\bar{\tau}$ , the value function  $v$  fulfills the dynamic

programming principle

$$V(u) = \sup_{(\bar{D}, \bar{R}) \in \Pi_u^{D, R}} \mathbb{E}_u \left[ \int_0^{\bar{\tau} \wedge \tau^{\bar{D}, \bar{R}}} e^{-\delta s} dD_s^b + e^{-\delta(\bar{\tau} \wedge \tau^{\bar{D}, \bar{R}})} V \left( U_{\bar{\tau} \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}} \right) \right]$$

### 3.4.2 HJB equation

We now proceed to adapt or recall the Hamilton-Jacobi-Bellman (HJB) equation for the optimization problem (3.3.7) as derived by Azcue and Muler (2014:14-15). The HJB equation is a non-linear *first-order* partial differential equation in the *deterministic* case, while in the *stochastic* case it is a non-linear *second-order* PDE (Yong and Zhou, 1999). Thus, we obtain a second-order derivative-constrained integrodifferential equation which is the infinitesimal version of the DPP and is satisfied for all time-state pairs  $(t, u)$  by the value or cost-to-go function  $v(u) = V_u^{(\bar{D}, \bar{R})^*} = V_u^*$ . The HJB equation is derived on the assumption that the value function  $V$  is continuously differentiable at  $u$  and follows from Lemma 3.4.1 and from the theorem below. In most cases, it is impossible to solve the HJB equation analytically but if it can be solved, it provides an optimal control policy in state-variable feedback or closed-loop form (Lewis *et al.*, 2012).

#### Theorem 3.4.2

The HJB equation corresponding to the optimization problem under consideration is

$$\max\{1 - V'(u), \sup_{R \in \mathcal{R}} \tilde{\mathcal{L}}(V)(u)\} = 0, \quad (3.4.1)$$

with  $V(0) = 0$ . The infinitesimal generator  $\tilde{\mathcal{L}}(V)(u)$  is defined by

$$\tilde{\mathcal{L}}(V)(u) = \frac{1}{2} \sigma^2 V''(u) + c^{\bar{R}} V'(u) - (\lambda + \delta) V(u) + \lambda \int_0^u V(u - kx \wedge a) dF(x) \quad (3.4.2)$$

and  $\mathcal{R}$  is the set of all reinsurance policies.

*Proof.* See Appendix 2. □

The function  $V(u)$  will satisfy (3.4.2) only if  $V(u)$  is strictly increasing, strictly concave and

twice continuously differentiable.

### 3.4.3 Integrodifferential equation

Belhaj (2010) argues that the optimal surplus level is a result of the trade-off between the cost of postponing dividends and the threat of ruin. Jeanblanc-Picqué and Shiryaev (1995) have shown that, for the pure diffusion model, the optimal policy is a barrier policy. In particular, the optimal dividend policy depends only on the level of cash reserves. The firm pays no dividends when  $u < b$ , where  $b$  is the dividend barrier level, and distributes as dividends any cash in excess of  $b$ . The reserves process is reflected at the barrier  $b$ . At this level of cash reserves, the marginal value of paying dividends is equal to the marginal value of retaining earnings inside the firm.

#### Theorem 3.4.3

Let  $\bar{D} = (D_t^b)$  denote the dividend policy corresponding to the barrier level  $b$  and let  $V_b(u) = V_b^{\bar{D}, \bar{R}}(u) = V^{\bar{D}, \bar{R}}(u; b)$ . If  $V_b(u)$  solves

$$\tilde{\mathcal{L}}(V_b)(u) = \delta V_b(u) \quad (3.4.3)$$

on  $[0, b]$  for some dividend barrier  $b$ , together with the conditions  $V_b(u) = 0$  on  $u < 0$ ,  $V_b(0) = 0$  if  $\sigma^2 > 0$ ,  $V_b'(b) = 1$  and  $V_b(u) = V_b(b) + u - b$  on  $u > b$ , then, for  $0 < u \leq \infty$ , the HJB equation (3.4.1) takes the form

$$\sup \tilde{\mathcal{L}}(V_b)(u) = 0. \quad (3.4.4)$$

*Proof.* The equation  $\sup \tilde{\mathcal{L}}(V_b)(u) = 0$  is derived using Itô's formula and as motivated by Schmidli (2008) which can be consulted for details.  $\square$

The following integrodifferential equation follows from Theorem 3.4.3 and refers to equation (3.4.2):

$$\frac{1}{2}\sigma^2 V_b''(u) + c^{\bar{R}} V_b'(u) - (\lambda + \delta) V_b(u) + \lambda \int_0^u V_b(u - kx \wedge a) dF(x) = 0 \quad (3.4.5)$$

This is an integrodifferential equation of Volterra type (VIDE), with the boundary condition  $V_b(u) = 0$  on  $u < 0$ . Solution of this equation will require that it is transformed into a Volterra integral equation (VIE) of the second kind using successive integration by parts.

An optimal strategy is derived from a solution  $(V_b(u), (\bar{D}, \bar{R})^*)$  of equation (3.4.4), with  $\bar{D}^* = b^*$  and  $\bar{R}^* = (k^*, a^*)$ , where  $(k^*, a^*)$  is the point at which the supremum in (3.4.4) is attained. The insurance company has a non-negative net premium income if

$$c \geq (1 + \theta)\lambda\mathbb{E}[(kX - a)^+] \quad (3.4.6)$$

Let  $\underline{a}$  be the excess-of-loss retention value where equality holds:

$$c = (1 + \theta)\lambda\mathbb{E}[(kX - \underline{a})^+] \quad (3.4.7)$$

Recall that  $(kX - a)^+ = X - kX \wedge a = X - \min\{kX, a\}$  is the part of the claim paid by the reinsurance company. Since we are looking for a non-decreasing solution of (3.4.4), we can rewrite it as

$$\sup_{\substack{k \in [0,1], \alpha \in [0,\infty) \\ a > \underline{a}}} \left\{ \frac{1}{2}\sigma^2 V_b''(u) + c\bar{R} V_b'(u) - (\lambda + \delta)V_b(u) + \lambda \int_0^u V_b(u - kx \wedge a) dF(x) \right\} = 0 \quad (3.4.8)$$

### 3.4.4 Integral equation

To compute numerical solutions, we first need to convert the Volterra integrodifferential equation (VIDE) (3.4.5) into a Volterra integral equation (VIE) of the second kind using successive integration by parts. A VIE of the second kind is, in fact, a special case of a general linear integral equation of the form

$$f(u)g(u) + \int_a^{b(u)} K(u, x)g(x)dx = \alpha(u). \quad (3.4.9)$$

where  $K(u, x)$  is the *kernel* and  $\alpha(u)$  the *forcing function* of the integral equation. Both  $K$  and  $\alpha$  are known continuous functions. In this study,  $u \in [0, \infty)$ , so that  $K : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$

and  $\alpha : [0, \infty) \rightarrow \mathbb{R}$ . Also,  $g : [0, \infty) \rightarrow \mathbb{R}$  is the unknown function to be determined. If  $f(u) = 1$  and  $b(u) = u$ , then (3.4.9) becomes a linear Volterra integral equation of the second kind which is the focus of this study and of the following theorem.

**Theorem 3.4.4**

The integrodifferential equation (3.4.5) can be represented as a Volterra integral equation of the second kind

$$V_b(u) + \int_0^u K(u, x)V_b(x)dx = \alpha(u), \tag{3.4.10}$$

where

(i) For  $u \leq \underline{a} < a$ , we have

$$K(u, x) = -\frac{\delta + \lambda\bar{F}(u - kx)}{c^{\bar{R}}} \tag{3.4.11}$$

$$\alpha(u) = V_b(0)$$

with  $\bar{F}(x) = 1 - F(x)$ , when there is no diffusion (that is, when  $\sigma^2 = 0$ ), and

$$K(u, x) = \frac{2\left(c^{\bar{R}} + \lambda G(x, u) - (\lambda + \delta)(u - kx)\right)}{\sigma^2} \tag{3.4.12}$$

$$\alpha(u) = uV_b'(0) \text{ if } \sigma^2 > 0$$

when there is diffusion.

(ii) For  $\underline{a} < a < u$ , we have

$$K(u, x) = -\frac{\delta + \lambda H_1(x, u)}{c^{\bar{R}}} \tag{3.4.13}$$

$$\alpha(u) = V_b(0)$$

with

$$H_1(x, u) = \begin{cases} \bar{F}(u - kx) & kx < a \\ 1 - (F(kx + a) - F(a)) & kx \geq a \end{cases}$$

and  $V_b(u) = V_b(b) + u - b$ , for  $u > b$  when there is no diffusion, and

$$K(u, x) = \frac{2 \left( c^{\bar{R}} + \lambda H_2(x, u) - (\lambda + \delta)(u - kx) \right)}{\sigma^2} \quad (3.4.14)$$

$$\alpha(u) = uV'_b(0) \text{ if } \sigma^2 > 0$$

with

$$H_2(x, u) = \begin{cases} G(u - kx) & kx < a \\ (F(kx + a) - F(a))(u - kx) & kx \geq a \end{cases}$$

and  $G(x) = \int_0^x F(v)dv$  when there is diffusion.

*Proof.* The proof for the case  $u \leq \underline{a} < a$  is similar to the proof of Theorem 2.2 in Paulsen *et al.* (2005) but with  $r = \sigma_R^2 = 0$ ,  $k = 1$  and  $p = c^{\bar{R}}$ . Here, we present the main steps of the proof for the case  $\underline{a} < a < u$  (a complete proof is presented in *Appendix 3*).

Integrating (3.4.5) on  $[0, z]$  w.r.t.  $u$  gives

$$0 = \frac{1}{2}\sigma^2[V'_b(z) - V'_b(0)] + c^{\bar{R}}[V_b(z) - V_b(0)] - (\lambda + \delta) \int_0^z V_b(u)du$$

$$+ \lambda \int_0^z \int_0^u V_b(u - kx \wedge a) f(x) dx du \quad (3.4.15)$$

To simplify the double integral in (3.4.15), we again use integration by parts and Fubini's theorem (Schmidli, 2008) to switch the order of integration and change the properties of the convolution integral. Thus,

$$\int_0^z \int_0^u V_b(u - kx \wedge a) f(x) dx du = \int_0^a F(z - kx) V_b(x) dx + \int_a^z V_b(\nu) [F(\nu + a) - F(a)] d\nu \quad (3.4.16)$$

where  $\nu = u - kx$ . Substituting into (3.4.15) gives

$$\begin{aligned}
\frac{1}{2}\sigma^2 V_b'(z) - \frac{1}{2}\sigma^2 V_b'(0) &+ c^{\bar{R}}V_b(z) - c^{\bar{R}}V_b(0) - (\lambda + \delta) \int_0^z V_b(u)du \\
&+ \lambda \left[ \int_0^a F(z - kx)V_b(x)dx + \int_a^z V_b(\nu)[F(\nu + a) - F(a)]d\nu \right] = 0
\end{aligned} \tag{3.4.17}$$

Replacing  $z$  with  $u$ ,  $\nu$  and  $u$  with  $x$  and  $F(\nu + a)$  with  $F(kx + a)$  gives

$$\begin{aligned}
\frac{1}{2}\sigma^2 V_b'(u) - \frac{1}{2}\sigma^2 V_b'(0) &+ c^{\bar{R}}V_b(u) - c^{\bar{R}}V_b(0) - (\lambda + \delta) \int_0^u V_b(x)dx \\
&+ \lambda \int_0^a F(u - kx)V_b(x)dx + \lambda \int_a^u [F(kx + a) - F(a)]V_b(x)dx = 0
\end{aligned} \tag{3.4.18}$$

Setting  $\sigma^2 = 0$  in (3.4.18) yields the case without diffusion. Consequently, equation (3.4.18) can be written as

$$\begin{aligned}
c^{\bar{R}}V_b(u) - c^{\bar{R}}V_b(0) - (\lambda + \delta) \int_0^u V_b(x)dx &+ \lambda \int_0^a F(u - kx)V_b(x)dx \\
&+ \lambda \int_a^u [F(kx + a) - F(a)]V_b(x)dx = 0
\end{aligned} \tag{3.4.19}$$

Dividing by  $c^{\bar{R}}$  and simplifying gives

$$\begin{aligned}
V_b(u) - \frac{\delta}{c^{\bar{R}}} \int_0^u V_b(x)dx &- \frac{\lambda}{c^{\bar{R}}} \int_0^a \bar{F}(u - kx)V_b(x)dx \\
&- \frac{\lambda}{c^{\bar{R}}} \int_a^u [1 - (F(kx + a) - F(a))]V_b(x)dx = V_b(0)
\end{aligned} \tag{3.4.20}$$

from which the kernel is  $K(u, x) = -\frac{\delta + \lambda H_1(x, u)}{c^{\bar{R}}}$  with

$$H_1(x, u) = \begin{cases} \bar{F}(u - kx) & x < a \\ 1 - (F(kx + a) - F(a)) & x \geq a \end{cases}$$

and the forcing function is  $\alpha(u) = V_b(0)$  as given by (3.4.13).

For the case with diffusion, repeated integration by parts of equation (3.4.17) on  $[0, u]$  w.r.t.  $z$  yields

$$\begin{aligned} \frac{1}{2}\sigma^2 V_b(u) &- \frac{1}{2}\sigma^2 V_b(0) - \left[ \frac{1}{2}\sigma^2 V_b'(0) + c^{\bar{R}} V_b(0) \right] u + c^{\bar{R}} \int_0^u V_b(z) dz \\ &- (\lambda + \delta) \int_0^u \int_0^z V_b(x) dx dz + \lambda \int_0^u \int_0^a F(z - kx) V_b(x) dx \\ &+ \lambda \int_0^u \int_a^z [F(\nu + a) - F(a)] V_b(\nu) d\nu dz = 0 \end{aligned} \quad (3.4.21)$$

where  $\nu = u - kx$ . Further simplification of (3.4.21) gives

$$\begin{aligned} \frac{1}{2}\sigma^2 V_b(u) &- \frac{1}{2}\sigma^2 V_b(0) - \left[ \frac{1}{2}\sigma^2 V_b'(0) + c^{\bar{R}} V_b(0) \right] u + c^{\bar{R}} \int_0^u V_b(z) dz \\ &- (\lambda + \delta) \int_0^u \int_x^u dz V_b(x) dx + \lambda \int_0^a \int_x^u F(z - kx) V_b(x) dz dx \\ &+ \lambda \int_a^u \int_\nu^u [F(\nu + a) - F(a)] V_b(\nu) dz d\nu = 0 \end{aligned} \quad (3.4.22)$$

Let  $G(x) := \int_0^x F(v) dv$ . Then (3.4.22) becomes

$$\begin{aligned} \frac{1}{2}\sigma^2 V_b(u) &- \frac{1}{2}\sigma^2 V_b(0) - \left[ \frac{1}{2}\sigma^2 V_b'(0) + c^{\bar{R}} V_b(0) \right] u + c^{\bar{R}} \int_0^u V_b(z) dz \\ &- (\lambda + \delta) \int_0^u (u - z) V_b(z) dz + \lambda \int_0^a G(u - kx) V_b(x) dx \\ &+ \lambda \int_a^u [F(\nu + a) - F(a)] (u - \nu) V_b(\nu) d\nu = 0 \end{aligned} \quad (3.4.23)$$

Replacing  $z$  and  $\nu$  with  $x$  in (3.4.23) results in

$$\begin{aligned} \frac{1}{2}\sigma^2 V_b(u) &- \frac{1}{2}\sigma^2 V_b(0) - \left[ \frac{1}{2}\sigma^2 V_b'(0) + c^{\bar{R}} V_b(0) \right] u + c^{\bar{R}} \int_0^u V_b(x) dx \\ &- (\lambda + \delta) \int_0^u (u - x) V_b(x) dx + \lambda \int_0^a G(u - kx) V_b(x) dx \\ &+ \lambda \int_a^u [F(x + a) - F(a)] (u - x) V_b(x) dx = 0 \end{aligned} \quad (3.4.24)$$

which simply leads back to (3.4.10) and (3.4.14) when we replace  $u - x$  with  $u - kx$  in the fifth and seventh terms and  $F(x + a)$  with  $F(kx + a)$  in the seventh term (to take into account

proportional reinsurance with retention level  $k \in [0, 1]$ ) and multiply through by  $\frac{2}{\sigma^2}$ . That is,

$$\begin{aligned}
V_b(u) &+ \frac{2}{\sigma^2} \int_0^u (c^{\bar{R}} - (\lambda + \delta)(u - kx))V_b(x)dx \\
&+ \frac{2\lambda}{\sigma^2} \left[ \int_0^a G(u - kx)V_b(x)dx + \int_a^u [F(kx + a) - F(a)](u - kx)V_b(x)dx \right] \\
&= \frac{\sigma^2(V_b(0) + uV_b'(0)) + 2c^{\bar{R}}uV_b(0)}{\sigma^2}
\end{aligned} \tag{3.4.25}$$

which is a linear VIE of the second kind with  $K(u, x)$  and  $\alpha(u)$  as given in (3.4.14), since  $V_b(0) = 0$  for  $\sigma^2 > 0$  (see Theorem 3.4.3).  $\square$

### 3.5 Ruin probability models

#### 3.5.1 Probability of ultimate ruin

Let  $\tau_u$  denote the first time the surplus process  $U$  becomes negative, that is,  $\tau_u = \{t | U_t < 0\}$  with  $\tau_u = \infty$  if  $U$  always stays positive.

##### Definition 3.5.1

**Ruin probability:** The *probability of ruin* is given by  $\psi(u) = \mathbb{P}(\tau_u < \infty)$ , where  $\tau_u$  is the time of ruin defined above. At an initial surplus  $u$ , the probability of ruin occurring before time horizon  $T$  is  $\psi(t, u) = \mathbb{P}(\tau_u < T)$ .

##### Definition 3.5.2

**Ruin probability at barrier level  $b$ :** Let the surplus process be  $U_t^{\bar{D}, \bar{R}}$ , where  $(\bar{D}, \bar{R}) \in \Pi_u^{D, R}$  is an admissible dividend and reinsurance strategy, and let  $\bar{D}$  incorporate a dividend barrier  $b$ . Then the *probability of ruin at barrier level  $b$*  is defined as

$$\psi(\tau_b^{\bar{D}, \bar{R}} < T) = \mathbb{P}(\tau_b^{\bar{D}, \bar{R}} < T). \tag{3.5.1}$$

At time horizon  $T$  and a ruin tolerance  $\epsilon > 0$ , the ruin probability at barrier level  $b$  is defined as

$$\psi_b(T, b) := \psi_b(T, u)|_{u=b} = \mathbb{P}(\tau_b^{\bar{D}, \bar{R}} < T) = \epsilon \tag{3.5.2}$$

### Definition 3.5.3

**Ultimate ruin probability:** The *probability of ultimate ruin* is the probability that the surplus process ever falls below zero and is mathematically represented as

$$\psi(\tau_u < \infty | U_0 = u) = 1 - \phi(u), \quad (3.5.3)$$

where  $\phi(u)$  is the probability of survival.

The ruin probability is an indication of the long-term viability of the insurance company. Paulsen and Gjessing (1997) have shown that if the equation  $\tilde{\mathcal{L}}(\psi)(u) = 0$ , where  $\tilde{\mathcal{L}}$  is the infinitesimal generator defined in Theorem 3.4.2, has a solution satisfying the boundary conditions

$$\begin{aligned} \psi(u) &= 1 \text{ on } u < 0 \\ \psi(0) &= 1 \text{ if } \sigma^2 > 0 \end{aligned} \quad (3.5.4)$$

$$\lim_{u \rightarrow \infty} \psi(u) = 0$$

then the solution to the equation  $\tilde{\mathcal{L}}(\psi)(u) = 0$  is the probability of ruin.

Minimizing the probability of ultimate ruin  $\psi(u)$  is the same as maximizing the probability of ultimate survival  $\phi(u)$  s.t.  $\tilde{\mathcal{L}}(\phi)(u) = \tilde{\mathcal{L}}(1 - \psi(u)) = 0$  with the boundary conditions

$$\begin{aligned} \phi(u) &= 0 \text{ on } u < 0 \\ \phi(0) &= 0 \text{ if } \sigma^2 > 0 \end{aligned} \quad (3.5.5)$$

$$\lim_{u \rightarrow \infty} \phi(u) = 1$$

## 3.5.2 HJB, integrodifferential and integral equations

### Theorem 3.5.1

Assume that the survival probability  $\phi(u)$  defined by (3.3.10) is twice continuously differentiable on  $(0, \infty)$ . Then  $\phi(u)$  satisfies the HJB equation

$$\sup_{R \in \mathcal{R}} \left\{ \frac{1}{2} \sigma^2 \phi''(u) + c^{\bar{R}} \phi'(u) + \lambda \int_0^u [\phi(u - kx \wedge a) - \phi(u)] dF(x) \right\} = 0, \quad u > 0, \quad (3.5.6)$$

where  $\mathcal{R}$  is the set of all reinsurance policies.

*Proof.* The proof of this lemma is standard (see, for example, Øksendal and Sulem (2005) or Schmidli (2008)).  $\square$

We now present the verification theorem which is essential for solving the associated stochastic control problem.

**Theorem 3.5.2**

Suppose  $\Phi \in C^2$  is an increasing strictly concave function satisfying the HJB equation (3.5.6) subject to the boundary conditions

$$\begin{aligned} \Phi(u) &= 0 \text{ on } u < 0 \\ \Phi(0) &= 0 \text{ if } \sigma^2 > 0 \\ \lim_{u \rightarrow \infty} \Phi(u) &= 1 \end{aligned} \tag{3.5.7}$$

for  $0 \leq u < \infty$ . Then the maximal survival probability  $\phi(u)$  given by (3.3.10) coincides with  $\Phi$ . Furthermore, if  $(\bar{R})^* = (k^*, a^*)$  satisfies

$$\frac{1}{2}\sigma^2\Phi''(u) + c\bar{R}^*\Phi'(u) + \lambda \int_0^u [\Phi(u - k^*x \wedge a^*) - \Phi(u)]dF(x) = 0 \text{ when } 0 \leq u < \infty \tag{3.5.8}$$

then the policy  $(\bar{R})^*$  is an optimal policy, that is,  $\Phi(u) = \phi(u) = \phi^{\bar{R}^*}(u)$ .

*Proof.* Let  $\bar{R}$  be an arbitrary reinsurance strategy and let  $U^*$  be the surplus process when  $\bar{R} = \bar{R}^*$ . Choose  $n > u$  and define  $T = \mathcal{T}_n = \inf\{t|U_t \notin [0, n]\}$ . Note that  $U_{T \wedge t} \in (-\infty, n]$  because the jumps are downwards. The process

$$M_t^1 = \sum_{i=1}^{N_{T \wedge t}} [\Phi(U_{T_i}) - \Phi(U_{T_i-})] - \lambda \int_0^{T \wedge t} \left[ \int_0^{U_s} \Phi(U_s - kx \wedge a) dF(x) - \Phi(U_s) \right] ds \tag{3.5.9}$$

is a martingale. We write

$$\begin{aligned}
\Phi(U_{T \wedge t}) &= \Phi(u) + \Phi(U_{T \wedge t}) - \Phi(U_{T_{N_{T \wedge t}}}) + \sum_{i=1}^{N_{T \wedge t}} [\Phi(U_{T_i^-}) - \Phi(U_{T_{i-1}})] \\
&+ M_t^1 + \lambda \int_0^{T \wedge t} \left[ \int_0^{U_s} \Phi(U_s - kx \wedge a) dF(x) - \Phi(U_s) \right] ds
\end{aligned} \tag{3.5.10}$$

By Itô's formula,

$$\Phi(U_{T_i^-}) - \Phi(U_{T_{i-1}}) = \int_{T_{i-1}}^{T_i} \left[ \frac{1}{2} \sigma^2 \Phi''(U_s) + c \bar{R} \Phi'(U_s) \right] ds + \int_{T_{i-1}}^{T_i} \sigma \Phi'(U_s) dW_s \tag{3.5.11}$$

The corresponding result holds for  $\Phi(U_{T \wedge t}) - \Phi(U_{T_{N_{T \wedge t}}})$ . Thus,

$$\begin{aligned}
\Phi(U_{T \wedge t}) &= \Phi(u) + \int_0^{T \wedge t} \left[ \frac{1}{2} \sigma^2 \Phi''(U_s) + c \bar{R} \Phi'(U_s) \right. \\
&\quad \left. + \lambda \left( \int_0^{U_s} \Phi(U_s - kx \wedge a) dF(x) - \Phi(U_s) \right) \right] ds \\
&+ \int_0^{T \wedge t} \sigma \Phi'(U_s) dW_s + M_t^1
\end{aligned} \tag{3.5.12}$$

Using the HJB equation (3.5.6), we find that

$$\Phi(U_{T \wedge t}) \leq \Phi(u) + \int_0^{T \wedge t} \sigma \Phi'(U_s) dW_s + M_t^1 \tag{3.5.13}$$

and equality holds for  $U^*$ . Let  $\{\mathcal{S}_m\}$  be a localization sequence of the stochastic integral, and set  $\mathcal{T}_n^m = \mathcal{T}_n \wedge \mathcal{S}_m$ . Taking expectations yields

$$\mathbb{E} [\Phi(U_{\mathcal{T}_n^m \wedge t})] \leq \Phi(u) \tag{3.5.14}$$

By bounded convergence, letting  $m \rightarrow \infty$  and then  $t \rightarrow \infty$ , we have  $\mathbb{E} [\Phi(U_{\mathcal{T}_n})] \leq \Phi(u)$ .

Thus, for  $\Phi(0) = 0$

$$\mathbb{P}(\tau < \mathcal{T}_n, U_\tau = 0) + \Phi(n) \mathbb{P}(\mathcal{T}_n < \tau) = \mathbb{E} [\Phi(U_{\mathcal{T}_n})] \leq \Phi(u) \tag{3.5.15}$$

Note that  $\mathbb{P}(\mathcal{T}_n < \tau) \geq \phi^{\bar{R}}(u)$ . Because there is a strategy with  $\phi^{\bar{R}}(u) > 0$ , it follows that  $\Phi(u)$  is bounded. We therefore let  $n \rightarrow \infty$ , yielding  $\mathbb{E}[\Phi(U_\tau)] \leq \Phi(u)$ . In particular, we obtain

$$\phi^{\bar{R}}(u)\Phi(\infty) \leq \phi^{\bar{R}}(u)\Phi(\infty) + \mathbb{P}(\tau < \infty, U_\tau = 0) \leq \Phi(u) \quad (3.5.16)$$

which simplifies to

$$\phi^{\bar{R}}(u) \leq \phi^{\bar{R}}(u) + \mathbb{P}(\tau < \infty, U_\tau = 0) \leq \Phi(u) \quad (3.5.17)$$

since  $\Phi(\infty) = 1$ . For  $U^*$  we obtain an equality. In particular,  $\{\Phi(U_t^*)\}$  is a martingale. It remains to show that  $\mathbb{P}(U_\tau^* \neq 0) = 1$ . Note first from the HJB equation (5.4.12) that  $F(x)$  must be continuous; if not, the integral in (5.4.12) is not continuous. Choose  $\varepsilon > 0$  and consider the strategy  $\bar{R} = \bar{R}^* \mathbf{1}_{u \geq \varepsilon}$ . Let  $T_\varepsilon = \inf\{t | U_t^* < \varepsilon\}$ . By the martingale property,  $\Phi(u) = \Phi(\infty)\mathbb{P}(T_\varepsilon = \infty) + \mathbb{E}[\Phi(T_\varepsilon), T_\varepsilon < \tau < \infty]$  which reduces to

$$\Phi(u) = \mathbb{P}(T_\varepsilon = \infty) + \mathbb{E}[\Phi(T_\varepsilon), T_\varepsilon < \tau < \infty] \quad (3.5.18)$$

the last term of which is bounded by  $\Phi(\varepsilon)\mathbb{P}(T_\varepsilon < \tau < \infty)$ . Since  $F(x)$  is continuous, it must converge to zero as  $\varepsilon \rightarrow 0$ . Because  $\mathbb{P}(T_\varepsilon = \infty) \rightarrow \phi^*(u)$ , it follows that  $\Phi(u) = \phi^*(u)\Phi(\infty)$  or  $\Phi(u) = \phi^*(u) = \phi(u)$ . That is,  $\Phi(u)$  is the optimal value function and  $\bar{R}^* = (\bar{R})^*$  is an optimal policy.  $\square$

The integrodifferential equation corresponding to the optimization problem (3.3.10) immediately follows from Theorem 3.5.2 as:

$$\frac{1}{2}\sigma^2\phi''(u) + c^{\bar{R}}\phi'(u) + \lambda \int_0^u [\phi(u - kx \wedge a) - \phi(u)]dF(x) = 0 \text{ for } 0 \leq u < \infty. \quad (3.5.19)$$

This is a Volterra integrodifferential equation (VIDE) the solution of which will require that it is transformed into a Volterra integral equation (VIE) of the second kind using successive integration by parts. Hence the following theorem.

**Theorem 3.5.3**

The integrodifferential equation (3.5.19) can be represented as a Volterra integral equation of the second kind

$$\phi(u) + \int_0^u K(u, x)\phi(x)dx = \alpha(u), \quad (3.5.20)$$

where

(i) For  $u \leq \underline{a} < a$ , we have

$$K(u, x) = -\frac{\lambda \bar{F}(u - kx)}{c\bar{R}} \quad (3.5.21)$$

$$\alpha(u) = \phi(0)$$

with  $\bar{F}(x) = 1 - F(x)$ , when there is no diffusion (that is, when  $\sigma^2 = 0$ ), and

$$K(u, x) = \frac{2 \left( c\bar{R} + \lambda G(x, u) - \lambda(u - kx) \right)}{\sigma^2} \quad (3.5.22)$$

$$\alpha(u) = u\phi'(0) \text{ if } \sigma^2 > 0$$

when there is diffusion.

(ii) For  $\underline{a} < a < u$ , we have

$$K(u, x) = -\frac{\lambda H_1(x, u)}{c\bar{R}} \quad (3.5.23)$$

$$\alpha(u) = \phi(0)$$

with

$$H_1(x, u) = \begin{cases} \bar{F}(u - kx) & kx < a \\ 1 - (F(kx + a) - F(a)) & kx \geq a \end{cases}$$

when there is no diffusion, and

$$K(u, x) = \frac{2 \left( c\bar{R} + \lambda H_2(x, u) - \lambda(u - kx) \right)}{\sigma^2} \quad (3.5.24)$$

$$\alpha(u) = u\phi'(0) \text{ if } \sigma^2 > 0$$

with

$$H_2(x, u) = \begin{cases} G(u - kx) & kx < a \\ (F(kx + a) - F(a))(u - kx) & kx \geq a \end{cases}$$

and  $G(x) = \int_0^x F(v)dv$  when there is diffusion.

*Proof.* The proof of this theorem is similar to that of Theorem 3.4.4. □

### 3.6 Numerical solution of (3.4.10) and (3.5.20)

This section discusses the numerical method to be applied in finding numerical solutions of the VIEs (3.4.10) and (3.5.20) using a fixed grid  $u = 0, h, 2h, \dots$ . The numerical solution of the general linear Volterra integral equation of the second kind

$$g(u) + \int_0^u K(u, x)g(x)dx = \alpha(u), \quad (3.6.1)$$

where the kernel  $K(u, x)$  and the forcing function  $\alpha(u)$  are known functions and  $g(u)$  is the unknown function to be determined, is of the form

$$g_n + h \sum_{i=1}^n w_i K_{n,i} g_i = \alpha_n, \quad (3.6.2)$$

where  $g_i$  is the numerical approximation to  $g(ih)$ ,  $K_{n,i} = K(nh, ih)$ ,  $g_n = g(nh)$  and  $\alpha_n = \alpha(nh)$ . The  $w_i$  are the integration weights. The forcing function  $g(u)$  refers to the value function which may be

- (i)  $g(u) = V_b(u)$ : The dividend value function defined by (3.3.7) and obtained as the solution of the VIE (3.4.10).
- (ii)  $g(u) = \phi(u)$ : The ultimate survival probability defined by (3.3.10) and obtained as the solution of the VIE (3.5.20).

Here, the block-by-block method will be used in conjunction with Simpson's rule of integration (which is known to have an error of order 4, see Theorem 3.6.2 and Remark 3.6.3) to obtain solutions in blocks of two values.

To briefly describe the method, Simpson's rule gives, for any  $k \in C^4[ih, (i+2)h]$ ,

$$\int_{ih}^{(i+2)h} k(x)dx = \frac{h}{3} [k(ih) + 4k((i+1)h) + k((i+2)h)] + O(h^5). \quad (3.6.3)$$

Thus, equation (3.6.2) becomes

$$g_2 + \frac{h}{3} [K_{2,0}g_0 + 4K_{2,1}g_1 + K_{2,2}g_2] = \alpha_1. \quad (3.6.4)$$

Here,  $g_1$  is unknown, but using the same rule with gridsize  $\frac{h}{2}$ , Simpson's rule gives

$$g_1 + \frac{h}{6} [K_{1,0}g_0 + 4K_{1,\frac{1}{2}}g_{\frac{1}{2}} + K_{1,1}g_1] = \alpha_1. \quad (3.6.5)$$

Quadratic interpolation gives that  $g_{\frac{1}{2}} \approx \frac{3}{8}g_0 + \frac{3}{4}g_1 - \frac{1}{8}g_2$ . Substituting this into (3.6.5) yields

$$g_1 + \frac{h}{6} \left[ K_{1,0}g_0 + 4K_{1,\frac{1}{2}} \left( \frac{3}{8}g_0 + \frac{3}{4}g_1 - \frac{1}{8}g_2 \right) + K_{1,1}g_1 \right] = \alpha_1, \quad (3.6.6)$$

that is,

$$g_1 + \frac{h}{6} \left[ \left( K_{1,0} + \frac{3}{2}K_{1,\frac{1}{2}} \right) g_0 + \left( K_{1,1} + 3K_{1,\frac{1}{2}} \right) g_1 - \frac{1}{2}K_{1,\frac{1}{2}}g_2 \right] = \alpha_1. \quad (3.6.7)$$

Equations (3.6.4) and (3.6.7) can be solved for  $g_1$  and  $g_2$ . Proceeding in this manner in blocks of 2 with  $w_i = \frac{1}{3}\{1, 4, 2, \dots, 2, 4, 1\}$ , and  $i = 0, 1, \dots, 2m$ , we get

$$\begin{aligned} g_{2m+2} &+ h \sum_{i=0}^{2m} w_i K_{2m+2,i} g_i + \frac{h}{3} (K_{2m+2,2m}g_{2m} + 4K_{2m+2,2m+1}g_{2m+1} + K_{2m+2,2m+2}g_{2m+2}) \\ &= \alpha_{2m+2} \end{aligned} \quad (3.6.8)$$

and

$$\begin{aligned} g_{2m+1} &+ h \sum_{i=0}^{2m} w_i K_{2m+1,i} g_i + \frac{h}{6} \left( K_{2m+1,2m}g_{2m} + 4K_{2m+1,2m+\frac{1}{2}}g_{2m+\frac{1}{2}} + K_{2m+1,2m+1}g_{2m+1} \right) \\ &= \alpha_{2m+1} \end{aligned} \quad (3.6.9)$$

Now,  $g_{2m+\frac{1}{2}}$  is not known, but it may be estimated using quadratic interpolation. Thus, approximating  $g_{2m+\frac{1}{2}}$  by  $\frac{8}{3}g_{2m} + \frac{3}{4}g_{2m+1} - \frac{1}{8}g_{2m+2}$  and inserting this into (3.6.9) yields a pair of linear equations for  $g_{2m+1}$  and  $g_{2m+2}$ .

To solve each block from (3.6.8) and (3.6.9) for  $g_{2m+1}$  and  $g_{2m+2}$ , we make use of their form  $AG = \alpha$ , where the  $2 \times 2$  matrix  $A$  has entries

$$\begin{aligned} a_{11} &= 1 - \frac{h}{2}K_{2m+1,2m+\frac{1}{2}} - \frac{h}{6}K_{2m+1,2m+1} \\ a_{12} &= \frac{h}{12}K_{2m+1,2m+\frac{1}{2}} \\ a_{21} &= -\frac{4h}{3}K_{2m+1,2m+2} \\ a_{22} &= 1 - \frac{h}{3}K_{2m+2,2m+2} \end{aligned} \tag{3.6.10}$$

and vector  $\mathbf{G} = (g_{2m+1}, g_{2m+2})^T$  and vector  $\alpha = (\alpha_1, \alpha_2)^T$  with

$$\begin{aligned} \alpha_1 &= g_{2m+1} + h \sum_{i=0}^{2m} w_i K_{2m+1,i} g_i + \frac{h}{6} \left( K_{2m+1,2m} g_{2m} + \frac{3}{2} K_{2m+1,2m+\frac{1}{2}} g_{2m+\frac{1}{2}} \right) \\ \alpha_2 &= g_{2m+2} + h \sum_{i=0}^{2m} w_i K_{2m+2,i} g_i + \frac{h}{3} K_{2m+2,2m} g_{2m} \end{aligned} \tag{3.6.11}$$

Let  $d = \det A = a_{11}a_{22} - a_{12}a_{21} \neq 0$ . Then

$$\begin{aligned} \mathbf{G} &= A^{-1}\alpha \\ (g_{2m+1}, g_{2m+2})^T &= \frac{1}{d}(\alpha_1 a_{22} - \alpha_2 a_{12}, \alpha_2 a_{11} - \alpha_1 a_{21})^T \end{aligned} \tag{3.6.12}$$

That is,

$$\begin{aligned} g_{2m+1} &= \frac{1}{d}(\alpha_1 a_{22} - \alpha_2 a_{12}) \\ g_{2m+2} &= \frac{1}{d}(\alpha_2 a_{11} - \alpha_1 a_{21}) \end{aligned} \tag{3.6.13}$$

### Definition 3.6.1

**Convergence:** Let  $g_0(h), g_1(h), \dots$  denote the approximation obtained by a given method using step-size  $h$ . Then a method is said to be convergent iff

$$|g_i(h) - g(u_i)| \rightarrow 0, \text{ for } i = 0, 1, 2, \dots, N \tag{3.6.14}$$

as  $h \rightarrow 0, N \rightarrow \infty$ , s.t.  $Nh = a$ .

**Definition 3.6.2**

**Order of convergence:** A method is said to be of order  $q$  if  $q$  is the largest number for which there exists a finite constant  $C$  s.t.

$$|g_i(h) - g(u_i)| \leq Ch^q, \quad i = 0, 1, 2, \dots, \text{ for all } h > 0. \quad (3.6.15)$$

We need to show that the method we use converges and also establish its order of convergence. The following lemma is required as it forms the basis for the theorem that follows (Linz, 1969).

**Lemma 3.6.1**

If  $|\xi_n| \leq A \sum_{i=0}^{n-1} |\xi_i| + B, A > 0, B > 0$ , then  $|\xi_n| \leq B(1 + A)^n$ .

The proof follows immediately by induction. As a corollary we have that, if  $A = hK$  and  $u = nh$ , then

$$|\xi_n| \leq Be^{Ku} \quad (3.6.16)$$

**Theorem 3.6.2**

The block-by-block method is convergent and its order of convergence is four.

The proof of Theorem 3.6.2 is given by Linz (1969).

**Remark 3.6.3**

By *Theorem 3.1* in Paulsen *et al.* (2005) and from results in Chapter 7 of Linz (1985), it follows that for a fixed  $u$  so that  $nh = u$ , the solution satisfies

$$|g_n - g(u)| = O(h^4), \quad (3.6.17)$$

provided that  $g$  is four times continuously differentiable as is the case here by Theorem 2.4 in Paulsen *et al.* (2005). On the other hand, for the block-by-block method  $|g_{2m+2} - g_{2m+1}| = O(h^4)$  as well.

### 3.7 Materials

1. All the data simulations in this dissertation were performed using a Samsung Series 3 PC with an Intel Celeron 847 processor at 1.10GHz and 6.0GB of RAM.
2. To reduce computing time, the numerical method described in Section 3.6 was implemented using the FORTRAN programming language, taking advantage of its DOUBLE PRECISION feature which gives a high degree of accuracy.
3. All the figures were constructed using MATLAB R2016a.

### 3.8 Conclusion

In this chapter, the model for the study was formulated using the basic diffusion-perturbed classical risk process (equation (3.3.1)) as the starting point. Based on this model, the model with proportional and excess-of-loss reinsurance (equation (3.3.2)) and the controlled risk model incorporating reinsurance and dividends (equation (3.3.3)) were formulated. The optimal value functions were also defined: (3.3.7) for the dividend maximization problem and (3.3.10) for the problem of minimizing the ultimate ruin probability. Furthermore, stochastic control theory was used to derive the HJB equations from which the Volterra integrodifferential equations (VIDEs) were deduced. Successive integration by parts was then used to transform these VIDEs into Volterra integral equations (VIEs) of the second kind which were then solved using the block-by-block method based on Simpson's rule. This method was described in Section 3.6. The method transforms the VIEs into systems of algebraic equations which are then solved by matrix methods. Chapter 4 will present numerical results based on light- and heavy-tailed claim size distributions introduced in Section 1.1.2.

## CHAPTER FOUR

### RESULTS AND DISCUSSION

#### 4.1 Introduction

In this chapter, numerical solutions of the VIEs from Theorems 3.4.4 and 3.5.3 are presented for light- and heavy-tailed distributions. Regardless of whether they are light- or heavy-tailed (see Section 1.1.2), probability distributions are characterized by different parameters depending on the particular distribution. A *location parameter* shifts the distribution to the right or left without changing the shape or the volatility. A *scale parameter* and its inverse quantify the dispersion and precision of the random variable, respectively. A *shape parameter* is any parameter that is not changed by changes in the location or scale parameters. It describes the shape of the graph for particular distributions. Often the skewness or tail weight of a distribution can be specified by the shape parameters (Ruppert and Matteson, 2015). The chapter also presents a discussion of the results.

#### 4.2 Numerical results

For a given initial surplus  $u$ , let  $g_h(u)$  be the numerical value of the value function (where  $g$  can be either the dividend value function  $V_b$  or the ruin probability  $\psi$ ) (see Definition 3.5.1) computed when a gridsize  $h$  is used. The  $\text{Exp}(\beta)$  distribution is described by a single shape parameter  $\beta > 0$ . This distribution, which is a special case of the  $\text{Weibull}(\alpha, \beta)$  distribution, has density  $f(x) = \beta e^{-\beta x}$ , with corresponding distribution function  $F(x) = 1 - e^{-\beta x}$ . The mean excess function for the exponential distribution is  $e_F(x) = \frac{1}{\beta}$ ; thus,  $G(x) = x - \frac{1}{\beta}F(x)$ .

Since in the case of the exponential distribution, the exact solution  $g(u)$  is known for the Cramér-Lundberg model (CLM), it is possible to compute the absolute percentage relative

error as

$$D_h(u) = \left| 100 \cdot \frac{g_h(u) - g(u)}{g(u)} \right| \quad (4.2.1)$$

For  $g = V_b$ , we have, for  $u \leq b$ ,

$$V_b(u) = \frac{f(u)}{f'(b)}, \quad (4.2.2)$$

where  $f(u) = (\beta + r_1)e^{r_1 u} - (\beta + r_2)e^{r_2 u}$  (Kasozzi *et al.*, 2011), with the roots  $r_1 < 0 < r_2$  given by

$$r_{1,2} = \frac{-(kc\beta - \lambda - \delta) \pm \sqrt{(kc\beta - \lambda - \delta)^2 + 4kc\beta\delta}}{2kc} \quad (4.2.3)$$

and are derived in Section 4.2.2. Equation (4.2.2) above is the exact solution for exponential claims for the dividend problem on  $u \leq b$  in the CLM for  $k = 1$ , while on  $u > b$  the value function is given by  $V_b(b) + u - b$ .

For  $g = \psi$ , the exact solution for  $\text{Exp}(\beta)$  distributed claims is given in Rolski (1998) and Hipp (2004) by the following equations:

$$\psi(u) = \frac{1}{1 + \Lambda} e^{-\rho u}; \psi(0) = \frac{1}{1 + \Lambda}, \quad (4.2.4)$$

where  $\Lambda = \frac{c\beta}{\lambda} - 1$ . The quantity  $\rho = \beta \frac{\Lambda}{1 + \Lambda}$  is called the *adjustment coefficient* or *Lundberg's constant*. Equation (4.2.4) shows that  $\psi(0)$  depends only on the expected claim size and not on the specific claim size distribution  $F$ . This distribution makes it possible for us to assess the accuracy of the block-by-block method. Hence the exponential distribution is used extensively for small claims in the examples that follow.

However, for modelling large claims, the Pareto distribution will be used. The Pareto( $\alpha, \kappa$ ) distribution, with shape and scale parameters  $\alpha > 0$  and  $\kappa = \alpha - 1 > 0$ , respectively, is a special case of the three-parameter Burr( $\alpha, \kappa, \tau$ ) distribution. This distribution has density  $f(x) = \frac{\alpha\kappa^\alpha}{(\kappa+x)^{\alpha+1}}$  and distribution function  $F(x) = 1 - \left(\frac{\kappa}{\kappa+x}\right)^\alpha$ . The Pareto tail distribution is  $\bar{F}(x) = \left(\frac{\kappa}{\kappa+x}\right)^\alpha$  and  $G(x) = x - 1 + \left(\frac{\kappa}{\kappa+x}\right)^\kappa$ .

In the numerical examples given in this section, we assume  $\text{Exp}(0.5)$  and  $\text{Pareto}(3,2)$  claim sizes. Table 3 shows the parameter values used in these examples.

**Table 3:** Model parameter values

| Parameter | Value         | Reference                      |
|-----------|---------------|--------------------------------|
| $c$       | 6             | Kasozi and Paulsen (2005a)     |
| $\sigma$  | 0.02 (ruin)   | Joseph (2013)                  |
|           | 1 (dividends) | Nansubuga <i>et al.</i> (2016) |
| $\lambda$ | 2             | Kasozi and Paulsen (2005a)     |
| $\delta$  | 0.1           | Nansubuga <i>et al.</i> (2016) |

#### 4.2.1 Dividend models

In addition to managing the company's risk through reinsurance, management is allowed to pay dividends to the shareholders. It is known that under some reasonable assumptions, optimality in the jump-diffusion setting is achieved by using a barrier strategy (see, for example, Yin and Wang, 2009; Kyprianou *et al.*, 2010; Yuen and Yin, 2011). For this reason, throughout this study, as in most papers in the literature, we work with a barrier strategy. We assume that dividends are paid out continuously to the shareholders according to a barrier strategy with level  $b > 0$  and only until ruin. Whenever the surplus exceeds the barrier level  $b$ , the excess or 'overflow' is immediately paid out as dividends. No dividends are paid out when the surplus falls below the barrier level  $b$ . Once the surplus is negative, the company experiences ruin and the process therefore stops. Thus, mathematically, we denote by  $\bar{D} = (D_t^b)_{t \in \mathbb{R}^+} \in \Pi_u^D$  the barrier strategy at level  $b$  which is defined by  $D_0^b = 0$  and

$$D_t^b = \left( \sup_{0 \leq s < t} U_s - b \right) \vee 0 \text{ for } t > 0. \quad (4.2.5)$$

While the term 'dividend' may be understood in a broad sense of including any amount taken from the reserves for such purposes as investment and share repurchases (Rotar, 2015), in this study it is used in the narrower and more ordinary sense of payments periodically made to the shareholders in order to enhance their value in the firm. Shareholder value is enhanced through maximization of the total expected discounted dividend payouts made to the shareholders. Definition 1.1.5 aligns precisely with this understanding.

The various cases of dividend payouts with reinsurance can be derived from equations (3.4.10)-(3.4.14) (Theorem 3.4.4), which represent dividend models compounded by proportional and

excess-of-loss reinsurance, with appropriate values of  $k$ ,  $a$  and  $\sigma$ . For purposes of reference, the controlled surplus process incorporating dividends and reinsurance (see (3.3.3)) is again presented here:

$$U_t^{\bar{D}, \bar{R}} = u + c\bar{R}t + \sigma W_t - \sum_{i=1}^{N_t} kX_i \wedge a - D_t^b. \quad (4.2.6)$$

#### 4.2.2 Dividends in the Cramér-Lundberg model: exponential claim sizes

If  $\sigma = 0$  (no diffusion),  $k = 1$  (no quota-share (QS) reinsurance) and  $a = \infty$  (no XL reinsurance) in the controlled surplus process, then we have dividends in the CLM

$$U_t^{\bar{D}} = u + ct - \sum_{i=1}^{N_t} X_i - D_t^b. \quad (4.2.7)$$

The IDE corresponding to the model (4.2.7), which follows from Itô's formula and the HJB equation of this optimization problem, is

$$cV_b'(u) - (\lambda + \delta)V_b(u) + \lambda \int_0^u V_b(u-x)dF(x) = 0, \quad (4.2.8)$$

By integrating (4.2.8) by parts over  $[0, z]$  w.r.t.  $u$ , it can easily be shown that this IDE transforms into a VIE of the second kind with kernel  $K(u, x) = -\frac{\delta + \lambda \bar{F}(u-x)}{c}$  and forcing function  $\alpha(u) = V_b(0)$ . This is simply equations (3.4.10) and (3.4.13) with  $k = 1$ ,  $a = \infty$  and  $c^{\bar{R}} = c$ . The numerical solution of (3.4.10) (with  $k = 1$  and  $a = \infty$ ) for  $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$  for Exp(0.5) has been given in Nansubuga *et al.* (2016) as well as in Kasozi and Paulsen (2005a). The numerically computed optimal dividend barrier was found to be  $b^* = 10.27344$ .

Assuming that claims are exponentially distributed with parameter  $\beta$ , then by the technique of Belhaj (2010) the IDE (4.2.8) becomes

$$kcV_b'(u) - (\lambda + \delta)V_b(u) + \lambda\beta \int_0^u V_b(u-kx)e^{-\beta x} = 0. \quad (4.2.9)$$

Letting  $\mathcal{I}_e(V_b)(u) = \int_0^u V_b(u - kx)e^{-\beta x} dx$  changes (4.2.9) into

$$kcV_b'(u) - (\lambda + \delta)V_b(u) + \lambda\beta\mathcal{I}_e(V_b)(u) = 0. \quad (4.2.10)$$

Differentiating (4.2.10) w.r.t.  $u$  yields

$$kcV_b''(u) - (\lambda + \delta)V_b'(u) + \lambda\beta\mathcal{I}_e'(V_b)(u) = 0. \quad (4.2.11)$$

But  $\mathcal{I}_e'(V_b)(u) = -\beta\mathcal{I}_e(V_b)(u) + V_b(u)$ , so that (4.2.11) becomes

$$kcV_b''(u) - (\lambda + \delta)V_b'(u) - \lambda\beta[\beta\mathcal{I}_e(V_b)(u) - V_b(u)] = 0. \quad (4.2.12)$$

Multiplying (4.2.10) by  $\beta$  gives

$$kc\beta V_b'(u) - (\lambda + \delta)\beta V_b(u) + \lambda\beta^2\mathcal{I}_e(V_b)(u) = 0. \quad (4.2.13)$$

Adding (4.2.12) and (4.2.13) gives

$$\begin{aligned} kcV_b''(u) + (kc\beta - \lambda - \delta)V_b'(u) - (\lambda + \delta)\beta V_b(u) \\ - \lambda\beta^2\mathcal{I}_e(V_b)(u) + \lambda\beta V_b(u) + \lambda\beta^2\mathcal{I}_e(V_b)(u) = 0. \end{aligned} \quad (4.2.14)$$

which, on simplification, becomes

$$kcV_b''(u) + (kc\beta - \lambda - \delta)V_b'(u) - \delta\beta V_b(u) = 0. \quad (4.2.15)$$

This is a second-order ODE in  $V_b$  with constant coefficients. Its characteristic polynomial is  $P$  given by

$$P(r) = kcr^2 + (kc\beta - \lambda - \delta)r - \delta\beta. \quad (4.2.16)$$

$P(r) = 0$  is a quadratic equation in  $r$  with two roots  $r_1$  and  $r_2$  given by

$$r_{1,2} = \frac{-(kc\beta - \lambda - \delta) \pm \sqrt{(kc\beta - \lambda - \delta)^2 + 4kc\beta\delta}}{2kc}. \quad (4.2.17)$$

The total expected present value of dividends is given by the value function

$$V_b(u) = \begin{cases} \frac{f(u)}{f'(b)} & u \leq b \\ \frac{f(b)}{f'(b)} + u - b & u > b \end{cases} \quad (4.2.18)$$

where  $f(u) = (\beta + r_1)e^{r_1 u} - (\beta + r_2)e^{r_2 u}$  (Kasozi *et al.*, 2011). It is easy to see that the roots of the quadratic equation  $P(r) = 0$  satisfy the relation  $r_1 > 0 > r_2$ . The optimal barrier  $b^*$  is obtained by solving the equation  $f''(b^*) = 0$ , that is,  $(\beta + r_1)r_1^2 e^{r_1 b^*} - (\beta + r_2)r_2^2 e^{r_2 b^*} = 0$ . For any arbitrary starting point  $f(0)$ ,  $f(u)$  is the  $O(h^4)$  numerical solution obtained using the block-by-block method. To find  $f'(b)$ , we use the approximation  $f'(b) \approx \lim_{h \rightarrow 0} \frac{f(b+h) - f(b-h)}{2h}$ , where  $h$  is the grid size. We have solved (4.2.18) for several values of  $b$ . Using a FORTRAN program which, at each run, gives the  $O(h^4)$  solution to the Volterra equations of the second kind (Theorem 3.4.4), we have computed the values of  $f'(b)$ . The results indicate that for any two barriers  $b_1$  and  $b_2$ , with  $0 < b_1 < b_2 < \infty$ , the inequality  $f'(b_1) > f'(b_2)$  holds. Eventually, some interval  $[b_1, b_2]$  gives  $f'(b_1) < f'(b_2)$  for the first time. This interval contains the optimal  $b^*$  which gives the optimal value function  $V_b(b^*)$ .

Kasozi *et al.* (2011) give the optimal value of  $b$ , that is, the value that maximizes the expected discounted dividend payouts, as

$$b^* = \frac{1}{r_1 - r_2} \log \left[ \frac{r_2^2(\beta + r_2)}{r_1^2(\beta + r_1)} \right]. \quad (4.2.19)$$

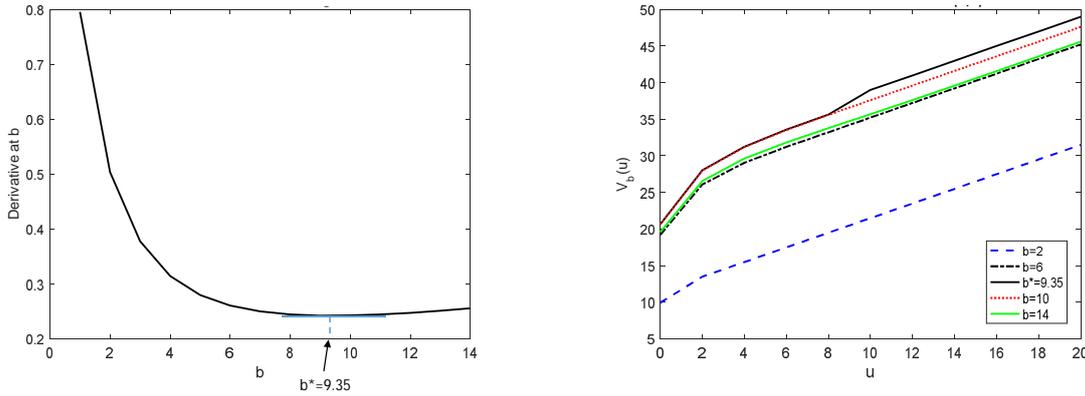
Equation (4.2.18) is the exact solution for exponential claims for the dividend problem in the Cramér-Lundberg model for  $k = 1$  (that is, without proportional reinsurance). The analytical results for Exp(0.5) claims show that the optimal dividend barrier is  $b^* = 10.27011$  as given by Kasozi and Paulsen (2005a). The latter also compared the numerical value function  $V_{b^*}^N(u)$  based on the numerical value of  $b^*$  with the exact value function  $V_{b^*}^E(u)$  based on the exact optimal barrier  $b^*$  by computing the absolute percentage relative error

$$D(u) = \left| 100 \cdot \frac{V_{b^*}^N(u) - V_{b^*}^E(u)}{V_{b^*}^E(u)} \right|. \quad (4.2.20)$$

This comparison showed the excellent performance of the block-by-block method which has also been used in this study.

### 4.2.3 Dividends in the Cramér-Lundberg model: Pareto claim sizes

Using the FORTRAN program BLOCK.for in *Appendix 6*, we obtained the derivatives  $f'(b)$  for Pareto(3,2) claim sizes and plotted them against the values of  $b$ . We then approximated the value of  $b$  at which the derivative was a minimum, as shown in Fig. 3(a). The optimal dividend barrier was found as  $b^* = 9.35$  since it corresponds to the lowest derivative  $f'(b)$ . The dividend value functions for Pareto(3,2) distributed claim sizes are given in Fig. 3(b).



(a) Optimal dividend barrier: Pareto (3,2) claims      (b) Dividend value functions: Pareto(3,2) claims

**Figure 3:** Optimal dividend barrier and value functions for large claims in CLM,  $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$

### 4.2.4 Dividends in the Cramér-Lundberg model compounded by proportional reinsurance

If  $\sigma = 0$  (no diffusion) and  $a = \infty$  (no XL reinsurance) in the controlled surplus process, then we have dividends in the CLM with pure quota-share reinsurance

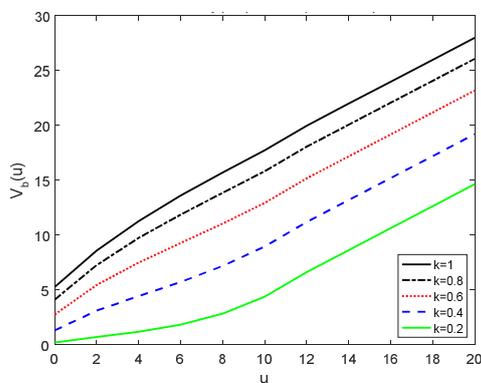
$$U_t^{\bar{D}, \bar{R}} = u + kct - \sum_{i=1}^{N_t} kX_i - D_t^b. \quad (4.2.21)$$

The IDE corresponding to the model (4.2.21), which follows from Itô's formula and the HJB equation of this optimization problem, is

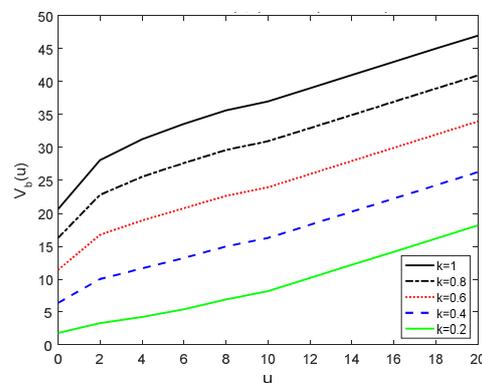
$$kcV_b'(u) - (\lambda + \delta)V_b(u) + \lambda \int_0^u V_b(u - kx)dF(x) = 0. \quad (4.2.22)$$

By integrating (4.2.22) by parts over  $[0, z]$  w.r.t.  $u$ , it can easily be shown that this IDE transforms into a VIE of the second kind with kernel  $K(u, x) = -\frac{\delta + \lambda \bar{F}(u - kx)}{kc}$  and forcing function  $\alpha(u) = V_b(0)$ . This is simply equations (3.4.10) and (3.4.13) with  $a = \infty$  and  $c^{\bar{R}} = kc$ .

- (i) **Exponential claims.** Using the FORTRAN program BLOCK.for, and recalling that the optimal dividend barrier for exponential claims in the CLM is  $b^* = 10.27$ , we found the dividend values as shown in Fig. 4(a). These results show that the optimal retention for quota-share reinsurance is  $k^* = 1$ , that is, it is optimal for the insurance company, in the small claim case, not to take proportional reinsurance.
- (ii) **Pareto claims.** For Pareto claims, the optimal dividend barrier for the CLM was found as  $b^* = 9.35$  and the dividend values for the CLM compounded by proportional reinsurance were found using the FORTRAN program BLOCK.for as shown in Fig.4(b) from which the optimal retention for large claims in the CLM compounded by proportional reinsurance was found to be  $k^* = 1$ . This means that the company should not take proportional reinsurance in the large claim case either.



(a) Dividends in CLM for Exp(0.5) claims



(b) Dividends in CLM for Pareto(3,2) claims

**Figure 4:** Dividend value functions for small and large claims for CLM with QS reins.,  
 $c = 6, \lambda = 2, \delta = 0.1$

#### 4.2.5 Dividends in the Cramér-Lundberg model with excess-of-loss reinsurance

This is the case where there is no diffusion ( $\sigma = 0$ ) and no proportional reinsurance ( $k = 1$ ), so we have dividends in the CLM with pure XL reinsurance, that is,

$$U_t^{\bar{D}, \bar{R}} = u + c^{\bar{R}}t - \sum_{i=1}^{N_t} X_i \wedge a - D_t^b, \quad (4.2.23)$$

where  $c^{\bar{R}} = c - (1 + \theta)\lambda\mathbb{E}[(X_i - a)^+]$ . The corresponding IDE follows from Itô's formula as

$$c^{\bar{R}}V_b'(u) - (\lambda + \delta)V_b(u) + \lambda \int_0^u V_b(u - x \wedge a)dF(x) = 0 \quad (4.2.24)$$

which, by integration by parts, transforms into a VIE of the second kind with kernel  $K(u, x) = -\frac{\delta + \lambda H_1(x, u)}{c^{\bar{R}}}$  and forcing function  $\alpha(u) = V_b(0)$ , with

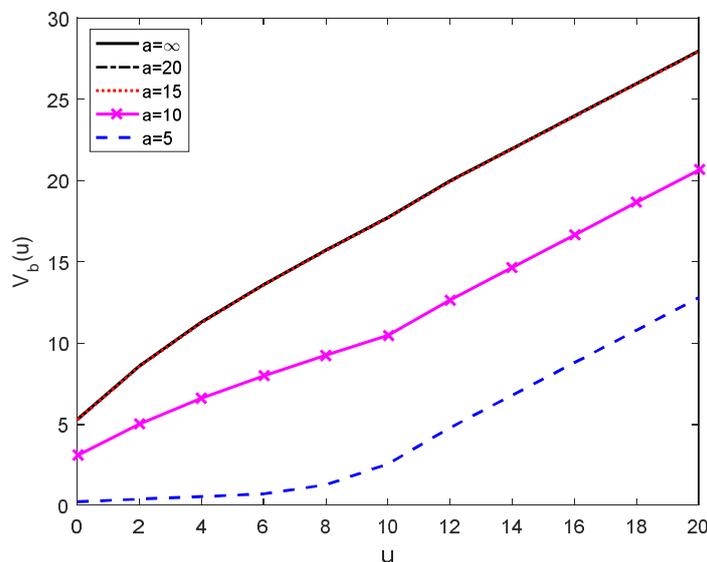
$$H_1(x, u) = \begin{cases} \bar{F}(u - x) & x < a \\ 1 - (F(x + a) - F(a)) & x \geq a \end{cases}$$

- (i) **Exponential claims.** Using the FORTRAN program BLOCK.for with appropriate minor modifications, we obtained the dividend values for the CLM compounded by XL reinsurance. The results are shown in Table 4 which shows that the retention level  $a = \infty$  gives the highest dividend values. Therefore it is optimal not to take XL reinsurance in the CLM when the claim sizes are exponentially distributed. This is also confirmed by Fig. 5, although the graphs for  $a = \infty$ ,  $a = 20$  and  $a = 15$  seem to coincide. In reality, though,  $a = \infty$  yields the highest dividend values, as shown in Table 4.
- (ii) **Pareto claims.** The FORTRAN program BLOCK.for with slight adjustments was used to obtain dividend values for the CLM compounded by XL reinsurance. Figure 6 shows that the dividend values increase consistently up to  $a = 10$  and then drop when  $a = 5$ . Thus, the optimal XL retention  $a^* \in [5, 10]$ . But numerical computations show that the dividend values for  $5 < a < 10$  are much lower than those for  $a = 10$ . Therefore, the optimal retention for large claims in the CLM compounded with XL reinsurance is  $a^* = 10$ .

A comparison of Fig. 6 with Fig. 4(b) shows that, for large claims in the CLM, XL

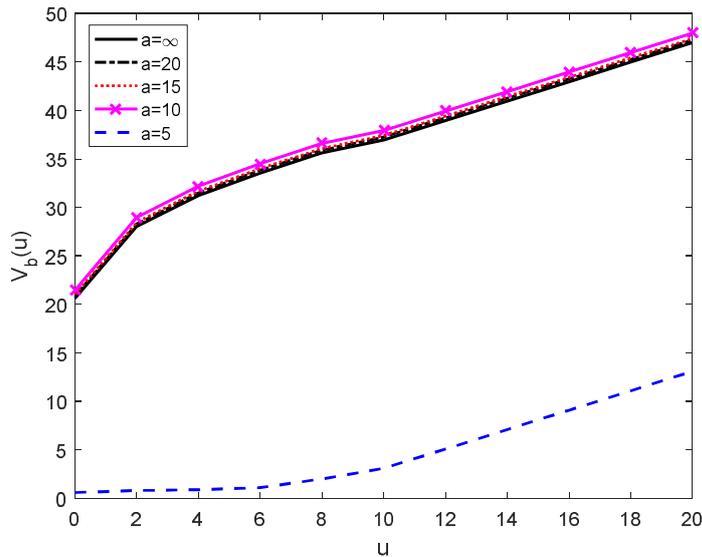
**Table 4:** Dividends in CLM with XL reins.: Exp(0.5) claims ( $\lambda = 2$ ,  $c = 6$ ,  $\delta = 0.1$ )

| $u$ | $a = \infty$ | $a = 20$ | $a = 15$ | $a = 10$ | $a = 5$ |
|-----|--------------|----------|----------|----------|---------|
| 0   | 5.2932       | 5.2873   | 5.2708   | 3.0703   | 0.2264  |
| 2   | 8.5923       | 8.5860   | 8.5650   | 5.0147   | 0.3942  |
| 4   | 11.2844      | 11.2772  | 11.2529  | 6.6093   | 0.5425  |
| 6   | 13.6047      | 13.5970  | 13.5711  | 7.9921   | 0.7155  |
| 8   | 15.7165      | 15.7086  | 15.6822  | 9.2574   | 1.2759  |
| 10  | 17.7348      | 17.7145  | 17.7009  | 10.4720  | 2.5626  |
| 12  | 19.9749      | 19.9670  | 19.9411  | 12.6569  | 4.7925  |
| 14  | 21.9749      | 21.9670  | 21.9411  | 14.6569  | 6.7925  |
| 16  | 23.9749      | 23.9670  | 23.9411  | 16.6569  | 8.7925  |
| 18  | 25.9749      | 25.9670  | 25.9411  | 18.6569  | 10.7925 |
| 20  | 27.9749      | 27.9670  | 27.9411  | 20.6569  | 12.7925 |



**Figure 5:** Dividend value functions for Exp(0.5) claims for CLM with XL reins.,  $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$

reinsurance with  $a = 10$  yields higher dividend values than QS reinsurance with  $k = 1$ . Therefore, it is optimal to take XL reinsurance, that is, for Pareto(3,2) claims in the CLM, the optimal reinsurance policy for dividend maximization is  $(k^*, a^*) = (1, 10)$ . In other words, the optimal policy is a pure XL reinsurance with  $a^* = 10$ .



**Figure 6:** Dividend value functions for Pareto(3,2) claims for CLM with XL reins.,  
 $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$

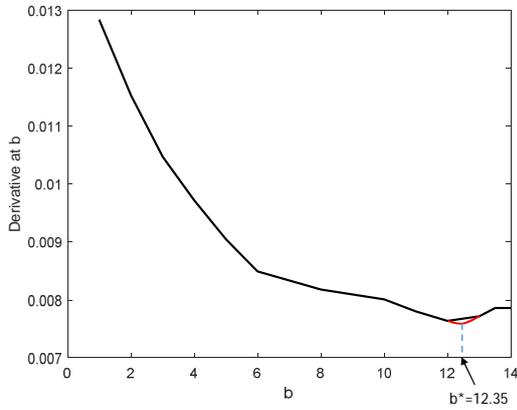
#### 4.2.6 Optimal dividend barriers in the diffusion-perturbed model

Before considering the diffusion-perturbed model compounded by reinsurance, it is important to obtain the optimal dividend barriers for small and large claims. The dividend values for Exp(0.5) claims were obtained using the FORTRAN program BLOCK.for in *Appendix 6*, with slight modifications in the claim-size distribution and model, and are shown in Table 5. In this case, the optimal barrier  $b$  lies in the interval  $[12, 14]$  for small claims. The actual optimal barrier was obtained as  $b^* = 12.35$  by plotting the values of  $f'(b)$  at different values of  $b$  and approximating the value of  $b$  at which the derivative was a minimum as shown in Fig. 7(a).

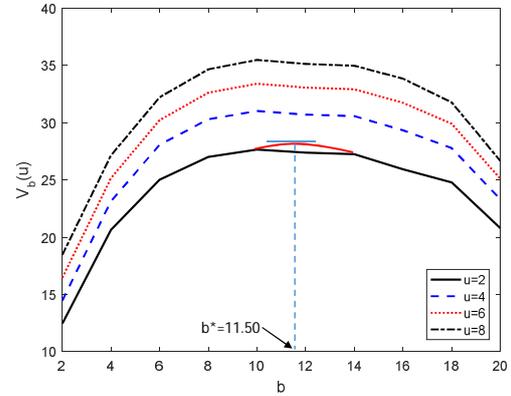
For Pareto(3,2) claims, the optimal barrier  $b^* \in [10, 12]$  with the actual  $b^*$  found as 11.50, as shown in Fig. 7(b). The dividend values for various barrier levels  $b$  (including the optimal barrier) are given in Fig. 8 for small and large claim sizes. In both cases, the optimal barrier gives the highest values of dividend payouts as expected.

**Table 5:** Dividends in DPM (no reins.): Exp(0.5) claims ( $\lambda = 2, c = 6, \delta = 0.1$ )

| $b$           | $u = 2$ | $u = 4$ | $u = 6$ | $u = 8$ | $u = 10$ | $u = 12$ | $u = 14$ |
|---------------|---------|---------|---------|---------|----------|----------|----------|
| 2             | 5.6462  | 7.6462  | 9.6462  | 11.6462 | 13.6462  | 15.6462  | 17.6462  |
| 4             | 6.6968  | 8.8651  | 10.8651 | 12.8651 | 14.8651  | 16.8651  | 18.8651  |
| 6             | 7.6659  | 10.1478 | 12.2855 | 14.2855 | 16.2855  | 18.2855  | 20.2855  |
| 8             | 7.9578  | 10.5342 | 12.7533 | 14.7682 | 16.7682  | 18.7682  | 20.7682  |
| 10            | 8.1250  | 10.7556 | 13.0213 | 15.0786 | 17.0385  | 19.0385  | 21.0385  |
| 12            | 8.5155  | 11.2725 | 13.6471 | 15.8032 | 17.8573  | 19.8924  | 21.8924  |
| 14            | 8.2853  | 10.9678 | 13.2782 | 15.3760 | 17.3746  | 19.3547  | 21.3774  |
| 16            | 7.2210  | 9.5590  | 11.5726 | 13.4010 | 15.1428  | 16.8686  | 18.6314  |
| 18            | 6.8983  | 9.1318  | 11.0554 | 12.8020 | 14.4660  | 16.1146  | 17.7987  |
| 20            | 6.7561  | 8.8729  | 10.6974 | 12.3579 | 13.9448  | 15.5232  | 17.1408  |
| $b^* = 12.35$ | 8.6865  | 11.4989 | 13.9212 | 16.1206 | 18.2160  | 20.2919  | 22.6480  |



(a) Optimal barrier: Exp(0.5) claims



(b) Optimal barrier: Pareto (3,2) claims

**Figure 7:** Optimal dividend barriers for DPM for small and large claims

Now, fixing  $b = b^*$ , we computed the dividend values for small and large claims in the DPM. The results are presented in the following sections for varying retention levels for quota-share and XL reinsurance.

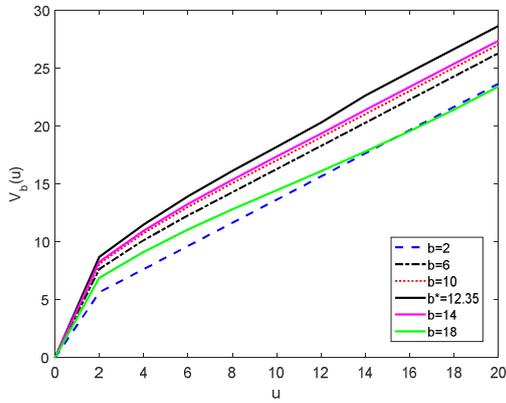
#### 4.2.7 Dividends in the diffusion-perturbed model with proportional reinsurance

Here,  $\sigma > 0, a = \infty$  and so the model is

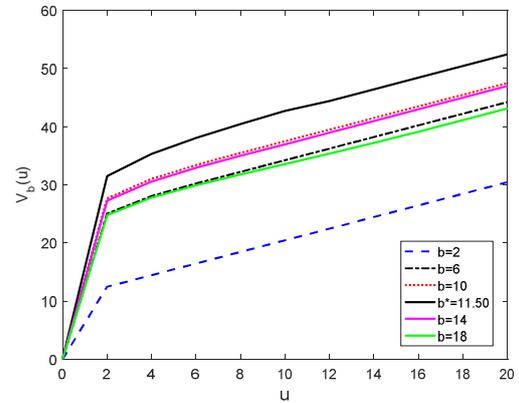
$$U_t^{\overline{D}, \overline{R}} = u + kct + \sigma W_t - \sum_{i=1}^{N_t} kX_i - D_t^b. \quad (4.2.25)$$

**Table 6:** Dividends in DPM (no reins.): Pareto(3,2) claims ( $\lambda = 2$ ,  $c = 6$ ,  $\delta = 0.1$ ,  $\sigma = 1.0$ )

| $b$           | $u = 2$ | $u = 4$ | $u = 6$ | $u = 8$ | $u = 10$ | $u = 12$ | $u = 14$ |
|---------------|---------|---------|---------|---------|----------|----------|----------|
| 2             | 12.4488 | 14.4488 | 16.4488 | 18.4488 | 20.4488  | 22.4488  | 24.4488  |
| 4             | 20.6442 | 23.1525 | 25.1525 | 27.1525 | 29.1525  | 31.1525  | 33.1525  |
| 6             | 25.0237 | 28.0642 | 30.2186 | 32.2186 | 34.2186  | 36.2186  | 38.2186  |
| 8             | 27.0098 | 30.2917 | 32.6171 | 34.6541 | 36.6541  | 38.6541  | 40.6541  |
| 10            | 27.6548 | 31.0150 | 33.3960 | 35.4816 | 37.4898  | 39.4898  | 41.4898  |
| 12            | 27.3761 | 30.7024 | 33.0594 | 35.1240 | 37.1120  | 39.1035  | 41.1035  |
| 14            | 27.2562 | 30.5680 | 32.9146 | 34.9702 | 36.9495  | 38.9322  | 40.9698  |
| 16            | 25.9341 | 29.3282 | 31.7332 | 33.8399 | 35.8684  | 37.9004  | 39.9887  |
| 18            | 24.7624 | 27.7711 | 29.9030 | 31.7705 | 33.5687  | 35.3701  | 37.2212  |
| 20            | 20.7777 | 23.3022 | 25.0911 | 26.6581 | 28.1669  | 29.6784  | 31.2317  |
| $b^* = 11.50$ | 31.5039 | 35.3317 | 38.0441 | 40.4200 | 42.7077  | 44.4236  | 46.4236  |



(a) DPM: Dividends for Exp(0.5) claims



(b) DPM: Dividends for Pareto(3,2) claims

**Figure 8:** Dividends in DPM (no reinsurance),  $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$ ,  $\sigma = 1.0$

By Itô's formula, the IDE for this model is

$$\frac{1}{2}\sigma^2 V_b''(u) + kcV_b'(u) - (\lambda + \delta)V_b(u) + \lambda \int_0^u V_b(u - kx)dF(x) = 0. \quad (4.2.26)$$

Integrating (4.2.26) by parts over  $[0, z]$  w.r.t.  $u$  gives

$$\begin{aligned} 0 &= \frac{1}{2}\sigma^2 \int_0^z V_b''(u)du + kc \int_0^z V_b'(u)du - (\lambda + \delta) \int_0^z V_b(u)du \\ &\quad + \lambda \int_0^z \int_0^u V_b(u - kx)dF(x)du \end{aligned} \quad (4.2.27)$$

That is,

$$\begin{aligned}
0 &= \frac{1}{2}\sigma^2[V_b'(u)]_0^z + kc[V_b(u)]_0^z - (\lambda + \delta) \int_0^z V_b(u)du \\
&\quad + \lambda \int_0^z \int_0^u V_b(u - kx)dF(x)du \\
&= \frac{1}{2}\sigma^2 V_b'(z) - \frac{1}{2}\sigma^2 V_b'(0) + kcV_b(z) - kcV_b(0) - (\lambda + \delta) \int_0^z V_b(u)du \\
&\quad + \lambda \int_0^z \int_0^u V_b(u - kx)f(x)dxdu \\
&= \frac{1}{2}\sigma^2 V_b'(z) - \left( kcV_b(0) + \frac{1}{2}\sigma^2 V_b'(0) \right) + kcV_b(z) - (\lambda + \delta) \int_0^z V_b(u)du \\
&\quad + \lambda \int_0^z \int_0^u V_b(\nu)f(u - \nu)d\nu du \quad (\nu := u - kx) \\
&= \frac{1}{2}\sigma^2 V_b'(z) - \left( kcV_b(0) + \frac{1}{2}\sigma^2 V_b'(0) \right) + kcV_b(z) - (\lambda + \delta) \int_0^z V_b(u)du \\
&\quad + \lambda \int_0^z \int_\nu^z f(u - \nu)du V_b(\nu)d\nu \\
&= \frac{1}{2}\sigma^2 V_b'(z) - \left( kcV_b(0) + \frac{1}{2}\sigma^2 V_b'(0) \right) + kcV_b(z) - (\lambda + \delta) \int_0^z V_b(\nu)d\nu \\
&\quad + \lambda \int_0^z f(z - \nu)V_b(\nu)d\nu \tag{4.2.28}
\end{aligned}$$

Integrating (4.2.27) by parts over  $[0, u]$  w.r.t.  $z$  gives

$$\begin{aligned}
0 &= \frac{1}{2}\sigma^2 \int_0^u V_b'(z)dz - \left( kcV_b(0) + \frac{1}{2}\sigma^2 V_b'(0) \right) u + kc \int_0^u V_b(z)dz \\
&\quad - (\lambda + \delta) \int_0^u \int_0^z V_b(\nu)d\nu dz + \lambda \int_0^u \int_0^z F(z - \nu)V_b(\nu)d\nu dz \\
&= \frac{1}{2}\sigma^2 [V_b(z)]_0^u - \left( kcV_b(0) + \frac{1}{2}\sigma^2 V_b'(0) \right) u + kc \int_0^u V_b(z)dz \\
&\quad - (\lambda + \delta) \int_0^u \int_\nu^u dz V_b(\nu)d\nu + \lambda \int_0^u \int_\nu^u F(z - \nu)dz V_b(\nu)d\nu
\end{aligned}$$

That is,

$$\begin{aligned}
0 &= \frac{1}{2}\sigma^2 V_b(u) - \frac{1}{2}\sigma^2 (V_b(0) + uV_b'(0)) - kcuV_b(0) \\
&\quad + \int_0^u [kc + (\lambda + \delta)z + \lambda G(u - z) - (\lambda + \delta)u]V_b(z)dz \tag{4.2.29}
\end{aligned}$$

which by rearrangement becomes

$$\begin{aligned} V_b(u) &+ \frac{2}{\sigma^2} \int_0^u [kc - (\lambda + \delta)(u - z) + \lambda G(u - z)] V_b(z) dz \\ &= \frac{\sigma^2(V_b(0) + uV_b'(0)) - 2kcuV_b(0)}{\sigma^2} \end{aligned} \quad (4.2.30)$$

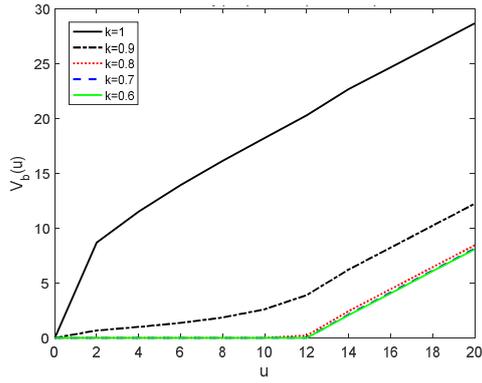
Replacing  $z$  with  $x$  and  $u - z$  with  $u - kx$  yields

$$\begin{aligned} V_b(u) &+ \frac{2}{\sigma^2} \int_0^u [kc - (\lambda + \delta)(u - kx) + \lambda G(u - kx)] V_b(x) dx \\ &= \frac{\sigma^2(V_b(0) + uV_b'(0)) - 2kcuV_b(0)}{\sigma^2} \end{aligned} \quad (4.2.31)$$

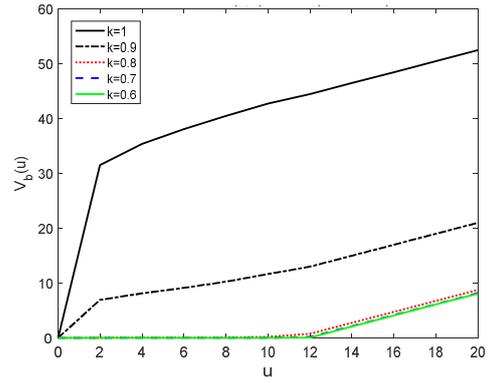
which is a VIE of the second kind with kernel  $K(u, x) = \frac{2[kc - (\lambda + \delta)(u - kx) + \lambda G(u - kx)]}{\sigma^2}$  and forcing function  $\alpha(u) = \frac{\sigma^2(V_b(0) + uV_b'(0)) - 2kcuV_b(0)}{\sigma^2}$ . But since  $V_b(0) = 0$ ,  $\alpha(u) = uV_b'(0)$  for  $\sigma^2 > 0$ . The FORTRAN program BLOCK.for in *Appendix 6*, with appropriate modifications in the claim-size distribution and model, was used to compute the dividend values in this case.

(i) **Exponential claims.** For the small claim case in the DPM, the optimal barrier was found as  $b^* = 12.35$ . The dividend values for small claims in the diffusion-perturbed model are shown in Fig. 9(a). The optimal retention level for quota-share reinsurance is  $k^* = 1$  because this is the retention value giving the highest dividend values. Thus, if the claim sizes are exponentially distributed it is optimal not to take proportional reinsurance in the diffusion-perturbed classical risk process. We also observe from Fig. 9 that in the small and large claim cases involving the DPM and for values of the retention  $k$  below 0.8, the company can only begin to pay dividends at higher values of the initial surplus  $u$ , in particular, for  $u \geq 12$ . This is because as the retention level reduces, the cession level increases, meaning that the company pays more to reinsurers for risk-sharing. Consequently, the funds available for dividend distribution are reduced, necessitating a higher initial capital to guarantee survival of the insurance company.

(ii) **Pareto claims.** The optimal barrier for large claims in the DPM was found to be  $b^* = 11.50$  and the dividend values for the DPM compounded by XL reinsurance are given in Fig. 9(b). In the large claim case involving the diffusion-perturbed classical risk model,



(a) DPM: Dividends for Exp(0.5) claims



(b) DPM: Dividends for Pareto(3,2) claims

**Figure 9:** Dividend value functions in DPM with QS reins.,  $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$ ,  $\sigma = 1.0$

the optimal retention level for quota-share reinsurance is  $k^* = 1$  since this is the retention value yielding the highest dividend values.

#### 4.2.8 Dividends in the diffusion-perturbed model with excess-of-loss reinsurance

This is a diffusion model compounded only by XL reinsurance (that is,  $\sigma > 0$ ,  $k = 1$ ), so we have

$$U_t^{\bar{D}, \bar{R}} = u + c^{\bar{R}}t + \sigma W_t - \sum_{i=1}^{N_t} X_i \wedge a - D_t^b, \quad (4.2.32)$$

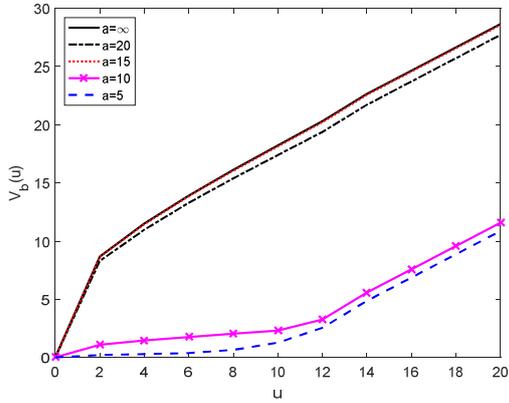
where  $c^{\bar{R}} = c - (1 + \theta)\lambda\mathbb{E}[(X_i - a)^+]$ . The corresponding IDE follows from Itô's formula as

$$\frac{1}{2}\sigma^2 V_b''(u) + c^{\bar{R}}V_b'(u) - (\lambda + \delta)V_b(u) + \lambda \int_0^u V_b(u - x \wedge a)dF(x) = 0 \quad (4.2.33)$$

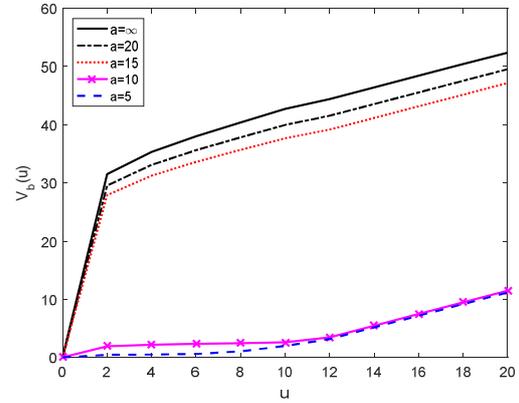
which transforms into a VIE of the second kind with  $K(u, x)$  and  $\alpha(u)$  as given in (3.4.10) and (3.4.14) but with  $k = 1$ . The FORTRAN program BLOCK.for, with appropriate modifications in the claim-size distribution and additional pieces of code to include excess-of-loss reinsurance, was used to obtain the results.

- (i) **Exponential claims.** As we have already noted, the optimal barrier for small claims in the DPM was found to be  $b^* = 12.35$  and the dividend values when XL reinsurance is used as a risk measure are given in Fig. 10(a) from which it can be concluded that it is

optimal not to take XL reinsurance, that is,  $a^* = \infty$ . This means that in the DPM for small claims, reinsurance does not serve the cause of dividend maximization. In other words, the insurance company should not reinsure as doing so reduces the funds available for distribution in the form of dividends.



(a) DPM: Dividends for Exp(0.5) claims



(b) DPM: Dividends for Pareto(3,2) claims

**Figure 10:** Dividend value functions for large claims in DPM with XL reins.,  
 $c = 6, \lambda = 2, \delta = 0.1, \sigma = 1.0$

- (ii) **Pareto claims.** Here, the optimal dividend barrier is  $b^* = 11.50$  and the dividend values for the DPM compounded by XL reinsurance for large claims is given in Fig. 10(b). As in the cases previously discussed, Fig. 10 displays clear evidences according to which it is optimal not to take XL reinsurance. Figure 10 shows that in both the small and large claim cases in the DPM, for values of the retention  $a$  below 15, the company will only be able to pay more in dividends at higher values of the initial surplus  $u$ .

#### 4.2.9 Optimal dividend and reinsurance strategies

- (i) **Cramér-Lundberg model.** From Fig. 4, we see that the optimal QS retention for small and large claims in the CLM is  $k^* = 1$ , meaning that it is optimal not to reinsure. In terms of XL reinsurance, Fig. 5 shows that the optimal XL retention level is  $a^* = \infty$  for small claims, that is, it is optimal not to reinsure, while Fig. 6 shows that the optimal retention is  $a^* = 10$  for large claims in the CLM. However, comparing Figs. 4 and 5 shows that QS reinsurance is optimal for Exp(0.5) claims in the CLM. But comparing

Figs. 4 and 6 shows that for large claims in the CLM, XL reinsurance with  $a = 10$  yields higher dividend values than QS reinsurance with  $k = 1$ . Therefore, it is optimal to take XL reinsurance with  $a^* = 10$ , so that the optimal reinsurance policy for large claims in the CLM is  $(k^*, a^*) = (1, 10)$ . The optimal dividend strategies for the CLM are  $b^* = 10.27$  for small claims and  $b^* = 9.35$  for large ones. This means that for small claims in the CLM the optimal policy is  $(\bar{D}, \bar{R}) = (10.27, (1, \infty))$ , while for large claims it is  $(\bar{D}, \bar{R}) = (9.35, (1, a^*))$ , where, for the chosen parameters,  $a^* = 10$ . In other words, the optimal combinational reinsurance policy here is a pure XL reinsurance.

- (ii) **Diffusion-perturbed model.** From the results presented in the previous sections, it can be concluded that at the optimal dividend barrier  $b^*$  for the DPM it is optimal not to take reinsurance. This applies both in the light- and heavy-tailed cases and the optimal retentions do not vary according to the claim size distribution used. Thus, for small claims the optimal policy is  $(\bar{D}, \bar{R}) = (12.35, (1, \infty))$ , while for large claims it is  $(\bar{D}, \bar{R}) = (11.50, (1, \infty))$ . For the chosen parameters, dividends are maximized when no reinsurance is taken. Thus, though the literature (e.g., Li *et al.*, 2017; Zhang and Liang, 2016) shows that in a combinational proportional and XL reinsurance scenario the optimal strategy is a pure XL, i.e.,  $(1, a^*)$ , this study has established that in the DPM neither QS nor XL reinsurance has any advantage over the other. This means that for purposes of dividend maximization, the company should consider using other risk measures such as investment, capital injections or refinancing, portfolio selection and premium control.

We now present results on ultimate ruin probabilities for the CLM and DPM with and without reinsurance.

#### 4.2.10 Ultimate ruin probability in the Cramér-Lundberg model

When  $\sigma = 0$  (that is, when there is *no diffusion*), the diffusion-perturbed risk model (3.3.1) reduces to the Cramér-Lundberg model:

$$U_t = u + ct - \sum_{i=1}^{N_t} X_i, \quad t \geq 0 \quad (4.2.34)$$

whose dynamics are given by

$$dU_t = cdt - dS_t, \text{ where } S_t = \sum_{i=1}^{N_t} X_i. \quad (4.2.35)$$

This model is obtained by considering the controlled surplus process  $U_t^{\bar{D}, \bar{R}}$  in (3.3.3) when there is no diffusion ( $\sigma^2 = 0$ ), no reinsurance ( $k = 1$  (no quota-share reinsurance),  $a = \infty$  (no excess-of-loss reinsurance)) and no dividends ( $\delta = 0$  and  $D_t^b = 0$ ). Since the HJB equation for the controlled process is

$$\max\{1 - V'(u), \sup \tilde{\mathcal{L}}(V)(u)\} = 0, \quad (4.2.36)$$

where the infinitesimal generator  $\tilde{\mathcal{L}}(V)(u)$  is given by Itô's formula as  $\tilde{\mathcal{L}}(V)(u) = \frac{1}{2}\sigma^2 V''(u) + c\bar{R}V'(u) - (\lambda + \delta)V(u) + \lambda \int_0^u V(u - kx \wedge a) dF(x)$ , it follows that with the above restrictions, the HJB equation for the Cramér-Lundberg model will be (4.2.36) with the infinitesimal generator  $\tilde{\mathcal{L}}(V)(u)$  defined by the first-order integrodifferential operator

$$\tilde{\mathcal{L}}(V)(u) = cV'(u) + \lambda \int_0^u (V(u - x) - V(u)) dF(x). \quad (4.2.37)$$

Since the value function is now the ultimate survival probability  $\phi(u)$ , its IDE which follows from the above HJB equation is given as

$$c\phi'(u) + \lambda \int_0^u (\phi(u - x) - \phi(u)) dF(x) = 0. \quad (4.2.38)$$

Integrating (4.2.38) by parts on  $[0, z]$  w.r.t.  $u$  and noting that  $F(\infty) = \int_0^\infty f(x) dx = 1$ , we obtain

$$c(\phi(z) - \phi(0)) + \lambda \int_0^z F(u - x)\phi(x) dx - \lambda \int_0^z \phi(x) dx = 0. \quad (4.2.39)$$

Replacing  $z$  with  $u$  results in

$$\begin{aligned}
c\phi(u) &= c\phi(0) + \lambda \int_0^u \phi(x)dx - \lambda \int_0^u F(u-x)\phi(x)dx \\
&= c\phi(0) + \lambda \int_0^u [1 - F(u-x)]\phi(x)dx \\
&= c\phi(0) + \lambda \int_0^u \bar{F}(u-x)\phi(x)dx
\end{aligned}$$

or

$$\phi(u) - \frac{\lambda}{c} \int_0^u \bar{F}(u-x)\phi(x)dx = \phi(0) \tag{4.2.40}$$

which is a VIE of the second kind

$$\phi(u) + \int_0^u K(u,x)\phi(x)dx = \alpha(u), \tag{4.2.41}$$

where  $\alpha(u) = \phi(0)$  is the forcing function and  $K(u,x) = -\frac{\lambda\bar{F}(u-x)}{c}$  is the kernel (with  $\bar{F}(x) = 1 - F(x)$ ). This is precisely equations (3.4.10) and (3.4.11) when  $\delta = 0$ ,  $k = 1$  and  $a = \infty$  and with  $V_b = \phi$ . The numerical results obtained using (4.2.40) are compared in Table 7 with the exact solution for exponentially distributed claims and show that the block-by-block numerical method used in this study produces very accurate results, as has been shown also by Paulsen *et al.* (2005) for a different choice of parameter values. The results were obtained using the FORTRAN program LBERG.for in *Appendix 8*.

Table 7 shows that the insurer's ruin probability is a function of the initial surplus and that it reduces as the initial surplus increases. This applies to the large claim case as well.

#### 4.2.11 Ultimate ruin probability in the Cramér-Lundberg model compounded by proportional reinsurance

Here, the surplus process takes the form

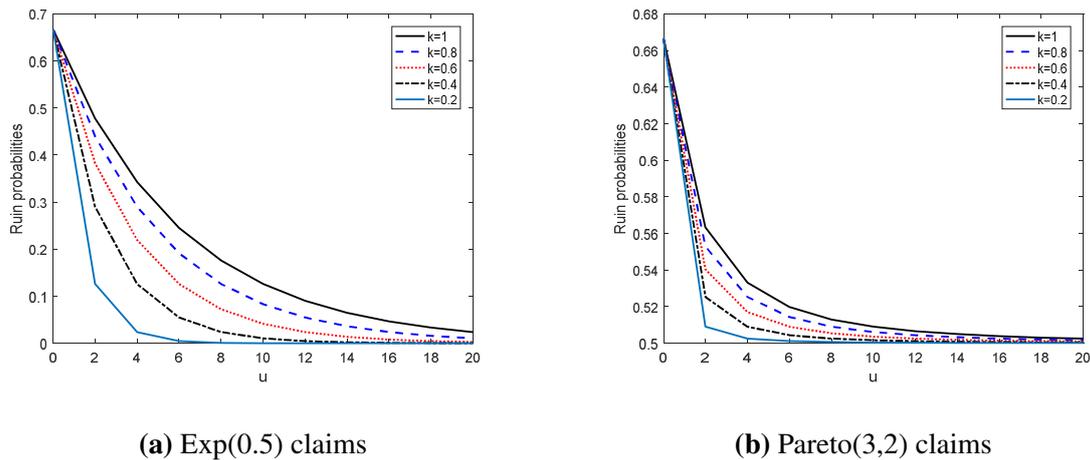
$$U_t^{\bar{R}} = u + kct - \sum_{i=1}^{N_t} kX_i. \tag{4.2.42}$$

**Table 7:** Exact and numerical ruin prob. in CLM: Exp(0.5) claims

| $u$ | $\psi(u)$ | $\psi_{0.01}(u)$ | $D_{0.01}(u)$ |
|-----|-----------|------------------|---------------|
| 0   | 0.6667    | 0.6667           | 0.0000        |
| 5   | 0.2897    | 0.2897           | 0.0000        |
| 10  | 0.1259    | 0.1259           | 0.0000        |
| 15  | 0.0547    | 0.0547           | 0.0000        |
| 20  | 0.0238    | 0.0238           | 0.0000        |
| 25  | 0.0103    | 0.0103           | 0.0000        |
| 30  | 0.0045    | 0.0045           | 0.0000        |
| 35  | 0.0020    | 0.0020           | 0.0000        |
| 40  | 0.0008    | 0.0008           | 0.0000        |
| 45  | 0.0004    | 0.0004           | 0.0000        |
| 50  | 0.0002    | 0.0002           | 0.0000        |

So, the survival probability  $\phi(u)$  satisfies equations (3.4.10) and (3.4.11) with  $\delta = 0$ ,  $a = \infty$ ,  $c^{\bar{R}} = kc$  and  $V_b = \phi$ , that is, it satisfies a VIE of the second kind with kernel  $K(u, x) = -\frac{\lambda \bar{F}(u-kx)}{kc}$  and forcing function  $\alpha(u) = \phi(0)$ .

The ruin probabilities  $\psi_k(u)$ , where  $k$  is the retention level for QS reinsurance, for the CLM compounded by proportional reinsurance for small and large claims were obtained using the FORTRAN program LBERG.for in *Appendix 8*, with appropriate changes to account for proportional reinsurance, and are given in Fig. 11.



**Figure 11:** Ultimate ruin prob. at different QS retention levels in CLM,  $\lambda = 2$ ,  $c = 6$

Figure 11 provides validity for the assertion that reinsurance does in fact reduce the ruin probability, thus increasing the insurance company's chances of survival. The results for the case

$k = 1$  (no reinsurance) are the same as those obtained in Paulsen *et al.* (2005). It can be seen that the ruin probabilities decay faster for large claims than for small ones.

#### 4.2.12 Ultimate ruin probability in the Cramér-Lundberg model compounded by excess-of-loss reinsurance

This is the case  $\delta = 0$ ,  $k = 1$  and  $\sigma = 0$ , so the surplus process is

$$U_t^{\bar{R}} = u + c^{\bar{R}}t - \sum_{i=1}^{N_t} X_i \wedge a, \quad (4.2.43)$$

where  $c^{\bar{R}} = c - (1 + \theta)\lambda E[(X_i - a)^+]$ . Applying Itô's formula to (4.2.43) gives the infinitesimal generator of  $U_t^{\bar{R}}$  as  $\tilde{\mathcal{L}}(V)(u) = c^{\bar{R}}V'(u) + \lambda \int_0^u [V(u - x \wedge a) - V(u)]dF(x)$ . Thus, the ultimate survival probability  $\phi(u)$  satisfies the VIDE

$$c^{\bar{R}}\phi'(u) + \lambda \int_0^u [\phi(u - x \wedge a) - \phi(u)]dF(x) = 0 \quad (4.2.44)$$

which reduces to a VIE of the second kind  $\phi(u) + \int_0^u K(u, x)\phi(x)dx = \alpha(u)$  with  $K(u, x)$  and  $\alpha(u)$  given by

$$\begin{aligned} K(u, x) &= -\frac{\lambda \bar{F}(u - x)}{c^{\bar{R}}} \\ \alpha(u) &= \phi(0) \end{aligned} \quad (4.2.45)$$

when  $u \leq a < \underline{a}$ , and by

$$\begin{aligned} K(u, x) &= -\frac{\lambda H(x, u)}{c^{\bar{R}}} \\ \alpha(u) &= \phi(0) \end{aligned} \quad (4.2.46)$$

with

$$H(x, u) = \begin{cases} \bar{F}(u - x) & x < a \\ 1 - (F(x + a) - F(a)) & x \geq a \end{cases}$$

when  $\underline{a} < a < u$ . This is simply equations (3.4.11) and (3.4.13) with  $\delta = 0$ ,  $k = 1$  and  $c^{\bar{R}} = c - (1 + \theta)\lambda E[(X_i - a)^+]$ .

Ruin probabilities for the Cramér-Lundberg model compounded by excess-of-loss (XL) rein-

surance were computed using the FORTRAN program LBERG.for, with minor changes to account for XL reinsurance, and are given in Table 8 for different values of the XL retention level  $a$  ranging from 20 to infinity. For Exp(0.5) claims, the ruin probabilities for the different retention levels reduce only very slightly as the retention level reduces.

**Table 8:** Ruin prob. for CLM with XL reins.: Exp(0.5) claims ( $\lambda = 2, c = 6$ )

| $u$ | $\psi_\infty(u)$ | $\psi_{35}(u)$ | $\psi_{30}(u)$ | $\psi_{25}(u)$ | $\psi_{20}(u)$ |
|-----|------------------|----------------|----------------|----------------|----------------|
| 0   | 0.6667           | 0.6667         | 0.6667         | 0.6667         | 0.6667         |
| 2   | 0.4777           | 0.4777         | 0.4777         | 0.4777         | 0.4777         |
| 4   | 0.3423           | 0.3423         | 0.3423         | 0.3423         | 0.3422         |
| 6   | 0.2453           | 0.2453         | 0.2453         | 0.2453         | 0.2453         |
| 8   | 0.1757           | 0.1757         | 0.1757         | 0.1757         | 0.1757         |
| 10  | 0.1259           | 0.1259         | 0.1259         | 0.1259         | 0.1258         |
| 12  | 0.0902           | 0.0902         | 0.0902         | 0.0902         | 0.0901         |
| 14  | 0.0646           | 0.0646         | 0.0646         | 0.0646         | 0.0646         |
| 16  | 0.0463           | 0.0463         | 0.0463         | 0.0463         | 0.0462         |
| 18  | 0.0332           | 0.0332         | 0.0332         | 0.0332         | 0.0331         |
| 20  | 0.0238           | 0.0238         | 0.0238         | 0.0238         | 0.0237         |

For Pareto (3,2) claims, the ruin probabilities increase slightly as the retention level reduces (as shown in Table 9), meaning that it is optimal not to reinsure. But comparing these probabilities with Fig. 11 leads to the conclusion that proportional reinsurance results in much lower ruin probabilities for the CLM as well as the perturbed model. Table 8 gives ultimate ruin probabilities  $\psi_a(u)$ , where  $a$  is the retention level for XL reinsurance, for Exp(0.5) claims in the CLM compounded by XL reinsurance, while Table 9 gives the probabilities for the same model for Pareto(3,2) claims.

#### 4.2.13 Ultimate ruin probability in the diffusion-perturbed model

The infinitesimal generator of the perturbed process

$$U_t = u + ct + \sigma W_t - \sum_{i=1}^{N_t} X_i, \quad t \geq 0 \quad (4.2.47)$$

**Table 9:** Ruin prob. for CLM with XL reins.: Par(3,2) claims ( $\lambda = 2, c = 6$ )

| $u$ | $\psi_\infty(u)$ | $\psi_{35}(u)$ | $\psi_{30}(u)$ | $\psi_{25}(u)$ | $\psi_{20}(u)$ |
|-----|------------------|----------------|----------------|----------------|----------------|
| 0   | 0.6667           | 0.6667         | 0.6667         | 0.6667         | 0.6667         |
| 2   | 0.5634           | 0.5636         | 0.5637         | 0.5639         | 0.5641         |
| 4   | 0.5331           | 0.5335         | 0.5336         | 0.5338         | 0.5341         |
| 6   | 0.5198           | 0.5202         | 0.5204         | 0.5206         | 0.5210         |
| 8   | 0.5130           | 0.5134         | 0.5135         | 0.5138         | 0.5142         |
| 10  | 0.5090           | 0.5095         | 0.5096         | 0.5099         | 0.5103         |
| 12  | 0.5066           | 0.5070         | 0.5072         | 0.5074         | 0.5079         |
| 14  | 0.5050           | 0.5054         | 0.5056         | 0.5058         | 0.5063         |
| 16  | 0.5039           | 0.5043         | 0.5045         | 0.5048         | 0.5052         |
| 18  | 0.5031           | 0.5036         | 0.5037         | 0.5040         | 0.5044         |
| 20  | 0.5025           | 0.5030         | 0.5032         | 0.5034         | 0.5039         |

is given by Itô's formula as

$$\tilde{\mathcal{L}}(V)(u) = \frac{1}{2}\sigma^2 V''(u) + cV'(u) + \lambda \left[ \int_0^u V(u-x)dF(x) - V(u) \right] \quad (4.2.48)$$

from which the IDE for the ultimate survival probability follows as

$$\frac{1}{2}\sigma^2 \phi''(u) + c\phi'(u) + \lambda \left[ \int_0^u \phi(u-x)dF(x) - \phi(u) \right] = 0. \quad (4.2.49)$$

Integrating (4.2.49) by parts over  $[0, z]$  w.r.t.  $u$  gives

$$\begin{aligned} 0 &= \frac{1}{2} \int_0^z \sigma^2 \phi''(u) du + c \int_0^z \phi'(u) du - \lambda \int_0^z \phi(u) du + \lambda \int_0^z \int_0^u \phi(u-x) dF(x) du \\ &= \frac{1}{2} \sigma^2 [\phi'(u)]_0^z + c[\phi(u)]_0^z - \lambda \int_0^z \phi(u) du + \lambda \int_0^z \int_0^u \phi(u-x) dF(x) du \\ &= \frac{1}{2} \sigma^2 \phi'(z) - \frac{1}{2} \sigma^2 \phi'(0) + c\phi(z) - c\phi(0) - \lambda \int_0^z \phi(u) du + \lambda \int_0^z \int_0^u \phi(u-x) f(x) dx du \\ &= \frac{1}{2} \sigma^2 \phi'(z) - \frac{1}{2} \sigma^2 \phi'(0) + c\phi(z) - c\phi(0) - \lambda \int_0^z \phi(u) du + \lambda \int_0^z \int_0^u \phi(\nu) f(u-\nu) d\nu du \\ &= \frac{1}{2} \sigma^2 \phi'(z) - \left( c\phi(0) + \frac{1}{2} \sigma^2 \phi'(0) \right) + c\phi(z) - \lambda \int_0^z \phi(\nu) d\nu + \lambda \int_0^z F(z-\nu) \phi(\nu) d\nu \end{aligned} \quad (4.2.50)$$

where  $\nu = u - x$ . Integrating (4.2.50) by parts over  $[0, u]$  w.r.t.  $z$  yields

$$\begin{aligned}
0 &= \frac{1}{2}\sigma^2 \int_0^u \phi'(z)dz - \left( c\phi(0) + \frac{1}{2}\sigma^2\phi'(0) \right) u + c \int_0^u \phi(z)dz - \lambda \int_0^u \int_0^z \phi(\nu)d\nu dz \\
&+ \lambda \int_0^u \int_0^z F(z - \nu)\phi(\nu)d\nu dz \\
&= \frac{1}{2}\sigma^2[\phi(z)]_0^u - \left( c\phi(0) + \frac{1}{2}\sigma^2\phi'(0) \right) u + c \int_0^u \phi(z)dz - \lambda \int_0^u \int_\nu^u dz\phi(\nu)d\nu \\
&+ \lambda \int_0^u \int_\nu^u F(z - \nu)dz\phi(\nu)d\nu \\
&= \frac{1}{2}\sigma^2\phi(u) - \frac{1}{2}\sigma^2(\phi(0) + u\phi'(0)) - uc\phi(0) + \int_0^u [\lambda z + c + \lambda G(u - z) - \lambda u]\phi(z)dz \\
&= \frac{1}{2}\sigma^2\phi(u) - \frac{1}{2}\sigma^2(\phi(0) + u\phi'(0)) - uc\phi(0) + \int_0^u [c - \lambda(u - z) + \lambda G(u - z)]\phi(z)dz
\end{aligned} \tag{4.2.51}$$

where  $G(x) = \int_0^x F(v)dv$ . Equation (4.2.51) can be expressed as

$$\phi(u) + \frac{2}{\sigma^2} \int_0^u [c - \lambda(u - z) + \lambda G(u - z)]\phi(z)dz = \frac{\sigma^2(\phi(0) + u\phi'(0)) + 2uc\phi(0)}{\sigma^2} \tag{4.2.52}$$

Equation (4.2.52) is a linear VIE of the second kind with kernel and forcing function given, respectively, by

$$\begin{aligned}
K(u, x) &= \frac{2[c - \lambda(u - x) + \lambda G(u - x)]}{\sigma^2} \\
\alpha(u) &= \frac{\sigma^2(\phi(0) + u\phi'(0)) + 2uc\phi(0)}{\sigma^2} = u\phi'(0) \text{ if } \sigma^2 > 0
\end{aligned} \tag{4.2.53}$$

when  $z$  is replaced with  $x$ . The FORTRAN program LBERG.for in *Appendix 8*, with appropriate changes to account for diffusion in the model, was used to obtain the results. For  $c = 6$ ,  $\lambda = 2$ ,  $\sigma = 0.02$ ,  $h = 0.01$ , the ruin probabilities are given in Table 10. These results indicate that the ultimate ruin probabilities for small claims are higher than those for large claims. But also, the ruin probabilities for Pareto(2,1) claims are much higher than those for Pareto(3,2) claims. However, in all cases covered by Table 10 the ruin probability is a decreasing function of the initial surplus  $u$ .

**Table 10:** Ultimate ruin prob. in DPM (no reins.)

| $u$ | Exp(0.5) | Pareto(2, 1) | Pareto(3, 2) |
|-----|----------|--------------|--------------|
| 0   | 1.0000   | 1.0000       | 1.0000       |
| 2   | 0.5159   | 0.2283       | 0.2029       |
| 4   | 0.3467   | 0.1083       | 0.0748       |
| 6   | 0.2458   | 0.0745       | 0.0405       |
| 8   | 0.1759   | 0.0580       | 0.0261       |
| 10  | 0.1257   | 0.0468       | 0.0175       |
| 12  | 0.0901   | 0.0396       | 0.0128       |
| 14  | 0.0646   | 0.0342       | 0.0097       |
| 16  | 0.0463   | 0.0300       | 0.0076       |
| 18  | 0.0333   | 0.0268       | 0.0062       |
| 20  | 0.0240   | 0.0244       | 0.0054       |

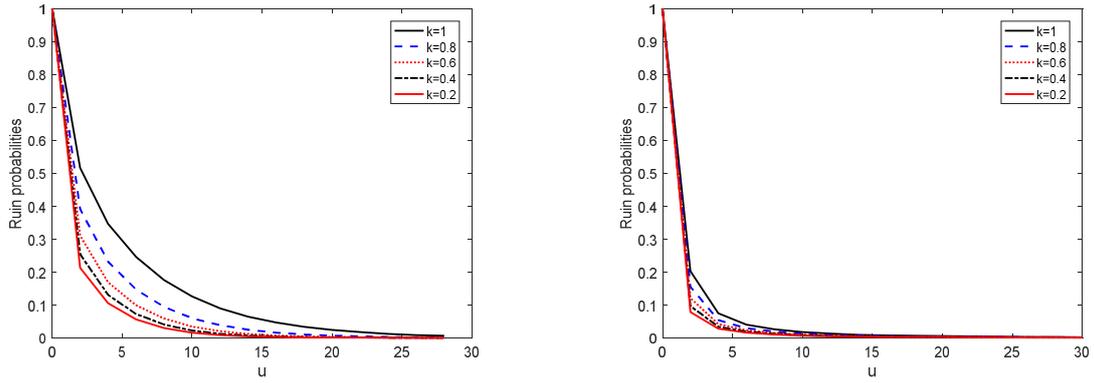
#### 4.2.14 Ultimate ruin probability in the diffusion-perturbed model compounded by proportional reinsurance

The survival probability  $\phi(u)$  satisfies equations (3.4.10) and (3.4.14) with  $\delta = 0$ ,  $a = \infty$  and  $V_b = \phi$ , that is,

$$\phi(u) + \frac{2}{\sigma^2} \int_0^u [kc - \lambda(u - kx) + \lambda G(u - kx)] \phi(x) dx = \frac{\sigma^2(\phi(0) + u\phi'(0)) - 2kcu\phi(0)}{\sigma^2} \quad (4.2.54)$$

which is a VIE of the second kind with kernel  $K(u, x) = \frac{2[kc - (\lambda + \delta)(u - kx) + \lambda G(u - kx)]}{\sigma^2}$  and forcing function  $\alpha(u) = \frac{\sigma^2(\phi(0) + u\phi'(0)) - 2kcu\phi(0)}{\sigma^2} = u\phi'(0)$ . The results were computed using the FORTRAN program LBERG.for with minor alterations to account for proportional reinsurance in the diffusion-perturbed model. For Exp(0.5) claims ( $c = 6$ ,  $\lambda = 2$ ,  $h = 0.01$ ), see Fig. 12(a) and for Pareto(3,2) claims ( $c = 6$ ,  $\lambda = 2$ ,  $h = 0.01$ ), see Fig 12(b).

Figure 12 shows the ruin probabilities for the diffusion-perturbed model compounded by proportional reinsurance for different retention levels ranging from  $k = 1$  (no reinsurance) to  $k = 0.2$  (80% reinsurance). In the case of both Exp(0.5) claims and Pareto(3,2) claims, applying proportional reinsurance significantly reduces the ultimate ruin probability of an insurance company. But, again, we see from Fig. 12 that the ruin probabilities decay faster towards zero for large claims than for small ones.



(a) Exp(0.5) claims

(b) Pareto(3,2) claims

**Figure 12:** Ultimate ruin probabilities at different proportional retention levels in the diffusion-perturbed model,  $\lambda = 2$ ,  $c = 6$ ,  $\sigma = 0.02$

#### 4.2.15 Ultimate ruin probability in the diffusion-perturbed model compounded by excess-of-loss reinsurance

This case was dealt with by Joseph (2013) who showed that the survival probability  $\phi(u)$  satisfies a VIE of the second kind with kernel  $K(u, x)$  as given in equations (3.4.12) (for the case  $u \leq \underline{a} < a$ ) and (3.4.14) (for the case  $\underline{a} < a < u$ ) with  $k = 1$ ,  $\delta = 0$  and  $V_b = \phi$ , and forcing function  $\alpha(u) = \frac{\sigma^2(\phi(0) + u\phi'(0)) + 2c\bar{R}u\phi(0)}{\sigma^2} = u\phi'(0)$  in both cases. That is,

For  $u \leq \underline{a} < a$ ,

$$K(u, x) = \frac{2[c\bar{R} + \lambda G(x, u) - \lambda(u - x)]}{\sigma^2} \quad (4.2.55)$$

For  $\underline{a} < a < u$ ,

$$K(u, x) = \frac{2[c\bar{R} + \lambda H_2(x, u) - \lambda(u - x)]}{\sigma^2} \quad (4.2.56)$$

with

$$H_2(x, u) = \begin{cases} G(u - x) & x < a \\ (F(x + a) - F(a))(u - x) & x \geq a \end{cases}$$

#### Remark 4.2.1

Equations (3.4.10)-(3.4.14) with  $\delta = 0$  and  $V_b = \phi$  represent the ruin probability models with *both* proportional and excess-of-loss reinsurance.

**Table 11:** Ruin prob. in DPM with XL reins.: Exp(0.5) claims ( $\lambda = 2$ ,  $c = 6$ ,  $\sigma = 0.02$ )

| $u$ | $\psi_{\infty}(u)$ | $\psi_{35}(u)$ | $\psi_{30}(u)$ | $\psi_{25}(u)$ | $\psi_{20}(u)$ |
|-----|--------------------|----------------|----------------|----------------|----------------|
| 0   | 1.0000             | 1.0000         | 1.0000         | 1.0000         | 1.0000         |
| 2   | 0.5159             | 0.5159         | 0.5159         | 0.5155         | 0.5109         |
| 4   | 0.3467             | 0.3467         | 0.3466         | 0.3461         | 0.3399         |
| 6   | 0.2458             | 0.2458         | 0.2457         | 0.2451         | 0.2380         |
| 8   | 0.1759             | 0.1759         | 0.1758         | 0.1752         | 0.1674         |
| 10  | 0.1257             | 0.1258         | 0.1257         | 0.1250         | 0.1167         |
| 12  | 0.0901             | 0.0901         | 0.0901         | 0.0893         | 0.0807         |
| 14  | 0.0646             | 0.0646         | 0.0645         | 0.0638         | 0.0550         |
| 16  | 0.0463             | 0.0463         | 0.0463         | 0.0455         | 0.0365         |
| 18  | 0.0333             | 0.0333         | 0.0332         | 0.0324         | 0.0233         |
| 20  | 0.0240             | 0.0241         | 0.0240         | 0.0232         | 0.0140         |

The results were obtained using the FORTRAN program LBERG.for in *Appendix 8*, with appropriate modifications to account for XL reinsurance in the diffusion-perturbed model. For ruin probabilities in the DPM compounded by XL reinsurance for Exp(0.5) claims ( $c = 6$ ,  $\lambda = 2$ ,  $h = 0.01$ ), see Table 11 and for probabilities for the same model for Pareto(3,2) claims, see Table 12. The results in both of these tables indicate that for small and large claims in the DPM the size of the XL retention level  $a$  has little impact on the ultimate ruin probabilities. In fact, the decay in the ruin probabilities is very small as the XL retention level reduces.

**Table 12:** Ruin prob. for DPM with XL reins.: Par(3,2) claims ( $\lambda = 2$ ,  $c = 6$ ,  $\sigma = 0.02$ )

| $u$ | $\psi_{\infty}(u)$ | $\psi_{200}(u)$ | $\psi_{150}(u)$ | $\psi_{100}(u)$ | $\psi_{50}(u)$ |
|-----|--------------------|-----------------|-----------------|-----------------|----------------|
| 0   | 1.0000             | 1.0000          | 1.0000          | 1.0000          | 1.0000         |
| 2   | 0.2026             | 0.2029          | 0.2027          | 0.2022          | 0.1973         |
| 4   | 0.0744             | 0.0747          | 0.0745          | 0.0740          | 0.0683         |
| 6   | 0.0401             | 0.0405          | 0.0403          | 0.0397          | 0.0338         |
| 8   | 0.0257             | 0.0260          | 0.0258          | 0.0252          | 0.0192         |
| 10  | 0.0171             | 0.0174          | 0.0172          | 0.0167          | 0.0106         |
| 12  | 0.0124             | 0.0127          | 0.0125          | 0.0119          | 0.0058         |
| 14  | 0.0093             | 0.0096          | 0.0094          | 0.0088          | 0.0027         |
| 16  | 0.0072             | 0.0075          | 0.0073          | 0.0067          | 0.0006         |
| 18  | 0.0058             | 0.0061          | 0.0059          | 0.0054          | 0.0008         |
| 20  | 0.0050             | 0.0054          | 0.0052          | 0.0042          | 0.0015         |

#### 4.2.16 Optimal reinsurance strategy: asymptotic ruin probabilities

It is known that the optimal quota-share retention  $k^*$  tends to the asymptotically optimal  $k^\rho$  that maximizes the adjustment coefficient  $\rho$  (Schmidli, 2008). Thus, we will use asymptotic ruin probabilities to determine the optimal QS retention  $k^*$ . For illustrative purposes, we will now find the optimal strategies only in the CLM for both the small and large claim cases.

- (i) **Exponential claims.** We note, as in Schmidli (2008), that for exponential claims the optimal choice of the quota-share retention  $k$  that maximizes the adjustment coefficient  $\rho(k)$  is given by

$$k^\rho = \min \left\{ \left(1 - \frac{\eta}{\theta}\right) \left(1 + \frac{1}{\sqrt{1+\theta}}\right), 1 \right\}, \quad (4.2.57)$$

where  $\theta$  and  $\eta$  are, respectively, the safety loadings of the reinsurer and insurer. Because maximizing the adjustment coefficient yields the asymptotically best strategy, we expect that the optimal retention  $k^*$  will tend to  $k^\rho$ . Since this study assumes cheap reinsurance (that is,  $\theta = \eta$ ), we have that  $k^\rho = 0$ . That is, it is optimal for the insurance company to reinsure the entire portfolio or to take full proportional reinsurance.

**Table 13:** Asympt. ruin prob. for CLM with QS reins.: Pareto claims ( $c = 6$ ,  $\lambda = 2$ ,  $\theta = \eta = 1$ )

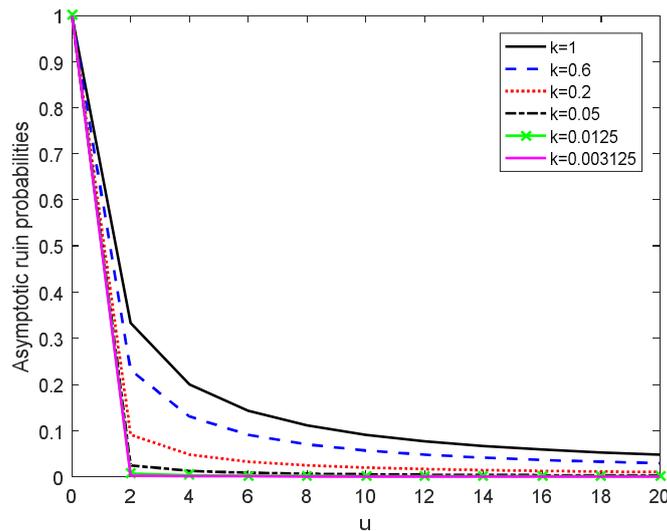
| $u$ | $\psi_1(u)$ | $\psi_{0.6}(u)$ | $\psi_{0.2}(u)$ | $\psi_{0.05}(u)$ | $\psi_{0.0125}(u)$ | $\psi_{0.003125}(u)$ |
|-----|-------------|-----------------|-----------------|------------------|--------------------|----------------------|
| 0   | 1.0000      | 1.0000          | 1.0000          | 1.0000           | 1.0000             | 1.0000               |
| 2   | 0.3333      | 0.2308          | 0.0909          | 0.0244           | 0.0062             | 0.0016               |
| 4   | 0.2000      | 0.1304          | 0.0476          | 0.0123           | 0.0031             | 0.0008               |
| 6   | 0.1429      | 0.0909          | 0.0323          | 0.0083           | 0.0021             | 0.0005               |
| 8   | 0.1111      | 0.0698          | 0.0244          | 0.0062           | 0.0016             | 0.0004               |
| 10  | 0.0909      | 0.0566          | 0.0196          | 0.0050           | 0.0012             | 0.0003               |
| 12  | 0.0769      | 0.0476          | 0.0164          | 0.0041           | 0.0010             | 0.0003               |
| 14  | 0.0667      | 0.0411          | 0.0141          | 0.0036           | 0.0009             | 0.0002               |
| 16  | 0.0588      | 0.0361          | 0.0123          | 0.0031           | 0.0008             | 0.0002               |
| 18  | 0.0526      | 0.0323          | 0.0110          | 0.0028           | 0.0007             | 0.0002               |
| 20  | 0.0476      | 0.0291          | 0.0099          | 0.0025           | 0.0006             | 0.0002               |

- (ii) **Pareto claims.** For a given initial surplus  $u$  and a retention level  $k \in [0, 1]$ , let the calculated ruin probability be given by  $\psi_k(u)$ . Then for large claims, the asymptotic

values of the ruin probability are given by

$$\psi_k(u) = \frac{1}{k\theta - (\theta - \eta)} \frac{k}{1 + \frac{u}{k}} \quad (4.2.58)$$

This ruin probability is minimized when  $k^\rho = \frac{2(\theta-\eta)u}{\theta u - (\theta-\eta)}$ . Thus, for Pareto-distributed claims, assuming  $\theta = \eta = 1$ , we find that  $\psi_k(u) = \frac{k}{k+u}$  and that  $k^\rho = 0$  as well. The insurance company should reinsure the entire portfolio of risks. The results for different values of  $k$  were computed using the FORTRAN program ASYMPROP.for in *Appendix 7*. These results are summarised in Table 13 and shown in Fig. 13. The results indicate that the ultimate ruin probabilities tend to zero as the retention level  $k$  tends to zero. The intuitive understanding is that the more risk the insurer passes to the reinsurer, the lower its ruin probabilities.



**Figure 13:** Asymptotic ruin prob. for large claims in CLM with QS reinsurance  
( $c = 6, \lambda = 2, \theta = \eta = 1$ )

It is evident from Fig. 13 that the ruin probabilities become smaller as  $k \rightarrow 0$ , meaning that the asymptotically optimal retention must be  $k^\rho = 0$ . This confirms the results shown in Fig. 11. And since the optimal retention  $k^*$  tends to the asymptotically optimal  $k^\rho$  that maximizes the adjustment coefficient, it follows that  $k^* = 0$ . This means that the insurance company must cede the entire portfolio of risks to a reinsurer. We can there-

fore conclude that the optimal combinational quota-share and XL reinsurance strategy is  $(k^*, a^*) = (0, \infty)$ .

#### 4.2.17 Ruin probability targets

The objective in this section is to maximize the total expected discounted dividends paid out to the shareholders until ruin

$$V^{\bar{D}, \bar{R}}(u) = \mathbb{E}_u \left[ \int_0^{\tau_b^{\bar{D}, \bar{R}}} e^{-\delta t} dD_t^b \right] \quad (4.2.59)$$

under a set ruin probability target

$$\psi^{\bar{D}, \bar{R}}(u) = \mathbb{P}(\tau_b^{\bar{D}, \bar{R}} < \infty) \leq \epsilon. \quad (4.2.60)$$

The quantity  $\delta > 0$  is the constant rate at which dividends are discounted and  $\mathbb{E}_u$  denotes expectation with respect to  $\mathbb{P}_u$ , the probability measure conditioned on the initial capital  $U_0^{\bar{D}, \bar{R}} = u$ . Thus, the optimal value function of this problem becomes

$$V_b(u) = V^{\bar{D}, \bar{R}}(u, \epsilon) = \sup_{(\bar{D}, \bar{R}) \in \Pi_u^{\bar{D}, \bar{R}}} \left\{ \mathbb{E}_u \left[ \int_0^{\tau_b^{\bar{D}, \bar{R}}} e^{-\delta t} dD_t^b \right] : \psi^{\bar{D}, \bar{R}}(u) \leq \epsilon \right\}, \quad (4.2.61)$$

where  $0 < \epsilon \leq 1$  is the permitted ruin probability and  $\psi^{\bar{D}, \bar{R}}(u)$  is the with-dividend-and-reinsurance ruin probability.

To solve for the survival probability  $\phi(u)$  (from which we obtain the ruin probability  $\psi(u) = 1 - \phi(u)$ ) and for the dividend value function  $V_b(u)$ , we use the fourth-order block-by-block method described in Chapter 3. To maximize dividends under a ruin probability target for a model with initial capital  $u$  and ruin probability tolerance  $\epsilon$ , the following calculations have been performed:

- (i) Using  $u$ , we solve the dividend maximization problem to determine the optimal barrier  $b^*$ .

- (ii) For each optimal dividend barrier  $b^*$ , we incorporate proportional and XL reinsurance into the CLM and the DPM.
- (iii) In the ultimate ruin problem, we choose  $b_0$  so that  $\psi(b_0) = \epsilon$ , which is the ultimate ruin probability at  $b_0$ . This means that dividends cannot be paid unless the survival probability  $1 - \epsilon$  is greater than  $\epsilon$ .

We now present the following results that are relevant to the case involving ruin probability targets.

**Theorem 4.2.2**

At every dividend barrier level  $b$ , there exists a unique probability  $\epsilon_b$  such that  $\mathbb{P}(\tau_b^{\overline{D}, \overline{R}} < T) = \epsilon_b$  and if  $b_1 < b_2$ , then  $\epsilon_{b_2} < \epsilon_{b_1}$ .

*Proof.* We note that  $\mathbb{P}(\tau_u^{\overline{D}, \overline{R}} < T)$  is defined  $\forall u > 0$ . This implies that  $\mathbb{P}(\tau_b^{\overline{D}, \overline{R}} < T)$  follows by setting  $u = b$  and since  $\mathbb{P}(\tau_b^{\overline{D}, \overline{R}} < T)$  is a decreasing function of  $u$ ,  $b_1 < b_2$  implies that  $\epsilon_{b_2} < \epsilon_{b_1}$ . □

With both  $b^*$  and  $b_0$ , the decision is based on Theorem 4.2.3 which now follows.

**Theorem 4.2.3**

Suppose that the barrier that solves the VIDE (3.4.5) is  $b^*$  and that the insurance company enforces the constraint  $\mathbb{P}(\tau^{\overline{D}, \overline{R}} < T) = \epsilon_b$ . Then

- (i) If  $b \leq b^*$ , then the optimal strategy is to pay dividends using barrier level  $b^*$ .
- (ii) If  $b > b^*$ , then the optimal strategy is to pay dividends using barrier level  $b$ .

*Proof.* See Appendix 4. □

Some numerical results are now presented based on the methods outlined in the previous chapter for the exponential and Pareto distributions.

**Table 14:** Ultimate ruin probabilities in the CLM for Exp(0.5) claims,  $c = 6$ ,  $\lambda = 2$ 

| $u$       | 0      | 4      | 8      | 10     | 12     | 14     | 16     | 18     | 20     |
|-----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\psi(u)$ | 0.6667 | 0.3423 | 0.1757 | 0.1259 | 0.0902 | 0.0646 | 0.0463 | 0.0332 | 0.0238 |

**(i) Ruin probability targets for the Cramér-Lundberg model: exponential claims**

As expected, increasing the initial capital  $u$  reduces the ruin probability  $\psi(u)$ . We now set ruin probabilities to obtain different values of initial capital to be used as ruin probability target values in the dividend model for the CLM without reinsurance, that is, with  $k = 1$  and  $a = \infty$ . Let  $\epsilon_i \equiv i$ -th ruin probability used. Then, choosing arbitrarily from Table 14,

$$\psi(b_0^1) = \epsilon_1 = 0.1259 \text{ gives } b_0^1 = 10.00$$

$$\psi(b_0^2) = \epsilon_2 = 0.0902 \text{ gives } b_0^2 = 12.00$$

$$\psi(b_0^3) = \epsilon_3 = 0.0646 \text{ gives } b_0^3 = 14.00$$

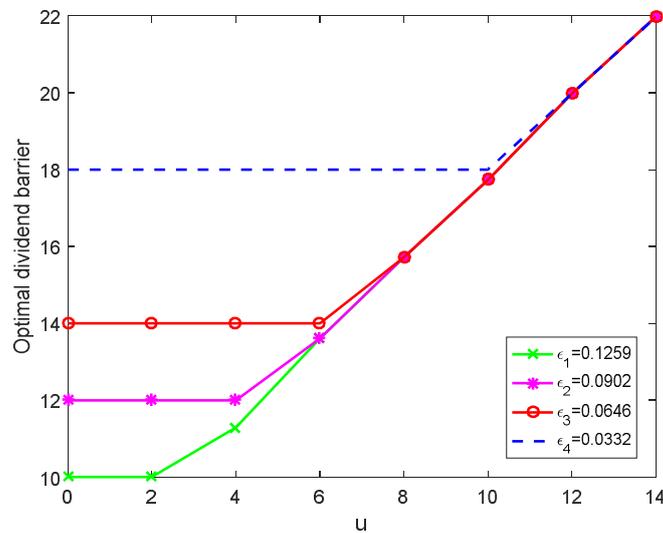
$$\psi(b_0^4) = \epsilon_4 = 0.0332 \text{ gives } b_0^4 = 18.00$$

**(ii) Dividends for the Cramér-Lundberg model: exponential claims****Table 15:** Dividends in the CLM with Exp(0.5) claims,  $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$ 

| $b$             | $u = 2$ | $u = 4$ | $u = 6$ | $u = 8$ | $u = 10$ | $u = 12$ | $u = 14$ |
|-----------------|---------|---------|---------|---------|----------|----------|----------|
| 2               | 5.8278  | 7.8278  | 9.8278  | 11.8278 | 13.8278  | 15.8278  | 17.8278  |
| 4               | 6.9460  | 9.1224  | 11.1224 | 13.1224 | 15.1224  | 17.1224  | 19.1224  |
| 6               | 7.8320  | 10.2860 | 12.4009 | 14.4009 | 16.4009  | 18.4009  | 20.4009  |
| 8               | 8.3886  | 11.0169 | 13.2822 | 15.3439 | 17.3439  | 19.3439  | 21.3439  |
| 10              | 8.5897  | 11.2810 | 13.6006 | 15.7117 | 17.7294  | 19.7294  | 21.7294  |
| 12              | 8.4894  | 11.1494 | 13.4419 | 15.5284 | 17.5225  | 19.5058  | 21.5058  |
| 14              | 8.1872  | 10.7525 | 12.9634 | 14.9757 | 16.8988  | 18.8115  | 20.7718  |
| 16              | 7.7567  | 10.1871 | 12.2817 | 14.1881 | 16.0101  | 17.8223  | 19.6794  |
| 18              | 7.2728  | 9.5516  | 11.5155 | 13.3030 | 15.0114  | 16.7105  | 18.4517  |
| 20              | 6.7561  | 8.8729  | 10.6974 | 12.3579 | 13.9448  | 15.5232  | 17.1408  |
| $b^* = 10.27$   | 8.5923  | 11.2844 | 13.6047 | 15.7165 | 17.7348  | 19.9749  | 21.9749  |
| $b_0^1 = 10.00$ | 10.0000 | 11.2844 | 13.6047 | 15.7165 | 17.7348  | 19.9749  | 21.9749  |
| $b_0^2 = 12.00$ | 12.0000 | 12.0000 | 13.6047 | 15.7165 | 17.7348  | 19.9749  | 21.9749  |
| $b_0^3 = 14.00$ | 14.0000 | 14.0000 | 14.0000 | 15.7165 | 17.7348  | 19.9749  | 21.9749  |
| $b_0^4 = 18.00$ | 18.0000 | 18.0000 | 18.0000 | 18.0000 | 18.0000  | 19.9749  | 21.9749  |

Using Theorem 4.2.3, we obtain the optimal barriers under ruin probability targets. For example, for initial capital  $u = 2$  the optimal dividend barrier is  $b^* = 8.5923$  and  $b_0^1 =$

10.0000. Since  $b_0^1 > b^*$ , we take 10.0000 as the optimal barrier. For  $u = 6$ ,  $b^* = 13.6047$  and  $b_0^1 = 10.0000$ . Since  $b^* > b_0^1$ , we take 13.6047 as the optimal barrier. The optimal barriers for other values of  $u$  can be obtained in a similar manner. The results are presented in Table 15 and Fig. 14. The company pays out dividends to the shareholders whenever  $b^* > b_0^1$  because of the ruin probability target. Figure 14 shows that as the ruin probability reduces, the optimal dividend barrier increases and this is precisely the goal of dividend maximization. It should be noted that for Pareto claim sizes, the company can pay dividends at all barrier levels, since  $b^* > b_0^i (i = 1, 2, 3, 4) \forall u > 0$ .



**Figure 14:** Numerical optimal barriers in CLM for Exp(0.5) claims,  $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$

**(iii) Dividends for the Cramér-Lundberg model compounded by proportional reinsurance: exponential claims**

Since it is optimal not to take proportional reinsurance in the small claim case involving the CLM (see Fig. 4(a)), the ruin probability targets and optimal barriers under proportional reinsurance are the same as shown in the immediately preceding sections (see Tables 14 and 15). The same can be said about the large claim case in the CLM (see Fig. 4(b)).

(iv) **Dividends for the Cramér-Lundberg model compounded by excess-of-loss reinsurance: exponential claims**

According to Table 4, it is optimal not to take XL reinsurance in the CLM for exponential claims. From this we deduce that the ruin probability targets and optimal dividend barriers for XL reinsurance are the same as those for QS reinsurance as shown in Tables 14 and 15.

(v) **Ruin probability targets for the Cramér-Lundberg model compounded by excess-of-loss reinsurance: Pareto claims**

Since in the large claim case involving the CLM the optimal policy is to take XL reinsurance with  $a^* = 10$ , we have to compute ruin probabilities for  $a = 10$ . These are shown in Table 16.

**Table 16:** Ultimate ruin prob. in the CLM for Par(3,2) claims,  $c = 6$ ,  $\lambda = 2$

| $u$       | 0      | 4      | 8      | 10     | 10.5   | 11     | 11.5   | 12     | 12.5   |
|-----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\psi(u)$ | 0.6667 | 0.5363 | 0.5169 | 0.5132 | 0.4888 | 0.4318 | 0.3512 | 0.2492 | 0.1251 |

The ruin probability  $\psi(u)$  reduces quite slowly as the initial capital  $u$  increases. We now set ruin probabilities to obtain different values of initial capital to be used as ruin probability target values in the dividend model for the CLM with optimal XL reinsurance retention  $a^* = 10$ . Let  $\epsilon_i \equiv i$ -th ruin probability used. Then, choosing arbitrarily from Table 16,

$$\psi(b_0^1) = \epsilon_1 = 0.5169 \text{ gives } b_0^1 = 8.00$$

$$\psi(b_0^2) = \epsilon_2 = 0.5132 \text{ gives } b_0^2 = 10.00$$

$$\psi(b_0^3) = \epsilon_3 = 0.4318 \text{ gives } b_0^3 = 11.00$$

$$\psi(b_0^4) = \epsilon_4 = 0.2492 \text{ gives } b_0^4 = 12.00$$

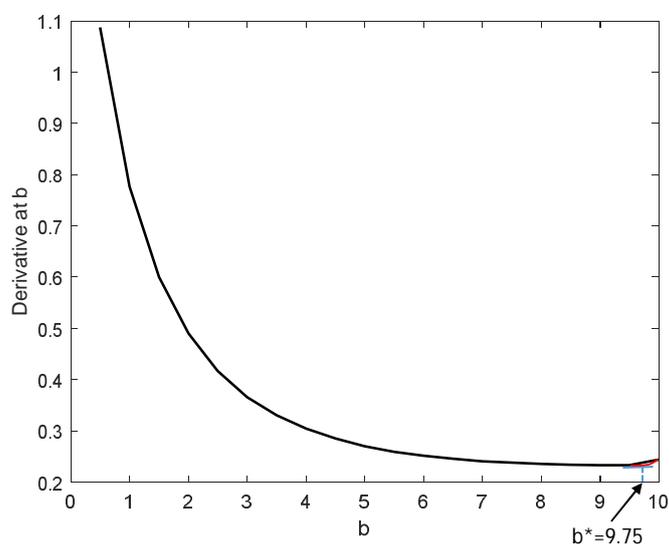
(vi) **Dividends for the Cramér-Lundberg model compounded by excess-of-loss reinsurance: Pareto claims**

The optimal dividend barrier for this model is found to be  $b^* \cong 9.75$  as shown in Fig. 15. It turns out that for Pareto(3,2) claims in the CLM compounded by XL reinsurance

**Table 17:** Dividends in the CLM with XL reinsurance: Par(3,2) claims ( $c = 6, \lambda = 2, \delta = 0.1$ )

| $b$          | $u = 2$ | $u = 4$ | $u = 6$ | $u = 8$ | $u = 10$ | $u = 12$ | $u = 14$ |
|--------------|---------|---------|---------|---------|----------|----------|----------|
| 2            | 13.7872 | 15.7872 | 17.7872 | 19.7872 | 21.7872  | 23.7872  | 25.7872  |
| 4            | 22.2092 | 24.6831 | 26.6831 | 28.6831 | 30.6831  | 32.6831  | 34.6831  |
| 6            | 26.8729 | 29.8662 | 32.0364 | 34.0364 | 36.0364  | 38.0364  | 40.0364  |
| 8            | 28.6866 | 31.8819 | 34.1985 | 36.2515 | 38.2515  | 40.2515  | 42.2515  |
| 10           | 27.6877 | 30.7718 | 33.0077 | 34.9893 | 36.9052  | 38.9052  | 40.9052  |
| $b^* = 9.75$ | 28.9684 | 32.1951 | 34.5345 | 36.6077 | 38.3522  | 40.3522  | 42.3522  |

$b^* > b_0^i (i = 1, 2, 3, 4) \forall u > 0$ . Therefore, the company can pay dividends at all dividend barrier levels.



**Figure 15:** Optimal dividend barrier in CLM for Par(3,2) claims,  $c = 6, \lambda = 2, \delta = 0.1$

(vii) **Ruin probability targets for the diffusion-perturbed model: exponential claims**

Since the ruin probabilities for the DPM in Sections 4.2.13 to 4.2.15 were obtained using  $\sigma = 0.02$ , it is necessary to recompute them using  $\sigma = 1.0$  so as to obtain ruin probability targets for the dividend maximization problem. These new probabilities are given in Table 18 which also shows the impact of proportional reinsurance.

**Table 18:** Ultimate ruin prob. for DPM with QS reins.: Exp(0.5) claims,  $c = 6$ ,  $\lambda = 2$ ,  $\sigma = 1.0$ 

| $u$ | $\psi_1(u)$ | $\psi_{0.8}(u)$ | $\psi_{0.6}(u)$ | $\psi_{0.4}(u)$ | $\psi_{0.2}(u)$ |
|-----|-------------|-----------------|-----------------|-----------------|-----------------|
| 0   | 1.0000      | 1.0000          | 1.0000          | 1.0000          | 1.0000          |
| 2   | 0.4952      | 0.2816          | 0.2119          | 0.1719          | 0.1330          |
| 4   | 0.3581      | 0.1630          | 0.1236          | 0.1100          | 0.0983          |
| 6   | 0.2589      | 0.0926          | 0.0717          | 0.0708          | 0.0738          |
| 8   | 0.1872      | 0.0517          | 0.0413          | 0.0458          | 0.0559          |
| 10  | 0.1354      | 0.0285          | 0.0237          | 0.0297          | 0.0427          |
| 12  | 0.0979      | 0.0155          | 0.0135          | 0.0193          | 0.0328          |
| 14  | 0.0708      | 0.0083          | 0.0077          | 0.0126          | 0.0254          |
| 16  | 0.0512      | 0.0044          | 0.0044          | 0.0083          | 0.0197          |
| 18  | 0.0370      | 0.0023          | 0.0025          | 0.0054          | 0.0153          |
| 20  | 0.0268      | 0.0012          | 0.0014          | 0.0035          | 0.0120          |

It should be noted from Table 18 that the ruin probabilities reduce as the initial surplus increases. Also, the ruin probabilities increase slightly for  $k = 0.4$  and  $k = 0.2$  and  $u > 6$ . Furthermore, since the optimal reinsurance policy for the dividend maximization problem is  $(k^*, a^*) = (1, \infty)$ , that is, do not reinsure, we use only the ruin probabilities  $\psi_{k=1}(u)$  which are the same as  $\psi_{a=\infty}(u)$  to choose ruin probability targets and set optimal dividend barriers under set ruin probability targets.

Thus, from Table 18, we arbitrarily choose

$$\psi(b_0^1) = \epsilon_1 = 0.1354, \text{ giving } b_0^1 = 10.00$$

$$\psi(b_0^2) = \epsilon_2 = 0.0708, \text{ giving } b_0^2 = 14.00$$

$$\psi(b_0^3) = \epsilon_3 = 0.0512, \text{ giving } b_0^3 = 16.00$$

$$\psi(b_0^4) = \epsilon_4 = 0.0268, \text{ giving } b_0^4 = 20.00$$

(viii) **Dividends for the diffusion-perturbed model compounded by proportional reinsurance: exponential claims**

The optimal barriers for varying initial surplus values are shown in Table 19 and Fig. 16.

**Table 19:** Dividends in DPM for Exp(0.5) claims,  $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$ ,  $\sigma = 1.0$ 

| $b$             | $u = 2$ | $u = 4$ | $u = 6$ | $u = 8$ | $u = 10$ | $u = 12$ | $u = 14$ |
|-----------------|---------|---------|---------|---------|----------|----------|----------|
| 2               | 5.6462  | 7.6462  | 9.6462  | 11.6462 | 13.6462  | 15.6462  | 17.6462  |
| 4               | 6.6968  | 8.8651  | 10.8651 | 12.8651 | 14.8651  | 16.8651  | 18.8651  |
| 6               | 7.6659  | 10.1478 | 12.2855 | 14.2855 | 16.2855  | 18.2855  | 20.2855  |
| 8               | 7.9578  | 10.5342 | 12.7533 | 14.7682 | 16.7682  | 18.7682  | 20.7682  |
| 10              | 8.1250  | 10.7556 | 13.0213 | 15.0786 | 17.0385  | 19.0385  | 21.0385  |
| 12              | 8.5155  | 11.2725 | 13.6471 | 15.8032 | 17.8573  | 19.8924  | 21.8924  |
| 14              | 8.2853  | 10.9678 | 13.2782 | 15.3760 | 17.3746  | 19.3547  | 21.3774  |
| 16              | 7.2210  | 9.5590  | 11.5726 | 13.4010 | 15.1428  | 16.8686  | 18.6314  |
| 18              | 6.8983  | 9.1318  | 11.0554 | 12.8020 | 14.4660  | 16.1146  | 17.7987  |
| 20              | 6.7561  | 8.8729  | 10.6974 | 12.3579 | 13.9448  | 15.5232  | 17.1408  |
| $b^* = 12.35$   | 8.6865  | 11.4989 | 13.9212 | 16.1206 | 18.2160  | 20.2919  | 22.6480  |
| $b_0^1 = 10.00$ | 10.0000 | 11.4989 | 13.9212 | 16.1206 | 18.2160  | 20.2919  | 22.6480  |
| $b_0^2 = 14.00$ | 14.0000 | 14.0000 | 14.0000 | 16.1206 | 18.2160  | 20.2919  | 22.6480  |
| $b_0^3 = 16.00$ | 16.0000 | 16.0000 | 16.0000 | 16.1206 | 18.2160  | 20.2919  | 22.6480  |
| $b_0^4 = 20.00$ | 20.0000 | 20.0000 | 20.0000 | 20.0000 | 20.0000  | 20.2919  | 22.6480  |

Figure 16 shows that set ruin probability targets have a positive impact on the dividend payouts because the optimal barriers increase with a decrease in the ultimate ruin probabilities. In this way, ruin probability targets enhance the survival of an insurance company. Instead of paying out dividends to the shareholders, say, at a barrier level of 12.35 for  $u = 2$ , the company can delay the payment until the surplus exceeds a barrier level of 14.00 because of the ruin probability target.

(ix) **Ruin probability targets for the diffusion-perturbed model: Pareto claims**

The infinite ruin probabilities for the DPM for Pareto(3,2) claim sizes are given in Table 20.

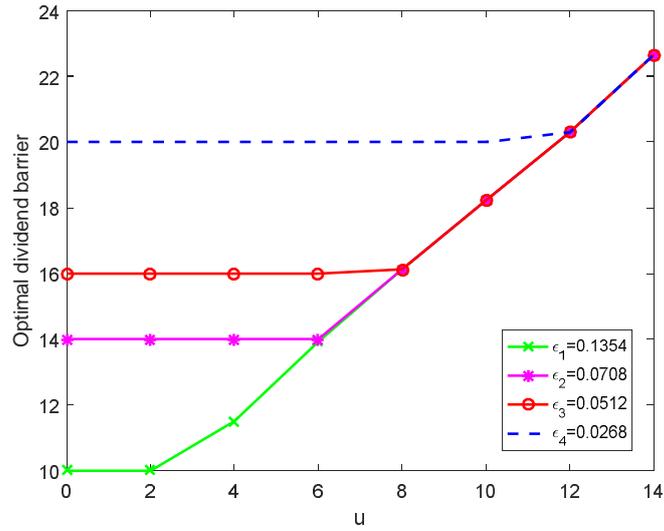
**Table 20:** Ultimate ruin prob. in DPM for Par(3,2) claims,  $c = 6$ ,  $\lambda = 2$ ,  $\sigma = 1.0$ 

| $u$       | 0      | 8      | 16     | 24     | 32     | 34     | 36     | 38     | 40     |
|-----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\psi(u)$ | 1.0000 | 0.0273 | 0.0080 | 0.0036 | 0.0020 | 0.0018 | 0.0016 | 0.0014 | 0.0012 |

Choosing arbitrarily from Table 20, we have

$$\psi(b_0^1) = \epsilon_1 = 0.0020, \text{ giving } b_0^1 = 32.00$$

$$\psi(b_0^2) = \epsilon_2 = 0.0016, \text{ giving } b_0^2 = 36.00$$



**Figure 16:** Numerical optimal barriers in DPM for Exp(0.5) claims,  $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$ ,  $\sigma = 1.0$

$$\psi(b_0^3) = \epsilon_3 = 0.0014, \text{ giving } b_0^3 = 38.00$$

$$\psi(b_0^4) = \epsilon_4 = 0.0012, \text{ giving } b_0^4 = 40.00$$

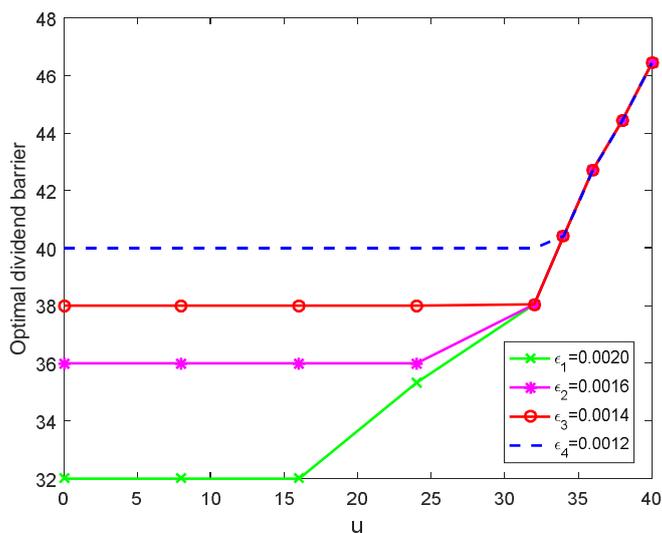
(x) **Dividends for the diffusion-perturbed model: Pareto claims**

The optimal barriers for Pareto claims are shown in Table 21 and Fig. 17 both of which also satisfy Theorem 4.2.3 and show selected optimal dividend barriers for large claims in the DPM against the initial capital  $u$ .

Table 21 shows that instead of making dividend payouts at a barrier level of 11.50, say for  $u = 2$ , the company can hold off payment until the surplus exceeds a barrier level of 32.00. The application of the ruin probability target makes this possible. It should also be noted that  $b^* > b_0^i$  ( $i = 1, 2, 3, 4$ )  $\forall u \geq 6$ . It means that as long as the initial surplus is at least 6, the company can pay dividends at all barrier levels. This is the point underscored by Fig. 17 as well.

**Table 21:** Dividends in DPM for Par(3,2) claims,  $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$ ,  $\sigma = 1.0$

| $b$             | $u = 2$ | $u = 4$ | $u = 6$ | $u = 8$ | $u = 10$ | $u = 12$ | $u = 14$ |
|-----------------|---------|---------|---------|---------|----------|----------|----------|
| 2               | 12.4488 | 14.4488 | 16.4488 | 18.4488 | 20.4488  | 22.4488  | 24.4488  |
| 4               | 20.6442 | 23.1525 | 25.1525 | 27.1525 | 29.1525  | 31.1525  | 33.1525  |
| 6               | 25.0237 | 28.0642 | 30.2186 | 32.2186 | 34.2186  | 36.2186  | 38.2186  |
| 8               | 27.0098 | 30.2917 | 32.6171 | 34.6541 | 36.6541  | 38.6541  | 40.6541  |
| 10              | 27.6548 | 31.0150 | 33.3960 | 35.4816 | 37.4898  | 39.4898  | 41.4898  |
| 12              | 27.3761 | 30.7024 | 33.0594 | 35.1240 | 37.1120  | 39.1035  | 41.1035  |
| 14              | 27.2562 | 30.5680 | 32.9146 | 34.9702 | 36.9495  | 38.9322  | 40.9698  |
| 16              | 25.9341 | 29.3282 | 31.7332 | 33.8399 | 35.8684  | 37.9004  | 39.9887  |
| 18              | 24.7624 | 27.7711 | 29.9030 | 31.7705 | 33.5687  | 35.3701  | 37.2212  |
| 20              | 20.7777 | 23.3022 | 25.0911 | 26.6581 | 28.1669  | 29.6784  | 31.2317  |
| $b^* = 11.50$   | 31.5039 | 35.3317 | 38.0441 | 40.4200 | 42.7077  | 44.4236  | 46.4236  |
| $b_0^1 = 32.00$ | 32.0000 | 35.3317 | 38.0441 | 40.4200 | 42.7077  | 44.4236  | 46.4236  |
| $b_0^2 = 36.00$ | 36.0000 | 36.0000 | 38.0441 | 40.4200 | 42.7077  | 44.4236  | 46.4236  |
| $b_0^3 = 38.00$ | 38.0000 | 38.0000 | 38.0441 | 40.4200 | 42.7077  | 44.4236  | 46.4236  |
| $b_0^4 = 40.00$ | 40.0000 | 40.0000 | 40.0000 | 40.4200 | 42.7077  | 44.4236  | 46.4236  |



**Figure 17:** Numerical optimal barriers in DPM for Par(3,2) claims,  $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$ ,  $\sigma = 1.0$

We now present a discussion of the results which serves to summarise the study's major findings.

### 4.3 Discussion of results

This study was conducted with the aim of developing and analysing a mathematical model for optimization of dividend payouts and reinsurance policies for a diffusion-perturbed classical risk process under a set ruin probability target. The study sought to achieve this aim by fulfilling four specific objectives as given in Section 1.4.2. The basic diffusion-perturbed model in Section 3.3.1 was modified by incorporating proportional and excess-of-loss reinsurance as well as dividend payouts (see Sections 3.3.2 and 3.3.3), thus fulfilling the first specific objective.

Analysis of the model was performed, resulting in derivation of the HJB, integrodifferential and integral equations corresponding to the dividend maximization and ruin probability minimization problems. The resulting Volterra integral equations were solved numerically using the block-by-block method as outlined in Chapter 3. Thus, the second objective was also attained. The third objective involved determining the optimal reinsurance policy that maximizes the dividend payouts as well as minimizes the ultimate ruin probabilities.

With regard to the ruin probability minimization problem, the results presented in the previous section show that proportional and XL reinsurance both result in a reduction in the ruin probabilities. However, the reduction is more drastic for Pareto than for exponential claims in both the Cramér-Lundberg and diffusion-perturbed models. While the literature shows that the optimal reinsurance strategy is a pure XL, i.e.,  $(1, a^*)$  (see, e.g., Li *et al.*, 2017; Zhang and Liang, 2016; Hu and Zhang, 2016), a comparison of the figures presented in the foregoing shows that proportional reinsurance results in lower ruin probabilities than XL reinsurance and is therefore optimal. The optimal quota-share retention was found as  $k^* = 0$ , meaning that in both the small and large claim cases in the Cramér-Lundberg model, it is optimal for the insurance company to reinsure the whole portfolio using proportional reinsurance. Going by the results in Fig. 13, the same conclusion can be drawn about the diffusion-perturbed model. Thus, the optimal combinational quota-share and XL reinsurance strategy is a pure quota-share reinsurance with  $k^* = 0$ , i.e.,  $(k^*, a^*) = (0, \infty)$ . It should be noted that full reinsurance is not ideal from the reinsurer's standpoint and this provides a strong argument for the use of non-cheap reinsurance.

On the dividend maximization problem, Fig. 4 shows that the optimal QS retention is  $k^* = 1$  (that is, do not reinsure) both for small and large claims in the CLM. In terms of XL reinsurance, Fig. 5 shows that the optimal XL retention level is  $a = \infty$  for small claims, that is, it is optimal not to reinsure, while Fig. 6 shows that the optimal retention  $a^* \in [5, 10]$  for large claims in the CLM. The optimal XL retention was, in fact, found as  $a^* = 10$  for Pareto claims in the CLM. However, comparing Figs. 4(a) and 5 shows that QS reinsurance is optimal for Exp(0.5) claims in the CLM. However, a comparison of Figs. 4(b) and 6 shows that for Pareto(3,2) claims in the CLM XL reinsurance is optimal at  $a^* = 10$ . The optimal dividend strategies for the CLM are  $b^* = 10.27$  for small claims and  $b^* = 9.35$  for large ones.

From the results presented in the previous sections, it can be concluded that at the optimal dividend barrier  $b^*$  for the DPM it is optimal not to take reinsurance. This applies both in the light- and heavy-tailed cases and the optimal retentions do not vary according to the claim size distribution used. Thus, for small claims the optimal policy is  $(\bar{D}, \bar{R}) = (12.35, (1, \infty))$ , while for large claims it is  $(\bar{D}, \bar{R}) = (11.50, (1, \infty))$ . For the chosen parameters, dividends are maximized when no reinsurance is taken. Thus, though the literature (e.g., Li *et al.*, 2017; Zhang and Liang, 2016) shows that in a combinational proportional and XL reinsurance scenario the optimal strategy is a pure XL, i.e.,  $(1, a^*)$ , this study has established that in the DPM neither QS nor XL reinsurance has any advantage over the other.

The fourth objective involved determining reinsurance and dividend strategies under ruin probability targets. In this regard, the study has established that the reinsurance strategies are no different than before imposing ruin probability targets but that the optimal dividend barriers to use for dividend maximization increase as the ruin probability reduces. Insurance companies should therefore work towards reducing their ruin probabilities using some risk measures as this has a desirable effect on the optimal dividend barriers to be used for dividend payouts.

#### 4.4 Summary

In this chapter, numerical results have been presented based on the numerical method outlined in Chapter 3 both for small and large claims. The results are based on the exponential and

Pareto distributions for small and large claims, respectively. The chapter has also analysed the results and provided some discussions and conclusions. The next chapter draws general conclusions, makes recommendations based on the research findings and proposes some open problems for future research.

## CHAPTER FIVE

### CONCLUSION AND RECOMMENDATIONS

#### 5.1 Introduction

This chapter draws conclusions based on the study's findings and makes recommendations which should be of interest to insurance companies in terms of policy measures to be taken in relation to ruin probability minimization, dividend maximization and the use of ruin probability targets. The chapter concludes by suggesting possible extensions to this work.

#### 5.2 Conclusion

- (i) *Purpose of the study:* The purpose of the study was to formulate and analyse a mathematical model for the optimization of dividend payouts and reinsurance policies for a diffusion-perturbed classical risk model under a set ruin probability target. The study had two major objective functions: the dividend value function and the ultimate ruin probability. The objectives were to maximize the dividend payouts to the company's shareholders and to minimize its ruin probabilities. Using the basic diffusion-perturbed compound Poisson model as a starting point, the controlled model incorporating reinsurance and dividends was formulated. The HJB, integrodifferential and integral equations were obtained. The proposed method of solution of the integral equations was the block-by-block method. This method was applied on the resulting Volterra integral equations of the second kind and the optimal value functions were determined. The effect of ruin probability targets on the optimal dividend and reinsurance strategies was also investigated.
- (ii) *Methodology used:* A gridsize of  $h = 0.01$  was used throughout in performing the data simulations. The simulations were performed using a Samsung Series 3 PC with an Intel Celeron 847 processor at 1.10GHz and 6.0GB RAM. To reduce computing time, the block-by-block numerical method was implemented using the FORTRAN programming language, taking advantage of its DOUBLE PRECISION feature which gives a high de-

gree of accuracy. Slower software such as R, MATLAB, Maple or Mathematica could have been used for coding the block-by-block method but at the expense of considerably longer computing time. All the figures in this study were constructed using MATLAB R2016a.

(iii) *Research findings:* The research findings presented in the preceding chapters are summarised as follows:

- (a) Results indicate that the optimal combinational QS and XL reinsurance policy in the ruin minimization problem is a pure QS with  $k^* = 0$ . This means that the company has the possibility of taking full reinsurance, that is, ceding its entire portfolio of risks to a reinsurer.
- (b) For dividend maximization in the CLM, the study established that  $(k^*, a^*) = (1, \infty)$  for small claims and that  $(k^*, a^*) = (1, 10)$  for large claims. That is, it is optimal not to reinsure in the small claim case and to take XL reinsurance in the large claim case (see Section 4.2.9).
- (c) For dividend maximization in the DPM, the results show that  $(k^*, a^*) = (1, \infty)$  for small and large claims (see Section 4.2.9). This means that it is optimal not to reinsure both for small and large claims, meaning that neither QS nor XL reinsurance has the advantage over the other.
- (d) Though the results have shown that a barrier strategy is optimal for payment of dividends to the shareholders, the company should use a higher optimal barrier for small claims than for large ones regardless of the risk model used, as shown in the summary in Table 22.

**Table 22:** Optimal dividend barriers

| <b>Model</b> | <b>Small claims</b> | <b>Large claims</b> |
|--------------|---------------------|---------------------|
| CLM          | 10.27               | 9.35                |
| DPM          | 12.35               | 11.50               |

- (e) The study has shown that as the ruin probability reduces the optimal dividend barrier to use for payment of dividends increases, and vice versa, as shown in Figs. 14, 16 and 17 in Chapter 4 (Section 4.2.17).

### 5.3 Recommendations

Based on the research findings presented in the previous chapter and summarised in Section 5.2, the following recommendations are made:

- (i) An insurance company should use reinsurance as a risk measure as it reduces the probability of ultimate ruin, thus enhancing the company's chances of survival as demonstrated by the results in Figs. 11 and 12 (see also Tables 8, 9, 11 and 12).
- (ii) Since the optimal combinational QS and XL reinsurance policy in the ruin minimization problem is a pure QS with  $k^* = 0$ , the company has the possibility of ceding its entire portfolio of risks to a reinsurer or taking full reinsurance. While this may be undesirable for the reinsurance company, it is highly advantageous for the insurance company as it can benefit from reinsurance cover without itself bearing any risks.
- (iii) Reinsurance plays no part in dividend maximization. In view of this, it is recommended that the insurance company should use other risk measures in order to maximize dividend payouts. These measures include investment, capital injections, portfolio selection and premium control.
- (iv) Based on the results obtained, the company should use a higher optimal barrier for small claims than for large ones regardless of the risk model used.
- (v) The use of ruin probability targets by the insurance company is highly desirable from the shareholders' point of view. This is because a lower ruin probability makes it possible for the insurance company to pay more in dividends to the shareholders (that is, to use a higher dividend value function or optimal barrier).

### 5.4 Possible extensions

The results obtained in this dissertation raise further interesting and challenging questions. Therefore, this study can be extended by the following:

- (i) Inclusion of *investments*: Investment can be at two levels: (a) in risk-free assets such as treasury bills and notes, bonds and fixed deposit accounts. These provide virtually guaranteed returns, since their future returns are known with certainty beforehand; (b) in risky assets, that is, the stock market.
- (ii) Considering *non-cheap reinsurance*: This is a reinsurance mechanism whereby, for a given risk, the reinsurer requires more premium, and therefore uses a higher safety loading, than the insurer. Thus, the problem is to investigate the case  $\theta > \eta$ , where  $\theta$  and  $\eta$  are the safety loadings of the reinsurer and cedent, respectively.
- (iii) Incorporating *transaction costs* when paying dividends: These could be proportional transaction costs incurred during the dividend payout process or generated by taxation. However, the fixed transaction costs in the dividends payout process can be omitted because of the ever-increasing efficiency with which the financial system is operated.
- (iv) Exploring optimality of *other dividend strategies* (e.g., threshold or band).
- (v) Incorporating *capital injections, equity issuance or refinancing*.
- (vi) Replacing the claim number process  $N$  by a *general renewal process* so that the surplus process becomes a Sparre-Andersen model.
- (vii) Using a *Lévy process* instead of a compound Poisson process for modelling the total claim amounts.

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## APPENDICES

### Appendix 1: Proof of Lemma 3.4.1

*Proof.* We prove the lemma for the case  $\bar{\tau} =$  some fixed  $T \geq 0$ . The general case follows using standard methods. We call

$$v(u, T) = \sup_{(\bar{D}, \bar{R}) \in \Pi_u^{D, R}} \mathbb{E}_u \left[ \int_0^{T \wedge \tau^{\bar{D}, \bar{R}}} e^{-\delta s} dD_s^b + e^{-\delta(T \wedge \tau^{\bar{D}, \bar{R}})} V \left( U_{T \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}} \right) \right] \quad (5.4.1)$$

and first prove that  $V_u \leq v(u, T)$ . Take any admissible dividend and reinsurance strategy  $(\bar{D}, \bar{R}) = (D_t^b, R_t) \in \Pi_u^{D, R}$ . We can write

$$\begin{aligned} V_u^{\bar{D}, \bar{R}} &= \mathbb{E}_u \left[ \int_0^{T \wedge \tau^{\bar{D}, \bar{R}}} e^{-\delta s} dD_s^b \right] + e^{-\delta T} \mathbb{E}_u \left[ \mathbf{1}_{\{\tau^{\bar{D}, \bar{R}} > T\}} \mathbb{E} \left( \int_0^{\tau^{\bar{D}, \bar{R}} - T} e^{-\delta s} dD_{s+T}^b \mid U_T^{\bar{D}, \bar{R}} \right) \right] \\ &\leq \mathbb{E}_u \left[ \int_0^{T \wedge \tau^{\bar{D}, \bar{R}}} e^{-\delta s} dD_s^b \right] + e^{-\delta T} \mathbb{E}_u \left[ \mathbf{1}_{\{\tau^{\bar{D}, \bar{R}} > T\}} V \left( U_T^{\bar{D}, \bar{R}} \right) \right] \\ &= \mathbb{E}_u \left[ \int_0^{T \wedge \tau^{\bar{D}, \bar{R}}} e^{-\delta s} dD_s^b \right] + \mathbb{E}_u \left[ e^{-\delta(T \wedge \tau^{\bar{D}, \bar{R}})} V \left( U_{T \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}} \right) \right] \\ &\leq v(u, T) \end{aligned} \quad (5.4.2)$$

Since the optimal dividend and reinsurance value function is defined by

$$V_u = \sup \{ V_u^{\bar{D}, \bar{R}} \text{ with } (\bar{D}, \bar{R}) \in \Pi_u^{D, R} \} \text{ for } u \geq 0 \quad (5.4.3)$$

we obtain the result. We now prove that  $V_u \geq v(u, T)$ . Given any  $\varepsilon > 0$ , take an admissible strategy  $(\bar{D}, \bar{R}) = (D_t^b, R_t) \in \Pi_u^{D, R}$  s.t.

$$\mathbb{E}_u \left[ \int_0^{T \wedge \tau^{\bar{D}, \bar{R}}} e^{-\delta s} dD_s^b + e^{-\delta(T \wedge \tau^{\bar{D}, \bar{R}})} V \left( U_{T \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}} \right) \right] \geq v(u, T) - \frac{\varepsilon}{2} \quad (5.4.4)$$

where  $U_t^{\bar{D}, \bar{R}}$  is the corresponding controlled risk process. Since  $V$  is increasing and continuous

in  $[0, +\infty)$ , we can find an increasing sequence  $(u_i)_{i \in \mathbb{N}}$  with  $u_1 = 0$  and  $\lim_{i \rightarrow \infty} u_i = \infty$  s.t. if  $y \in [u_i, u_{i+1})$  then

$$V(y) - V(u_i) < \frac{\varepsilon}{4} \text{ for } i \geq 0 \quad (5.4.5)$$

Take admissible strategies  $(\bar{D}_i, \bar{R}_i) = (D_t^i, R_t^i)_{t \geq 0} \in \Pi_u^{D, R}$  s.t.  $V(u_i) - V^{\bar{D}_i, \bar{R}_i}(u_i) \leq \frac{\varepsilon}{4}$ . We define a new strategy  $(\bar{D}, \bar{R})_* = (D_t^*, R_t^*)_{t \geq 0}$  as follows:

- If  $\tau^{\bar{D}, \bar{R}} \leq T$  take  $D_t^* = D_t^b \forall t \geq 0$ ;
- If  $\tau^{\bar{D}, \bar{R}} > T$  take  $D_t^* = D_t^b$  for  $t \in [0, T]$ ;
- If  $\tau^{\bar{D}, \bar{R}} > T$  and  $U_T^{\bar{D}, \bar{R}} \in [u_i, u_{i+1})$ , then pay immediately  $U_T^{\bar{D}, \bar{R}} - u_i$  as dividends at time  $T$ , that is, take  $D_{T+}^b - D_T^b = U_T^{\bar{D}, \bar{R}} - u_i$  and then follow the strategy  $\bar{D}_i$ .

By construction,  $(\bar{D}, \bar{R})_*$  is an admissible strategy and if  $U_T^{\bar{D}, \bar{R}} \in [u_i, u_{i+1})$ , we have

$$V^{(\bar{D}, \bar{R})_*}(U_T^{\bar{D}, \bar{R}}) = U_T^{\bar{D}, \bar{R}} - u_i + V^{\bar{D}, \bar{R}}(u_i) \geq V(u_i) - \frac{\varepsilon}{4} \quad (5.4.6)$$

Using (5.4.1)-(5.4.6), we obtain

$$v(u, T) - V^{(\bar{D}, \bar{R})_*}(u) \leq \mathbb{E}_u \left[ \int_0^{T \wedge \tau^{\bar{D}, \bar{R}}} e^{-\delta s} dD_s^b + e^{-\delta(T \wedge \tau^{\bar{D}, \bar{R}})} V \left( U_{T \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}} \right) \right] - V^{(\bar{D}, \bar{R})_*}(u) + \frac{\varepsilon}{2} < \varepsilon$$

and so the result follows. □

## Appendix 2: Proof of Theorem 3.4.2

*Proof.* Given any dividend payment rate  $l \geq 0$  and reinsurance function  $R$ , we consider the admissible strategy  $(\bar{D}, \bar{R}) = ((lt), (R))_{t \geq 0}$  which pays dividends at a constant rate  $l$  and takes reinsurance with constant retained function  $R(k, a, x) = kX_i \wedge a$ . Let the corresponding controlled surplus process be denoted by  $U_t^{\bar{D}, \bar{R}} = U_t^{\bar{R}} - lt$  and the corresponding ruin time by  $\tau^{\bar{D}, \bar{R}}$ . The surplus process  $U_{t \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}}$  stopped at the ruin time is a Markov process, so that, as in  $\mathcal{G} \left( (U_{t \wedge \tau^{\bar{D}, \bar{R}}})_{t \geq 0}, f \right) (u) = \frac{1}{2} \sigma^2 f''(u) + cf'(u) - (\lambda + \delta)f(u) + \lambda \mathcal{I}(f)(u)$  and *Remark 1.7* in Azcue and Muler (2014), if the company pays dividends at constant rate  $l \geq 0$ , we have

$$\tilde{\mathcal{G}} \left( U_{t \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}}, V \right) (u) = \begin{cases} (c^{\bar{R}} - l)V'(u) - (\lambda + \delta)V(u) + \lambda \mathcal{I}(V)(u) & l \leq c^{\bar{R}} \\ (c^{\bar{R}} - l)V'(u) - (\lambda + \delta)V(u) + \lambda \mathcal{I}(V)(u^-) & l > c^{\bar{R}} \end{cases}$$

where  $\tilde{\mathcal{G}}$  is the discounted infinitesimal generator defined by

$$\tilde{\mathcal{G}} \left( \left( U_{t \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}} \right)_{t \geq 0}, V \right) := \lim_{t \rightarrow 0} \frac{\mathbb{E}_u \left[ e^{-\delta t} V \left( U_{t \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}} \right) \right] - V(u)}{t} \quad (5.4.7)$$

and  $\mathcal{I}(V)$  is the integral operator

$$\mathcal{I}(V)(u) = \int_0^\infty V(u - R(k, a, x)) dF(x) = \int_0^u V(u - kx \wedge a) dF(x) \quad (5.4.8)$$

As in  $\sup_{l \geq 0} \left\{ l + \tilde{\mathcal{G}} \left( \left( U_{t \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}} \right)_{t \geq 0}, V \right) (u) \right\} \leq 0$  but using Lemma 3.4.1, we obtain the inequality

$$\sup_{l \geq 0, R \in \mathcal{R}} \left\{ l + \tilde{\mathcal{G}} \left( U_{t \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}}, V \right) (u) \right\} \leq 0 \quad (5.4.9)$$

so the HJB equation of this problem is

$$\sup_{l \geq 0, R \in \mathcal{R}} \left\{ l + \tilde{\mathcal{G}} \left( U_{t \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}}, V \right) (u) \right\} = 0 \quad (5.4.10)$$

which can be rewritten as

$$\max \left\{ 1 - V'(u), \sup_{R \in \mathcal{R}} \tilde{\mathcal{L}}(V)(u) \right\} = 0 \quad (5.4.11)$$

with  $V(0) = 0$ , where  $\tilde{\mathcal{L}}(V)(u) = \frac{1}{2}\sigma^2 V''(u) + c^{\bar{R}}V'(u) - (\lambda + \delta)V(u) + \lambda\mathcal{I}(V)(u)$  with the operator  $\mathcal{I}(V)$  as defined in (5.4.8) and with  $\sup_{R \in \mathcal{R}} \tilde{\mathcal{L}}(V)(u) = 0$  derived using Itô's formula (and as motivated by Schmidli, 2008) as follows:

Let  $(0, h]$  be a small interval, and suppose that for each surplus  $u(h) > 0$  at time  $h$  we have a dividend and reinsurance strategy  $(\bar{D}, \bar{R})_\varepsilon$  s.t.  $V^{(\bar{D}, \bar{R})_\varepsilon}(u(h)) > V(u(h)) - \varepsilon$ . Knowing that  $(\bar{D}, \bar{R}) = (D_t^b, R_t)$ , with  $R_t = kX \wedge a$ , where  $k$  and  $a$  are, respectively, the quota-share and excess-of-loss retention levels, we let  $k(t) = k \in (0, 1]$  and  $a(t) = a \in [0, \infty)$  for  $t \leq h$ . Then

$$\begin{aligned} V(u) \geq V_u^{(\bar{D}, \bar{R})} &= \mathbb{E} \left[ V^{(\bar{D}, \bar{R})_\varepsilon}(U^{\bar{D}, \bar{R}}(h)) \mathbf{1}_{\{\tau^{\bar{D}, \bar{R}} > h\}} \right] \\ &= \mathbb{E} \left[ V^{(\bar{D}, \bar{R})_\varepsilon}(U^{\bar{D}, \bar{R}}(\tau^{\bar{D}, \bar{R}} \wedge h)) \right] \\ &\geq \mathbb{E} \left[ V(U^{\bar{D}, \bar{R}}(\tau^{\bar{D}, \bar{R}} \wedge h)) \right] - \varepsilon \end{aligned} \quad (5.4.12)$$

Because  $\varepsilon$  is arbitrary, we can choose  $\varepsilon = 0$ , so that we have

$$V(u) \geq \mathbb{E} \left[ V(U^{\bar{D}, \bar{R}}(\tau^{\bar{D}, \bar{R}} \wedge h)) \right] \quad (5.4.13)$$

By Itô's formula, provided that  $V(u)$  is twice continuously differentiable, we have

$$\begin{aligned} V(U^{\bar{D}, \bar{R}}(\tau^{\bar{D}, \bar{R}} \wedge h)) &= V(u) + \int_0^{\tau^{\bar{D}, \bar{R}} \wedge h} \left\{ c^{\bar{R}}V'(U^{\bar{D}, \bar{R}}(x)) + \frac{1}{2}\sigma^2 V''(U^{\bar{D}, \bar{R}}(x)) \right. \\ &\quad \left. + \lambda \left[ \int_0^u V(U^{\bar{D}, \bar{R}}(x) - R(k, a, x))dF(x) - V(U^{\bar{D}, \bar{R}}(x)) \right] \right\} dx \\ &\quad - \delta V(U^{\bar{D}, \bar{R}}(x)) \end{aligned} \quad (5.4.14)$$

where  $R(k, a, x) = kx \wedge a$  denotes the part of the claim  $X_i$  paid by the cedent. Substituting (5.4.14) into the expected value (5.4.12) yields

$$\begin{aligned} &\mathbb{E} \left[ \int_0^{\tau^{\bar{D}, \bar{R}} \wedge h} \left\{ c^{\bar{R}}V'(U^{\bar{D}, \bar{R}}(x)) + \frac{1}{2}\sigma^2 V''(U^{\bar{D}, \bar{R}}(x)) \right. \right. \\ &\quad \left. \left. + \lambda \left[ \int_0^u V(U^{\bar{D}, \bar{R}}(x) - kx \wedge a)dF(x) - V(U^{\bar{D}, \bar{R}}(x)) \right] \right\} dx - \delta V(U^{\bar{D}, \bar{R}}(x)) \right] \leq \alpha \end{aligned} \quad (5.4.15)$$

Dividing through by  $h$  and letting  $h \rightarrow 0$  yields, provided that the limit and expectation are

interchangeable,

$$c^{\bar{R}}V'(u) + \frac{1}{2}\sigma^2V''(u) + \lambda \left[ \int_0^u V(u - kx \wedge a)dF(x) - V(u) \right] - \delta V(u) \leq 0 \quad (5.4.16)$$

This equation must hold  $\forall k \in (0, 1]$ ,  $a \in [0, \infty)$ , that is,

$$\sup_{R \in \mathcal{R}} \left\{ c^{\bar{R}}V'(u) + \frac{1}{2}\sigma^2V''(u) + \lambda \left[ \int_0^u V(u - kx \wedge a)dF(x) - V(u) \right] - \delta V(u) \right\} \leq 0 \quad (5.4.17)$$

Suppose that there is an optimal reinsurance strategy  $\bar{R}$  with  $k \in (0, 1]$ ,  $a \in [0, \infty)$  s.t.  $\lim_{t \downarrow 0} k(t) = k(0)$  and  $\lim_{t \downarrow 0} a(t) = a(0)$ . Then, as above,

$$\begin{aligned} & \mathbb{E} \left[ \int_0^{\tau^{\bar{D}, \bar{R}} \wedge h} \left\{ c^{\bar{R}}V'(U^{\bar{D}, \bar{R}}(x)) + \frac{1}{2}\sigma^2V''(U^{\bar{D}, \bar{R}}(x)) \right. \right. \\ & \left. \left. + \lambda \left[ \int_0^u V(U^{\bar{D}, \bar{R}}(x) - kx \wedge a)dF(x) - V(U^{\bar{D}, \bar{R}}(x)) \right] \right\} dx - \delta V(U^{\bar{D}, \bar{R}}(x)) \right] = 0 \end{aligned} \quad (5.4.18)$$

Dividing by  $h$  and letting  $h \rightarrow 0$  yields

$$\sup_{R \in \mathcal{R}} \left\{ c^{\bar{R}}V'(u) + \frac{1}{2}\sigma^2V''(u) + \lambda \left[ \int_0^u V(u - k_0x \wedge a_0)dF(x) - V(u) \right] - \delta V(u) \right\} = 0 \quad (5.4.19)$$

which motivates the equation  $\sup_{R \in \mathcal{R}} \tilde{\mathcal{L}}(V)(u) = 0$ , where

$$\tilde{\mathcal{L}}(V)(u) = c^{\bar{R}}V'(u) + \frac{1}{2}\sigma^2V''(u) + \lambda \left[ \int_0^u V(u - kx \wedge a)dF(x) - V(u) \right] - \delta V(u)$$

or

$$\tilde{\mathcal{L}}(V)(u) = \frac{1}{2}\sigma^2V''(u) + c^{\bar{R}}V'(u) - (\lambda + \delta)V(u) + \lambda \int_0^u V(u - kx \wedge a)dF(x) \quad (5.4.20)$$

with boundary condition  $V(u) = 0$  on  $u < 0$ . □

### Appendix 3: Complete proof of Theorem 3.4.4

*Proof.* The proof for the case  $u \leq \underline{a} < a$  is similar to the proof of Theorem 2.2 in Paulsen *et al.* (2005) but with  $r = \sigma_R^2 = 0$ ,  $k = 1$  and  $p = c^{\bar{R}}$ . Here, we present the proof for the case  $\underline{a} < a < u$ .

Integrating (3.4.5) on  $[0, z]$  w.r.t.  $u$  gives

$$\begin{aligned}
0 &= \frac{1}{2}\sigma^2 \int_0^z V_b''(u)du + c^{\bar{R}} \int_0^z V_b'(u)du - (\lambda + \delta) \int_0^z V_b(u)du \\
&\quad + \lambda \int_0^z \int_0^u V_b(u - kx \wedge a)f(x)dxdu \\
&= \frac{1}{2}\sigma^2 V_b'(u) \Big|_0^z + c^{\bar{R}} V_b(u) \Big|_0^z - (\lambda + \delta) \int_0^z V_b(u)du \\
&\quad + \lambda \int_0^z \int_0^u V_b(u - kx \wedge a)f(x)dxdu \\
&= \frac{1}{2}\sigma^2 [V_b'(z) - V_b'(0)] + c^{\bar{R}} [V_b(z) - V_b(0)] - (\lambda + \delta) \int_0^z V_b(u)du \\
&\quad + \lambda \int_0^z \int_0^u V_b(u - kx \wedge a)f(x)dxdu \\
&= \frac{1}{2}\sigma^2 V_b'(z) - \frac{1}{2}\sigma^2 V_b'(0) + c^{\bar{R}} V_b(z) - c^{\bar{R}} V_b(0) - (\lambda + \delta) \int_0^z V_b(u)du \\
&\quad + \lambda \int_0^z \int_0^u V_b(u - kx \wedge a)f(x)dxdu \tag{5.4.21}
\end{aligned}$$

To simplify the double integral in (5.4.21), we again use integration by parts and Fubini's theorem (Schmidli, 2008) to switch the order of integration and change the properties of the convolution integral. Thus,

$$\begin{aligned}
\int_0^z \int_0^u V_b(u - kx \wedge a)f(x)dxdu &= \int_0^z \left[ \int_0^a V_b(u - kx)f(x)dx + \int_a^u V_b(u - a)f(x)dx \right] du \\
&= \int_0^z \int_0^a V_b(u - kx)f(x)dxdu + \int_0^z \int_a^u V_b(u - a)f(x)dxdu \\
&= \int_0^z \int_0^u V_b(u - kx)f(x)dxdu - \int_0^z \int_a^u V_b(u - kx)f(x)dxdu \\
&\quad + \int_0^z \int_a^u V_b(u - a)f(x)dxdu \\
&= \int_0^z \int_0^u f(u - kx)V_b(x)dxdu - \int_0^z \int_a^u f(u - kx)V_b(x)dxdu \\
&\quad + \int_0^z \int_a^u V_b(u - a)f(x)dxdu \tag{5.4.22}
\end{aligned}$$

That is,

$$\begin{aligned}
\int_0^z \int_0^u V_b(u - kx \wedge a) f(x) dx du &= \int_0^z \int_x^z f(u - kx) V_b(x) du dx - \int_a^z \int_x^z f(u - kx) V_b(x) du dx \\
&+ \int_0^z V_b(u - a) [F(u) - F(a)] du \\
&= \int_0^z F(z - kx) V_b(x) dx - \int_a^z F(z - kx) V_b(x) dx \\
&+ \int_0^z V_b(u - a) [F(u) - F(a)] du \\
&= \int_0^a F(z - kx) V_b(x) dx + \int_a^z V_b(\nu) [F(\nu + a) - F(a)] d\nu
\end{aligned} \tag{5.4.23}$$

where  $\nu = u - kx$ . Substituting into (5.4.21) gives

$$\begin{aligned}
\frac{1}{2} \sigma^2 V_b'(z) - \frac{1}{2} \sigma^2 V_b'(0) &+ c^{\bar{R}} V_b(z) - c^{\bar{R}} V_b(0) - (\lambda + \delta) \int_0^z V_b(u) du \\
&+ \lambda \left[ \int_0^a F(z - kx) V_b(x) dx + \int_a^z V_b(\nu) [F(\nu + a) - F(a)] d\nu \right] = 0
\end{aligned} \tag{5.4.24}$$

Replacing  $z$  with  $u$ ,  $\nu$  and  $u$  with  $x$  and  $F(\nu + a)$  with  $F(kx + a)$  gives

$$\begin{aligned}
\frac{1}{2} \sigma^2 V_b'(u) - \frac{1}{2} \sigma^2 V_b'(0) &+ c^{\bar{R}} V_b(u) - c^{\bar{R}} V_b(0) - (\lambda + \delta) \int_0^u V_b(x) dx \\
&+ \lambda \int_0^a F(u - kx) V_b(x) dx + \lambda \int_a^u [F(kx + a) - F(a)] V_b(x) dx = 0
\end{aligned} \tag{5.4.25}$$

Setting  $\sigma^2 = 0$  in (5.4.25) yields the case without diffusion. Consequently, equation (5.4.25) can be written as

$$\begin{aligned}
c^{\bar{R}} V_b(u) - c^{\bar{R}} V_b(0) - (\lambda + \delta) \int_0^u V_b(x) dx &+ \lambda \int_0^a F(u - kx) V_b(x) dx \\
&+ \lambda \int_a^u [F(kx + a) - F(a)] V_b(x) dx = 0
\end{aligned} \tag{5.4.26}$$

Dividing by  $c^{\bar{R}}$  gives

$$\begin{aligned} V_b(u) - V_b(0) - \frac{(\lambda + \delta)}{c^{\bar{R}}} \int_0^u V_b(x) dx &+ \frac{\lambda}{c^{\bar{R}}} \int_0^a F(u - kx) V_b(x) dx \\ &+ \frac{\lambda}{c^{\bar{R}}} \int_a^u [F(kx + a) - F(a)] V_b(x) dx = 0 \end{aligned} \quad (5.4.27)$$

which can be written as

$$\begin{aligned} V_b(u) &- \frac{\delta}{c^{\bar{R}}} \int_0^u V_b(x) dx - \frac{\lambda}{c^{\bar{R}}} \left[ \int_0^a V_b(x) dx + \int_a^u V_b(x) dx \right] \\ &+ \frac{\lambda}{c^{\bar{R}}} \int_0^a F(u - kx) V_b(x) dx + \frac{\lambda}{c^{\bar{R}}} \int_a^u [F(kx + a) - F(a)] V_b(x) dx = V_b(0) \end{aligned} \quad (5.4.28)$$

This simplifies to

$$\begin{aligned} V_b(u) - \frac{\delta}{c^{\bar{R}}} \int_0^u V_b(x) dx &- \frac{\lambda}{c^{\bar{R}}} \int_0^a [1 - F(u - kx)] V_b(x) dx \\ &- \frac{\lambda}{c^{\bar{R}}} \int_a^u [1 - (F(kx + a) - F(a))] V_b(x) dx = V_b(0) \end{aligned} \quad (5.4.29)$$

or

$$\begin{aligned} V_b(u) - \frac{\delta}{c^{\bar{R}}} \int_0^u V_b(x) dx &- \frac{\lambda}{c^{\bar{R}}} \int_0^a \bar{F}(u - kx) V_b(x) dx \\ &- \frac{\lambda}{c^{\bar{R}}} \int_a^u [1 - (F(kx + a) - F(a))] V_b(x) dx = V_b(0) \end{aligned} \quad (5.4.30)$$

from which the kernel is  $K(u, x) = -\frac{\delta + \lambda H_1(x, u)}{c^{\bar{R}}}$  with

$$H_1(x, u) = \begin{cases} \bar{F}(u - kx) & x < a \\ 1 - (F(kx + a) - F(a)) & x \geq a \end{cases}$$

and the forcing function is  $\alpha(u) = V_b(0)$  as given by (3.4.13).

For the case with diffusion, repeated integration by parts of equation (5.4.24) on  $[0, u]$  w.r.t.  $z$  yields

$$\begin{aligned}
\frac{1}{2}\sigma^2 V_b(u) & - \frac{1}{2}\sigma^2 V_b(0) - \left[ \frac{1}{2}\sigma^2 V_b'(0) + c^{\bar{R}} V_b(0) \right] u + c^{\bar{R}} \int_0^u V_b(z) dz \\
& - (\lambda + \delta) \int_0^u \int_0^z V_b(x) dx dz + \lambda \int_0^u \int_0^a F(z - kx) V_b(x) dx \\
& + \lambda \int_0^u \int_a^z [F(\nu + a) - F(a)] V_b(\nu) d\nu dz = 0
\end{aligned} \tag{5.4.31}$$

where  $\nu = u - kx$ . Further simplification of (5.4.31) gives

$$\begin{aligned}
\frac{1}{2}\sigma^2 V_b(u) & - \frac{1}{2}\sigma^2 V_b(0) - \left[ \frac{1}{2}\sigma^2 V_b'(0) + c^{\bar{R}} V_b(0) \right] u + c^{\bar{R}} \int_0^u V_b(z) dz \\
& - (\lambda + \delta) \int_0^u \int_x^u dz V_b(x) dx + \lambda \int_0^a \int_x^u F(z - kx) V_b(x) dz dx \\
& + \lambda \int_a^u \int_\nu^u [F(\nu + a) - F(a)] V_b(\nu) dz d\nu = 0
\end{aligned} \tag{5.4.32}$$

Let  $G(x) := \int_0^x F(v) dv$ . Then (5.4.32) becomes

$$\begin{aligned}
\frac{1}{2}\sigma^2 V_b(u) & - \frac{1}{2}\sigma^2 V_b(0) - \left[ \frac{1}{2}\sigma^2 V_b'(0) + c^{\bar{R}} V_b(0) \right] u + c^{\bar{R}} \int_0^u V_b(z) dz \\
& - (\lambda + \delta) \int_0^u (u - z) V_b(z) dz + \lambda \int_0^a G(u - kx) V_b(x) dx \\
& + \lambda \int_a^u [F(\nu + a) - F(a)] (u - \nu) V_b(\nu) d\nu = 0
\end{aligned} \tag{5.4.33}$$

Replacing  $z$  and  $\nu$  with  $x$  in (5.4.33) results in

$$\begin{aligned}
\frac{1}{2}\sigma^2 V_b(u) & - \frac{1}{2}\sigma^2 V_b(0) - \left[ \frac{1}{2}\sigma^2 V_b'(0) + c^{\bar{R}} V_b(0) \right] u + c^{\bar{R}} \int_0^u V_b(x) dx \\
& - (\lambda + \delta) \int_0^u (u - x) V_b(x) dx + \lambda \int_0^a G(u - kx) V_b(x) dx \\
& + \lambda \int_a^u [F(x + a) - F(a)] (u - x) V_b(x) dx = 0
\end{aligned} \tag{5.4.34}$$

which simply leads back to (3.4.10) and (3.4.14) when we replace  $u - x$  with  $u - kx$  in the fifth and seventh terms and  $F(x + a)$  with  $F(kx + a)$  in the seventh term (to take into account proportional

reinsurance with retention level  $k \in [0, 1]$ ). That is,

$$\begin{aligned} \frac{1}{2}\sigma^2 V_b(u) &- \frac{1}{2}\sigma^2 (V_b(0) + uV_b'(0)) - c^{\bar{R}}uV_b(0) + \int_0^u (c^{\bar{R}} - (\lambda + \delta)(u - kx))V_b(x)dx \\ &+ \lambda \int_0^a G(u - kx)V_b(x)dx + \lambda \int_a^u [F(kx + a) - F(a)](u - kx)V_b(x)dx = 0 \end{aligned} \quad (5.4.35)$$

Multiplying through by  $\frac{2}{\sigma^2}$  gives

$$\begin{aligned} V_b(u) &- (V_b(0) + uV_b'(0)) - \frac{2}{\sigma^2}c^{\bar{R}}uV_b(0) + \frac{2}{\sigma^2} \int_0^u (c^{\bar{R}} - (\lambda + \delta)(u - kx))V_b(x)dx \\ &+ \frac{2\lambda}{\sigma^2} \int_0^a G(u - kx)V_b(x)dx + \frac{2\lambda}{\sigma^2} \int_a^u [F(kx + a) - F(a)](u - kx)V_b(x)dx = 0 \end{aligned} \quad (5.4.36)$$

or

$$\begin{aligned} V_b(u) &+ \frac{2}{\sigma^2} \int_0^u (c^{\bar{R}} - (\lambda + \delta)(u - kx))V_b(x)dx \\ &+ \frac{2\lambda}{\sigma^2} \left[ \int_0^a G(u - kx)V_b(x)dx + \int_a^u [F(kx + a) - F(a)](u - kx)V_b(x)dx \right] \\ &= \frac{\sigma^2(V_b(0) + uV_b'(0)) + 2c^{\bar{R}}uV_b(0)}{\sigma^2} \end{aligned} \quad (5.4.37)$$

which is a linear VIE of the second kind with  $K(u, x)$  and  $\alpha(u)$  as given in (3.4.14), since  $V_b(0) = 0$  for  $\sigma^2 > 0$  (see Theorem 3.4.3).  $\square$

### Appendix 4: Proof of Theorem 4.2.3

*Proof.* (a) This should be obvious and should always hold since the optimal policy  $b^*$  is feasible under the given constraint.

(b) If  $b > b^*$ , the management of the insurance company is prohibited from paying dividends at  $b^*$  according to the constraint. Applying the generalized Itô's formula (Theorem 4.2.1 in Øksendal (2003)) to  $V(U_t^b)$ , we have

$$\begin{aligned}
e^{-\delta(\tau_b^{\bar{D}, \bar{R}} \wedge t)} V\left(U_{\tau_b^{\bar{D}, \bar{R}} \wedge t}^b\right) &= V_b(u) + \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} \left[ \tilde{\mathcal{L}}(V_b)(U_s^b) - \delta V_b(U_s^b) \right] ds \\
&+ \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} \sigma V_b'(U_s^b) dW_s - \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} V_b'(U_s^b) \mathbf{1}_{\{U_{s-}^b \geq b\}} dD_s^b \\
&+ \sum_{\substack{0 \leq s \leq \tau_b^{\bar{D}, \bar{R}} \wedge t \\ U_s^b \neq U_{s-}^b}} e^{-\delta s} [V(U_s^b) - V(U_{s-}^b)] \\
&= V_b(u) + \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} \left[ \tilde{\mathcal{L}}(V_b)(U_s^b) - \delta V_b(U_s^b) \right] ds \\
&+ \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} \sigma V_b'(U_s^b) dW_s - \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} V_b'(U_s^b) \mathbf{1}_{\{U_{s-}^b \geq b\}} dD_s^b \\
&+ \sum_{\substack{0 \leq \tau_i \leq \tau_b^{\bar{D}, \bar{R}} \wedge t \\ U_{\tau_i}^b \neq U_{\tau_i-}^b}} e^{-\delta \tau_i} [V(U_{\tau_i}^b) - V(U_{\tau_i-}^b)] \\
&= V_b(u) + \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} \left[ \tilde{\mathcal{L}}(V_b)(U_s^b) - \delta V_b(U_s^b) \right] ds \\
&+ \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} \sigma V_b'(U_s^b) dW_s - \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} V_b'(U_s^b) \mathbf{1}_{\{U_{s-}^b \geq b\}} dD_s^b \\
&+ \sum_{\substack{0 \leq \tau_i \leq \tau_b^{\bar{D}, \bar{R}} \wedge t \\ U_{\tau_i}^b \neq U_{\tau_i-}^b}} e^{-\delta \tau_i} [V(U_{\tau_i}^b - kX_i \wedge a) - V(U_{\tau_i-}^b)] \quad (5.4.38)
\end{aligned}$$

But since  $V_b'(b) = 1$ , it follows by rearrangement that

$$\begin{aligned}
\int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} \mathbf{1}_{\{U_{s-}^b \geq b\}} dD_s^b &= -e^{-\delta(\tau_b^{\bar{D}, \bar{R}} \wedge t)} V \left( U_{\tau_b^{\bar{D}, \bar{R}} \wedge t}^b \right) \\
&+ V_b(u) + \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} \left[ \tilde{\mathcal{L}}(V_b)(U_s^b) - \delta V_b(U_s^b) \right] ds \\
&+ \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} \sigma V_b'(U_s^b) dW_s \\
&+ \sum_{\substack{0 \leq \tau_i \leq \tau_b^{\bar{D}, \bar{R}} \wedge t \\ U_s^b \neq U_{s-}^b}} e^{-\delta \tau_i} \left[ V(U_{\tau_i}^b - kX_i \wedge a) - V(U_{\tau_i-}^b) \right]
\end{aligned} \tag{5.4.39}$$

Since

$$\begin{aligned}
&\sum_{\substack{0 \leq \tau_i \leq \tau_b^{\bar{D}, \bar{R}} \wedge t \\ U_s^b \neq U_{s-}^b}} e^{-\delta \tau_i} \left[ V(U_{\tau_i}^b - kX_i \wedge a) - V(U_{\tau_i-}^b) \right] \\
&- \lambda \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} \int_0^{U_s^b} e^{-\delta s} \left[ V(U_{s-}^b - kx \wedge a) - V(U_{s-}^b) \right] dF(x) ds
\end{aligned}$$

and

$$\int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} \sigma V_b'(U_s^b) dW_s$$

are martingales with mean zero, taking expectations gives

$$\begin{aligned}
\mathbb{E} \left[ \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} dD_s^b \right] &= V_b(u) + \mathbb{E} \left[ -e^{-\delta(\tau_b^{\bar{D}, \bar{R}} \wedge t)} V \left( U_{\tau_b^{\bar{D}, \bar{R}} \wedge t}^b \right) \right] \\
&+ \mathbb{E} \left[ \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} \left[ \tilde{\mathcal{L}}(V_b)(U_s^b) - \delta V_b(U_s^b) \right] ds \right]
\end{aligned} \tag{5.4.40}$$

Combining (5.4.40) with (2.19), (2.26) and Lemma 2.4 in Nansubuga *et al.* (2016), and recalling that  $\tilde{\mathcal{L}}(V_b)(U_s^b) = \delta V_b(U_s^b)$ , we have

$$\mathbb{E} \left[ \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} dD_s^b \right] \leq V_b(u) - \mathbb{E} \left[ e^{-\delta(\tau_b^{\bar{D}, \bar{R}} \wedge t)} V \left( U_{\tau_b^{\bar{D}, \bar{R}} \wedge t}^b \right) \right] \tag{5.4.41}$$

But by definition of  $\tau_b^{\bar{D}, \bar{R}}$  and by the fact that  $V_b(0) = 0$ , we have that

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{-\delta(\tau_b^{\bar{D}, \bar{R}} \wedge t)} V_b \left( U_{\tau_b^{\bar{D}, \bar{R}} \wedge t}^b \right) &= e^{-\delta \tau_b^{\bar{D}, \bar{R}}} V_b(0) \mathbf{1}_{\{\tau_b^{\bar{D}, \bar{R}} < \infty\}} + \lim_{t \rightarrow \infty} e^{-\delta \tau_b^{\bar{D}, \bar{R}}} V_b(U_t) \mathbf{1}_{\{\tau_b^{\bar{D}, \bar{R}} = \infty\}} \\ &= 0 \end{aligned}$$

so that (5.4.41) becomes

$$\mathbb{E} \left[ \int_0^{\tau_b^{\bar{D}, \bar{R}} \wedge t} e^{-\delta s} dD_s^b \right] \leq V_b(u) \quad (5.4.42)$$

Letting  $t \rightarrow \infty$ , we have

$$\mathbb{E} \left[ \int_0^{\tau_b^{\bar{D}, \bar{R}}} e^{-\delta s} dD_s^b \right] < V_b(u) \quad (5.4.43)$$

and the result follows. □

## Appendix 5: MATLAB code *samppath.m* for simulating sample paths of a one-dimensional standard Brownian motion

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% this matlab program simulating a sample path of a one-
% dimensional standard brownian motion.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clf
t=0:0.002:1;
l=size(t);
white_noise=randn(l(1),l(2));
white_noise(1)=0;
set(gca, 'FontSize', 12);
x=xlabel('time');
set(x, 'FontSize', 12);
y=ylabel('white noise');
set(y, 'FontSize', 12);
%plot(t,white_noise)
dt=t(2)-t(1);
wiener_process=zeros(l(1),l(2));
wiener_process(1)=0;
for k=2:l(2)
wiener_process(k)=wiener_process(k-1)+sqrt(dt)*white_noise(k);
end
plot(t,wiener_process, 'r');
set(gca, 'FontSize', 12, 'YLim', [-1.5 1.5]);
x=xlabel('time');
set(x, 'FontSize', 12);
y=ylabel('W_t');
set(y, 'FontSize', 12);
```

**Appendix 6: The FORTRAN program BLOCK.for for computing dividend values for the Cramér-Lundberg and diffusion-perturbed models with and without proportional and XL reinsurance**

```

!*****
! LAST EDITED 21/08/2017 BY C. Kasumo
! DEVELOPED THUR 21/05/2004 BY J. Kasozi
! PROGRAM NAME BLOCK.for
! *THIS PROGRAM:
! *Uses external functions RK,G to solve any volterra
! *equation of the 2nd kind using the block-by-block
! *methods. The numerical integration formula is Simpson's
! *rule. *User supplied external function G(forcing
! *function) returns G(T), while RK(kernel) is another user
! *supplied external function that returns RK(T,S,F). The
! *function DKDF returns the derivative of the kernel w.r.t
! *F. Newton's method is used to solve each block
!*****
!
PROGRAM BLOCK
PARAMETER (MMAX=100,NMAX=4.5*mmax)
INTEGER M,NMAX,MMAX,I,COUNT,K
REAL FF(0:NMAX),W(0:NMAX),H,G,RK,T(0:NMAX),T0,EPS,DET
REAL A11,A12,A21,A22
REAL T00,T05,T1,T2,SS1,SS2,A1,A2,B,D,VBB,U,DER,
VBY,VB0,secd
EXTERNAL G,RK,DKDF
! ****open file to store results*****
OPEN(UNIT=10,FILE='AGKA2')
! **** check that NMAX is more than 2.5 times MMAX
! ****helps to avoid singular matrix in this program****
IF(NMAX.LE.2.5*MMAX) PAUSE 'You must increase NMAX now!!'
! ****T0 is the starting point of
integration*****
T0=0.0
! ****H is the step size to be
taken*****
H=0.01
!
FF(0)=G(0.0)
! ****generate the T(i)'s
*****
DO I=0,NMAX

```

```

      T(I) = T0+I*H
ENDDO
! **** we need T(2m), T(2m+0.5), T(2m+1) and T(2m+2)*****
DO M=0,MMAX
  T00 = T(2*M)
  T05 = T(2*M) + 0.5*H
  T1 = T(2*M+1)
  T2 = T(2*M+2)
! ****the W(i)'s are given by
1/3(1,4,2,.....,2,4,1)*****
! ****for
i=0,1,2,.....,2m*****
  W(0) = 1.0/3.0
  DO I = 2,2*M-2,2
    W(I) = 2.0/3.0
  ENDDO
  DO I = 1,2*M-1,2
    W(I) = 4.0/3.0
  ENDDO
  W(2*M) = 1.0/3.0
! **** initialise the sums
*****
  SS1 = 0.0
  SS2 = 0.0
! **** compute the sums as long as M is not zero
  IF (M.GT.0) THEN
    DO I = 0,2*M
      SS1 = SS1 + W(I)*RK(T1,T(I),FF(I))
      SS2 = SS2 + W(I)*RK(T2,T(I),FF(I))
    ENDDO
    SS1 = SS1 * H
    SS2 = SS2 * H
  ENDIF
! **** we need f(2m) and initial guess *****
  F00=FF(2*M)
  F10=FF(2*M)
  F20=FF(2*M)
! **** high precision is advantageous in the solution**
  EPS=MAX(ABS(F10),ABS(F20))
! **** check the precision and number of times to iterate

```

```

COUNT=0
DO WHILE (EPS.GT.1.0D-7.AND.COUNT.LT.200)
  COUNT=COUNT+1
  FV=(3.0*F00+6.0*F10-F20)/8.0
! * entries of the matrix A(2x2) which are partials wrt
! * f(2m+1) and f(2m+2)
  A11=1.0-H*(DKDF(T1,T1,F10)/6.0+DKDF(T1,T05,FV)/2.0)
  A12=H*DKDF(T1,T05,FV)/12.0
  A21=-4.0*H*DKDF(T2,T1,F10)/3.0
  A22=1.0-H*DKDF(T2,T2,F20)/3.0
! * the equations G1(f(2m+1),f(2m+2)) as A1
! * and G2(f(2m+1),f(2m+2)) as A2
  A1=F10-G(T1)-SS1-H*(RK(T1,T00,F00)+4.0*RK(T1,T05,FV)
    +RK(T1,T1,F10))/6.0
  A2=F20-G(T2)-SS2-H*(RK(T2,T00,F00)+4.0*RK(T2,T1,F10)
    +RK(T2,T2,F20))/3.0
! **** we compute the DET(not zero) for matrix A(2x2)
  DET = A11*A22-A12*A21
  IF(DET.EQ.0) PAUSE 'Singular matrix in general.for'
! **** compute inverse of A(2x2) times transpose(A1,A2) ****
  E1=(A1*A22-A2*A12)/DET
  E2=(A2*A11-A1*A21)/DET
  EPS=MAX(ABS(E1),ABS(E2))
! ** compute F(k+1)=F(k)-E(k) *****
  F10=F10-E1
  F20=F20-E2
  ENDDO
! ** repeat with other values of m till m=MMAX *****
  FF(2*M+1)=F10
  FF(2*M+2)=F20
ENDDO
! ** headers
! WRITE(*,*)
! WRITE(6,10)
! WRITE(10,10)
! WRITE(*,*)
! **** output FF values on the screen and in the file*****
! WRITE(6,12) 0,T0,FF(0)
! WRITE(10,12) 0,T0,FF(0)

```

```

! DO M=0,MMAX
!   WRITE(6,12) m,T(2*M+1),FF(2*M+1)
!   WRITE(6,12) m,T(2*M+2),FF(2*M+2)
!   WRITE(10,12) m,T(2*M+1),FF(2*M+1)
!   WRITE(10,12) m,T(2*M+2),FF(2*M+2)
! END DO
b=2.0
k=mmax-1
d=ff(2*k+1)
vbb=ff(2*k+2)
u=ff(2*k+3)
der=(u-d)/(2.0*h)
secd=abs(2*vbb-u-d)
write(6,13) k
write(10,13) k
write(6,8) b
write(10,8) b
write(6,8) d
write(10,8) d
write(6,8) vbb
write(10,8) vbb
write(6,8) u
write(10,8) u
write(6,8) der
write(10,8) der
write(6,9) secd
write(10,9) secd
! ** headers
WRITE(*,*)
WRITE(6,10)
WRITE(10,10)
WRITE(*,*)
WRITE(6,12) T0,FF(0),FF(0)/der
WRITE(10,12) T0,FF(0),FF(0)/der
DO M=0,MMAX-1
  WRITE(6,12) T(2*M+1),FF(2*M+1),FF(2*M+1)/der
  WRITE(6,12) T(2*M+2),FF(2*M+2),FF(2*M+2)/der
  WRITE(10,12) T(2*M+1),FF(2*M+1),FF(2*M+1)/der
  WRITE(10,12) T(2*M+2),FF(2*M+2),FF(2*M+2)/der
END DO

```

```

! ***Beyond the barrier: Vb(y) when y>b
do y=b,30,2
  vby=vbb/der+y-b
  write(6,12) y,vby,vby
  write(10,12) y,vby,vby
enddo
!
! ****here are the
formats*****
8  format(5X,F10.5)
9  format(5X,F10.8)
10 FORMAT(10X,'y',15X,'Fb(y)',10X,'Vb(y)')
12 FORMAT(5X,F10.2,5X,F10.4,5X,F10.4)
13 format(5X,I5)
14 format(5X,F10.4)
  CLOSE(UNIT=10,STATUS='KEEP')
  END PROGRAM BLOCK
!
*****
!
! **** the user supplied functions G, RK and DKDF *****
!
! ****Function G(T)*****
! **parameters are for exponential(0.5) so mu=2 *****
FUNCTION G(T)
REAL C,VB0,DLT,LMD,MU
C=6.0
VB0=5.0
DLT=0.1
LMD=2.0
MU=0.5
G=VB0
END FUNCTION G
!
*****
!
*****

```

```

! ****Function RK(T,S,F)*****
FUNCTION RK(T,S,F)
REAL DLT,LMD,MU,C,L,C_R,F1,F2
REAL KPPA,ALF
C=6.0
L=0.8
C_R=L*C
DLT=0.1
LMD=2.0
KPPA=2.0
ALF=3.0
MU=0.5
F1=EXP(-(T-L*S)*MU)*(LMD/C_R)
! F1=((KPPA/(KPPA+T-L*S))*ALF)*(LMD/C_R)
F2=DLT/C_R
RK=(F2+F1)*F
END FUNCTION RK

!
! *****
!
! ****Function DKDF(T,S,F)*****
FUNCTION DKDF(T,S,F)
REAL DLT,LMD,MU,C,L,C_R,F1,F2
REAL KPPA,ALF
C=6.0
L=0.8
C_R=L*C
DLT=0.1
LMD=2.0
MU=0.5
KPPA=2.0
ALF=3.0
F1=EXP(-(T-L*S)*MU)*(LMD/C_R)
! F1=((KPPA/(KPPA+T-L*S))*ALF)*(LMD/C_R)
F2=DLT/C_R
DKDF=(F2+F1)
END FUNCTION DKDF

!
!
! *****

```

```

! *****
! **Functions RK and DKDF for excess-of-loss reinsurance**
!
! *****
! ****Function RK(T,S,F)*****
! FUNCTION RK(T,S,F)
! REAL F,S,T,DLT,LMD,MU,C,L,C_R,N,BA,BB,BC,DD,FK
! REAL KPPA,ALF
! C=6.0
! L=5.0
! N=2.0
! DLT=0.1
! LMD=2.0
! BB=EXP(-MU*L)
! BA=EXP(-MU*(S+L))
! FK=EXP(-MU*(T-S))
! KPPA=2.0
! ALF=3.0
! MU=0.5
! C_R=C-(N*LMD*(BB/MU))
! DD=1.0+BA-BB
! BB=((KPPA**ALF)*((KPPA+L)**(1.0-ALF)))/(1.0-ALF)
! BC=(KPPA/(KPPA+L))**ALF
! BA=(KPPA/(KPPA+S+L))**ALF
!
! FK=(KPPA/(KPPA+T-S))**ALF
! DD=1.0+BA-BC
! C_R=C-(N*LMD*BB)
! IF (S.LT.L) THEN
! RK=((DLT+LMD*FK)/C_R)*F
! ELSE
! RK=((DLT+LMD*DD)/C_R)*F
! END IF
! END FUNCTION RK
!
!
! *****

```

```

! ****Function DKDF(T,S,F)*****
! FUNCTION DKDF(T,S,F)
! REAL F,S,T,DLT,LMD,MU,C,L,C_R,N,BA,BB,BC,DD,FK
! REAL KPPA,ALF
! C=6.0
! L=5.0
! N=2.0
! C_R=L*C
! DLT=0.1
! LMD=2.0
! MU=0.5
! BB=EXP(-MU*L)
! BA=EXP(-MU*(S+L))
! FK=EXP(-MU*(T-S))
! KPPA=2.0
! ALF=3.0
! C_R=C-(N*LMD*(BB/MU))
! DD=1.0+BA-BB
! BB=((KPPA**ALF)*((KPPA+L)**(1.0-ALF)))/(1.0-ALF)
! BC=(KPPA/(KPPA+L))**ALF
! BA=(KPPA/(KPPA+S+L))**ALF
! FK=(KPPA/(KPPA+T-S))**ALF
! DD=1.0+BA-BC
! C_R=C-(N*LMD*BB)
! IF (S.LT.L) THEN
! DKDF=(DLT+LMD*FK)/C_R
! ELSE
! DKDF=(DLT+LMD*DD)/C_R
! END IF
! END FUNCTION DKDF
!
! *****
! ****Function RK for excess-of-loss reinsurance*****
! *****
! ****Kernel Function RK(T,S,F)*****
! FUNCTION RK(T,S,F)
! REAL T,S,F,LMD,DLT,C,L,CL,N,MU,F1,SGM,BB,DD
! REAL KPPA,ALF,BA,BC
! LMD=2.0
! DLT=0.1

```

```

! SGM=1.0
! R=0.05
! C=6.0
! L=5.0
! MU=0.5
! N=2.0
! KPPA=2.0
! ALF=3.0
! BB=EXP(-MU*L)
! CL=C-(N*LMD*(BB/MU))
! DD=(EXP(-MU*L)-EXP(-MU*(S+L)))*(T-S)
! BB=((KPPA**ALF)*((KPPA+L)**(1.0-ALF)))/(1.0-ALF)
! BC=(KPPA/(KPPA+L))**ALF
! BA=(KPPA/(KPPA+S+L))**ALF
! DD=(BC-BA)*(T-S)
! CL=C-(N*LMD*BB)
! F1=-2.0/(SGM*SGM)
! IF (S.LT.L) THEN
!   RK=F1*(CL-(LMD+DLT)*(T-S)+LMD*F)
! ELSE
!   RK=F1*(CL-(LMD+DLT)*(T-S)+LMD*DD)
! END IF
! END FUNCTION RK
! *****

```

**Appendix 7: The FORTRAN program ASYMPROP.for for computing asymptotic ruin probabilities for large claims in the Cramér-Lundberg model compounded by proportional reinsurance**

```

! *****
! DEVELOPED SATURDAY 25/06/2011 BY C. KASUMO
! EDITED FRIDAY 12/01/2018 BY C. KASUMO
! PROGRAM NAME ASYMPROP.for
! *THIS PROGRAM:
! *computes the asymptotic ruin probabilities for large claims
! *(e.g., Pareto) in the Cramer-Lundberg model compounded by
! *proportional reinsurance. Thus, it provides a simple way of
! *determining the location of the optimal proportional
! *reinsurance strategy and of confirming the optimal retention
! *level obtained by the block-by-block method
! *****
PROGRAM ASYMPROP
PARAMETER (MMAX=4000,NMAX=2.5*MMAX)
INTEGER I,M,NMAX,MMAX
REAL ETA,THETA,B,H,H2,T(0:NMAX),FF(0:NMAX)
! ****open file to store results*****
OPEN(UNIT=10,FILE='ASYM11')
! ****H is the step size to be taken*****
T0=0.0
H=0.01
H2=H/2.0
!
! ****generate the T(i)'s *****
DO I=0,NMAX
    T(I) = T0+I*H
ENDDO
! ****provide process parameters*****
ETA=1.0
THETA=1.0
B=0.003125
EE=THETA-ETA
! ****compute the ruin probabilities*****
DO I=0,NMAX
    FF(I)=(1.0/(B*THETA-EE))*(B/(1.0+(T(I)/B)))
ENDDO

```

```

! **output B, H and MMAX on screen and in file*****
WRITE(*,*)
WRITE(6,6)
WRITE(10,6)
WRITE(*,*)
WRITE(6,8) B,H,MMAX
WRITE(10,8) B,H,MMAX
WRITE(6,10)
WRITE(10,10)
!
! ****output the ruin probabilities*****
WRITE(6,12) T0,FF(0)
WRITE(10,12) T0,FF(0)
DO M=0,MMAX
    WRITE(6,12) T(M+1),FF(M+1)
    WRITE(10,12) T(M+1),FF(M+1)
ENDDO
! ****here are the formats for presentation of results*****
6 FORMAT(10X,'H',10X,'MMAX')
8 FORMAT(4X,F10.4,6X,F10.4,6X,F10.2)
10 FORMAT(10X,'u',10X,'Psi(u)')
12 FORMAT(5X,F10.2,5X,F10.8)
CLOSE(UNIT=10,STATUS='KEEP')
END PROGRAM ASYMPROP
! *****

```

**Appendix 8: The FORTRAN program LBERG.for for computing ultimate ruin probabilities for the Cramér-Lundberg and diffusion-perturbed models with and without proportional and XL reinsurance**

```

! *****
! LAST CHANGE 08/06/2017 BY C. Kasumo at 01.30hrs
! PROGRAM NAME lberg.for
! *THIS PROGRAM:
! *Uses external functions RK,G to solve any volterra
! *equation of the 2nd kind using the block-by-block methods.
! *The numerical integration formula is Simpson's rule.
! *User supplied external function G(forcing function) returns
! *G(T), while RK(kernel) is another user supplied external
! *function that returns RK(T,F).
! ***** Cramer's method is used to solve each block *****
!
PROGRAM LBERG
PARAMETER (MMAX=10000,NMAX=4.5*MMAX,NEWNMAX=10*NMAX)
INTEGER M,NMAX,MMAX,I,COUNT,ADD,VV,PN,NEWNMAX
DOUBLE PRECISION,F(0:NMAX),W(0:NMAX),G,RK,T(0:NMAX),T0,
EPS,HINF
DOUBLE PRECISION SUM,PSI(0:NMAX),A,APPR(0:NMAX),RRE(0:NMAX)
DOUBLE PRECISION TR(0:NMAX),PICK(0:NMAX),H2,MU,V,PS0,ZI
DOUBLE PRECISION FKNEW(0:NEWNMAX),H,RN(0:NMAX),DET
DOUBLE PRECISION T00,T05,SS1,SS2,T1,T2,A1,A2,HF,EXACT,R,B
DOUBLE PRECISION FK(0:NMAX),KPPA,ALF,BETA,A11,A12,A21,A22
EXTERNAL G,RK,EXACT
! ****open file to store results*****
OPEN(UNIT=10,FILE='LBG1')
! **** check that NMAX is more than 4.5 times MMAX
! ****helps to avoid singular matrix in this program****
IF(NMAX.LE.4.1*MMAX) PAUSE 'You must increase NMAX now!!'
! ****T0 is the starting point of integration*****
T0=0.0D0
! ****H is the step size to be taken*****
H=0.01D0
A=0.5D0
VV=1/H
!
H2=H/2
HF=H/20
MU=0.5D0
! KPPA=2.0D0
! ALF=3.0D0
B=1.0D0
!
FK(0)=0.0D0
FKNEW(0)=0.0D0
!

```

```

FF(0)=G(0.0D0)
! ****generate the T(i)'s *****
DO I=0,NMAX
  T(I) = T0+I*H
ENDDO
! ** the claim-size distribution *****
DO I=1,NMAX
!   FK(I)=1.0D0-(KPPA/(KPPA+I*H2))*ALF
  FK(I)=1.0D0-EXP(-MU*I*H2)
ENDDO
! **** we need T(2m), T(2m+0.5), T(2m+1) and T(2m+2)*****
DO M=0,MMAX
  T00 = T(2*M)
  T05 = T(2*M) + 0.5*H
  T1 = T(2*M+1)
  T2 = T(2*M+2)
! **** we need the coefficients of FF(2m+1) and FF(2m+2)*****
  A11=1.0D0-H*(RK(T1,FK(0))/3.0D0+RK(T1,FK(1)))/2.0D0
  A12=H*RK(T1,FK(1))/12.0D0
  A21=-4.0D0*H*RK(T2,FK(2))/3.0D0
  A22=1.0D0-H*RK(T2,FK(0))/3.0D0
! ****the W(i)'s are given by 1/3(1,4,2,.....,2,4,1)
! ****for i=0,1,2,.....,2m
  W(0) = 1.0D0/3.0D0
  DO I = 2,2*M-2,2
    W(I) = 2.0D0/3.0D0
  ENDDO
  DO I = 1,2*M-1,2
    W(I) = 4.0D0/3.0D0
  ENDDO
  W(2*M) = 1.0D0/3.0D0
! **** initialise the sums *****
  SS1 = 0.0D0
  SS2 = 0.0D0
! **** compute the sums as long as M is not zero
  IF (M.GT.0) THEN
    DO I = 0,2*M
      SS1 = SS1 + W(I)*RK(T1,FK(2*(2*M+1-I)))*FF(I)
      SS2 = SS2 + W(I)*RK(T2,FK(2*(2*M+2-I)))*FF(I)
    ENDDO
    SS1 = SS1 * H
    SS2 = SS2 * H
  ENDIF
! **** compute the right-hand side of both equations*****
  A1=G(T1)+SS1+H*FF(2*M)*(RK(T1,FK(2))+1.5*RK(T1,FK(1)))/6.0D0
  A2=G(T2)+SS2+H*FF(2*M)*RK(T2,FK(4))/3.0D0
! *****

```

```

! **** we compute the DET and use Cramer's rule to get the
! **** FF values at 2m+1 and 2m+2 for DET not zero
      DET = A11*A22-A12*A21
      IF(DET.EQ.0) PAUSE 'Singular matrix in blaa5.for'
! ***** Cramer's rule *****
      FF(2*M+1) = (A1*A22-A2*A12)/DET
      FF(2*M+2) = (A2*A11-A1*A21)/DET
      ENDDO
!
! ** lets look for h(inf) by averaging last values
      SUM=0.0D0
      DO M=MMAX-500,MMAX-1
          SUM=SUM+FF(2*M+1)
          SUM=SUM+FF(2*M+2)
      ENDDO
!
! ** compute h(inf) *****
      HINF=SUM/1000.0D0
!
! ** output both A, and h(inf) on screen and in file *****
      WRITE(*,*)
      WRITE(6,6)
      WRITE(10,6)
      WRITE(*,*)
      WRITE(6,8) A,HINF,H,MMAX
      WRITE(10,8) A, HINF,H,MMAX
      WRITE(*,*)
      WRITE(*,*)
      WRITE(6,10)
      WRITE(10,10)
      WRITE(*,*)
! **** output FF and probabilities on the screen and in the file
! *note: ruin=1-PHI(y), abs to cater for negatives around 0
      WRITE(6,12) T0,1.0D0-(FF(0)/HINF),PS0,
((1.0D0-(FF(0)/HINF)-PS0)*100.0D0)/PS0
      WRITE(10,12) T0,1.0D0-(FF(0)/HINF),PS0,
((1.0D0-(FF(0)/HINF)-PS0)*100.0D0)/PS0
      DO M=0,MMAX
          WRITE(6,12) T(2*M+1),1.0D0-(FF(2*M+1)/HINF),EXACT(T(2*M+1)),
((1.0D0-(FF(2*M+1)/HINF)-
EXACT(T(2*M+1)))*100.0D0)/EXACT(T(2*M+1))
          WRITE(6,12) T(2*M+2),1.0D0-(FF(2*M+2)/HINF),EXACT(T(2*M+2)),
((1.0D0-(FF(2*M+2)/HINF)-
EXACT(T(2*M+2)))*100.0D0)/EXACT(T(2*M+2))

```

```

        WRITE(10,12) T(2*M+1),1.0D0-
(F F(2*M+1)/HINF),EXACT(T(2*M+1)),
((1.0D0-(F F(2*M+1)/HINF)-
EXACT(T(2*M+1)))*100.0D0)/EXACT(T(2*M+1))
        WRITE(10,12) T(2*M+2),1.0D0-
(F F(2*M+2)/HINF),EXACT(T(2*M+2)),
((1.0D0-(F F(2*M+2)/HINF)-
EXACT(T(2*M+2)))*100.0D0)/EXACT(T(2*M+2))
        ENDDO
!
! ****here are the formats*****
6 FORMAT(10X,'a',10X,'h(inf)',12X,'H',10X,'MMAX')
8 FORMAT(4X,F10.6,5X,F10.6,2X,F10.2,5X,I6)
10 FORMAT(5X,' u',8X,'Psi(u)',3X,' Exact Psi(u)',4X,'D(u)')
12 FORMAT(5X,F10.2,5X,F10.8,5X,F10.8,5X,F10.8)
14 FORMAT(4X,F4.1, 5X, F10.6, 5X, F10.6,5X,F10.6)
16 FORMAT(4X,'The ten consecutive values stablised to:')
18 FORMAT(4X,F10.6)
20 FORMAT(4X,I5,5X,F10.6)
        CLOSE(UNIT=10,STATUS='KEEP')
        END PROGRAM LBERG
!
*****
! **** the user supplied functions G, RK and EXACT *****
!
! ****Function EXACT(T)*****
FUNCTION EXACT(T)
DOUBLE PRECISION LMD,C,T,MU,PS0,ZI,V,MU_1,THTA
MU=0.5D0
LMD=2.0D0
MU_1=1.5D0
THTA=1.0D0
C=(1.0D0+THTA)*LMD*MU_1
V=((C*MU)/LMD)-1.0D0
PS0=1.0D0/(1.0D0+V)
ZI=(MU*V)/(1.0D0+V)
EXACT=PS0*EXP(-ZI*T)
END FUNCTION EXACT
!
! ****Function G(T)*****
FUNCTION G(T)
DOUBLE PRECISION T,C,A,LMD,MU,V,PS0,R
LMD=2.0D0
C=6.0D0
R=0.0D0
! A=0.5D0

```

```

MU=0.50D0
V=((C*MU)/LMD)-1.0D0
PS0=1.0D0/(1.0D0+V)
G=(C*PS0)/((R*T)+C)
! G=(C*A)/((R*T)+C)
END FUNCTION G
!
*****
!
! ****Function RK(T,S,F)*****
FUNCTION RK(T,F)
DOUBLE PRECISION T,F,LMD,C,R,S,B
LMD=2.0D0
C=6.0D0
R=0.0D0
B=1.0D0
RK=(R+(LMD*(1.0D0-F)))/(R*T+B*C)
END FUNCTION RK
! *****
! *****
!
! **Functions RK and DKDF for excess-of-loss reinsurance**
!
! ****Kernel Function RK(T,S,F)*****
! FUNCTION RK(T,S,F)
! REAL LMD,C,MU,T,S,F,B,MU_1,C_R,THTA,L,N,BA,BB,BC,DD,FK
! REAL KPPA,ALF
! LMD=2.0
! MU_1=1.5
! THTA=1.0
! B=1.0
! C=(1.0+THTA)*LMD*MU_1
! C_R=B*C
! MU=0.5
! L=5.0
! N=2.0
! BB=EXP(-MU*L)
! BA=EXP(-MU*(S+L))
! FK=EXP(-MU*(T-S))
! C_R=C-(N*LMD*(BB/MU))
! C_R=4.0*L
! DD=1.0+BA-BB
! KPPA=2.0
! ALF=3.0
! BC=(KPPA/(KPPA+L))*ALF
! BA=(KPPA/(KPPA+S+L))*ALF
! FK=(KPPA/(KPPA+T-S))*ALF

```

```

!   BB=((KPPA**ALF)*((KPPA+L)**(1.0D0-ALF)))/(1.0D0-ALF)
!   DD=1.0+BA-BC
!   C_R=C-(N*LMD*BB)
!   RK=((R+LMD*((KPPA/(KPPA+(T-B*S)))**ALF))/C_R)*F
!   IF (S.LT.L) THEN
!     RK=((LMD*FK)/C)*F
!   ELSE
!     RK=((LMD*DD)/C_R)*F
!   END IF
!   END FUNCTION RK
! *****
!
! ****Differentiate Kernel to get Function DKDF(T,S,F)*****
!   FUNCTION DKDF(T,S,F)
!   REAL LMD,C,MU,T,S,F,B,MU_1,C_R,THTA,L,N,BA,BB,BC,DD,FK
!   REAL KPPA,ALF
!   LMD=2.0
!   MU_1=1.5
!   THTA=1.0
!   B=1.0
!   C=(1.0+THTA)*LMD*MU_1
!   C_R=B*C
!   MU=0.5
!   L=5.0
!   N=2.0
!   BB=EXP(-MU*L)
!   BA=EXP(-MU*(S+L))
!   FK=EXP(-MU*(T-S))
!   C_R=C-(N*LMD*(BB/MU))
!   C_R=4.0*L
!   KPPA=2.0
!   ALF=3.0
!   BC=(KPPA/(KPPA+L))**ALF
!   BA=(KPPA/(KPPA+S+L))**ALF
!   FK=(KPPA/(KPPA+T-S))**ALF
!   BB=((KPPA**ALF)*((KPPA+L)**(1.0D0-ALF)))/(1.0D0-ALF)
!   DD=1.0+BA-BC
!   C_R=C-(N*LMD*BB)
!   DD=1.0+BA-BB
!   DKDF=(R+LMD*((KPPA/(KPPA+(T-B*S)))**ALF))/C_R
!   IF (S.LT.L) THEN
!     DKDF=(LMD*FK)/C
!   ELSE
!     DKDF=(LMD*DD)/C_R
!   END IF
!   END FUNCTION DKDF
!
! *****

```

## RESEARCH OUTPUTS

(i) **PUBLICATION 1:**

Kasumo, C., Kasozi, J. and Kuznetsov, D. (2018). On minimizing the ultimate ruin probability of an insurer by reinsurance. *Journal of Applied Mathematics*, Article ID 9180780, 2018, 1-11, doi:10.1155/2018/9180780

(ii) **PUBLICATION 2:**

Kasumo, C., Kasozi, J. and Kuznetsov, D. (2018). Dividend maximization in a diffusion-perturbed classical risk process compounded by proportional and excess-of-loss reinsurance. *International Journal of Applied Mathematics and Statistics*. 57(5), 68-83.

(iii) **DRAFT MANUSCRIPT:**

Kasumo, C., Kasozi, J. and Kuznetsov, D. (2018). Dividend maximization under a set ruin probability target in the presence of proportional and excess-of-loss reinsurance.

(iv) **POSTER PRESENTATION:**

Kasumo, C., Kasozi, J. and Kuznetsov, D. Minimizing the Ruin Probability in a Diffusion-Perturbed Model with Proportional and Excess-of-loss Reinsurance.

**Output 1: Paper on minimizing ruin probabilities by reinsurance**

## Research Article

# On Minimizing the Ultimate Ruin Probability of an Insurer by Reinsurance

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Received 28 November 2017; Revised 19 January 2018; Accepted 30 January 2018; Published 22 February 2018

Academic Editor: Saeid Abbasbandy

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We consider an insurance company whose reserves dynamics follow a diffusion-perturbed risk model. To reduce its risk, the company chooses to reinsure using proportional or excess-of-loss reinsurance. Using the Hamilton-Jacobi-Bellman (HJB) approach, we derive a second-order Volterra integrodifferential equation (VIDE) which we transform into a linear Volterra integral equation (VIE) of the second kind. We then proceed to solve this linear VIE numerically using the block-by-block method for the optimal reinsurance policy that minimizes the ultimate ruin probability for the chosen parameters. Numerical examples with both light- and heavy-tailed distributions are given. The results show that proportional reinsurance increases the survival of the company in both light- and heavy-tailed distributions for the Cramér-Lundberg and diffusion-perturbed models.

## 1. Introduction

When the surplus process of an insurance company falls below zero, the company is said to have experienced ruin. Insurance companies customarily take precautions to avoid ruin. These precautions are referred to as *control variables* and include investments, capital injections or refinancing, portfolio selection, and reinsurance arrangements, to mention but a few. This study focuses on reinsurance as a control measure. Reinsurance, sometimes referred to as “insurance for insurers,” is the transfer of risk from a direct insurer (the cedent) to a second insurance carrier (the reinsurer). With reinsurance, the cedent passes on some of its premium income to a reinsurer who, in turn, covers a certain proportion of the claims that occur. It has been argued in the literature that reinsurance plays an important role in risk reduction for cedents in that it offers additional underwriting capacity for them and reduces the probability of a direct insurer’s ruin. Apart from helping the cedent to manage financial risk, increase capacity, and achieve marketing goals, reinsurance also benefits policyholders by ensuring availability and affordability of necessary coverage.

Of interest in this paper are those studies which investigate more directly the effect of reinsurance on the ultimate ruin probability. The minimization of the probability of ruin for a company whose claim process evolves according to a Brownian motion with drift and is allowed to invest in a risky asset and to purchase quota-share reinsurance was considered in [1]. In this study, an analytical expression for the minimum ruin probability and the corresponding optimal controls were obtained. Kasozi et al. [2] studied the problem of controlling ultimate ruin probability by quota-share (QS) reinsurance arrangements. Under the assumption that the insurer could invest part of the surplus in a risk-free and risky asset, [2] found that quota-share reinsurance does reduce the probability of ruin and that for chosen parameter values the optimal QS retention  $b^* \in (0.2, 0.4)$ . This study also concluded that investment helps insurance companies to reduce their ruin probabilities but that the ruin probabilities increase when stock prices become more volatile. However, while Kasozi et al. [2] considered only quota-share reinsurance, this paper seeks to combine quota-share and excess-of-loss (XL) reinsurance for one and the same insurance portfolio, but in the absence of investment.

Liu and Yang [3] reconsidered the model in [4] and incorporated a risk-free interest rate. Since closed-form solutions could not be obtained in this case, they provided numerical results for optimal strategies for maximizing the survival probability under different claim-size distribution assumptions. Also using the results in [4], the problem of choosing a combination of investments and optimal dynamic proportional reinsurance to minimize ruin probabilities for an insurance company was investigated in [5] based on a controlled surplus process satisfying the stochastic differential equation  $dX_t^{Ab} = (c - c(b_t) + \mu A_t)dt + \sigma A_t dW_t - b_t dS_t$ , where  $b_t \in [0, 1]$  is a proportional reinsurance retention at time  $t$ ,  $c(b_t)$  is the dynamic reinsurance premium rate,  $\{A_t\}$  is the amount invested in a risky asset at time  $t$ , and  $S_t$  is the aggregate claims process. But while [5] uses proportional reinsurance in minimizing ruin probabilities in the Cramér-Lundberg model, this paper considers proportional and excess-of-loss reinsurance in the diffusion-perturbed model.

More recently, taking ruin probability as a risk measure for the insurer, [6] investigated a dynamic optimal reinsurance problem with both fixed and proportional transaction costs for an insurer whose surplus process is modelled by a Brownian motion with positive drift. Under the assumption that the insurer takes noncheap proportional reinsurance, they formulated the problem as a mixed regular control and optimal stopping problem and established that the optimal reinsurance strategy was to never take reinsurance if proportional costs were high and to wait to take the reinsurance when the surplus hits a level. Additionally, they obtained an explicit expression for the survival probability under the optimal reinsurance strategy and found it to be larger than that with the aforementioned strategies. Hu and Zhang [7] introduced a general risk model involving dependence structure with common Poisson shocks. Under a combined quota-share and excess-of-loss reinsurance arrangements, they studied the optimal reinsurance strategy for maximizing the insurer's adjustment coefficient and established that excess-of-loss reinsurance was optimal from the insurer's point of view. Zhang and Liang [8] studied the optimal retentions for an insurance company that intends to transfer risk by means of a layer reinsurance treaty. Under the criterion of maximizing the adjustment coefficient, they obtained the closed-form expressions of the optimal results for the Brownian motion as well as the compound Poisson risk models and concluded that under the expected value principle excess-of-loss reinsurance is better than any other layer reinsurance strategies while under the variance premium principle pure excess-of-loss reinsurance is no longer the optimal layer reinsurance strategy. Both of these studies, however, used the criterion of maximizing the adjustment coefficient rather than minimizing the insurer's ruin probability.

This paper aims at combining proportional and excess-of-loss reinsurance for one and the same insurance portfolio. In proportional or "pro rata" reinsurance, the reinsurer indemnifies the cedent for a predetermined portion of the claims or losses, while in excess-of-loss (XL) reinsurance, which is nonproportional, the reinsurer indemnifies the cedent for all claims or losses or for a specified portion of

them, but only if the claim sizes fall within a prespecified band. Excess-of-loss reinsurance has been defined in [9] as "a form of nonproportional reinsurance contract in which an insurer pays insurance claims up to a prefixed *retention level* and the rest are paid by a reinsurer." Mathematically, given retention level  $a$ , a claim of size  $X$  is divided into the cedent's payment  $X \wedge a$  and the reinsurer's payment  $X - X \wedge a$ . The combination of proportional and excess-of-loss reinsurance has been in fact widely used in the construction of reinsurance models (see, e.g., [10]).

The models in this paper result in Volterra integral equations (VIEs) of the second kind which are solved using the block-by-block method, generally considered as the best of the higher order methods for solving Volterra integral equations of the second kind. The block-by-block methods are essentially extrapolation procedures which produce a block of values at a time. These methods can be of high order and still be self-starting. They do not require special starting procedures, are simple to use, and allow for easy switching of step-size [11].

The rest of the paper is organized as follows. Section 2 presents the formulation of the model and assumptions, followed, in Section 3, by a derivation of the HJB, integrodifferential, and integral equations. In Section 4, we present numerical results for some ruin probability models with reinsurance, using the exponential distribution for small claims and the Pareto distribution for large ones. Some conclusions and possible extensions of this study are given in Section 5.

## 2. Model Formulation

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}^+}, \mathbb{P})$  be a filtered probability space containing all stochastic objects encountered in this paper and satisfying the usual conditions; that is,  $\{\mathcal{F}_t\}_{t \in \mathbb{R}^+}$  is right-continuous and  $\mathbb{P}$ -complete. In the absence of reinsurance, the surplus of an insurance company is governed by the diffusion-perturbed classical risk process:

$$U_t = u + ct + \sigma W_t - \sum_{i=1}^{N_t} X_i, \quad t \geq 0, \quad (1)$$

where  $u = U_0 \geq 0$  is the initial reserve,  $c = (1 + \theta)\lambda\mu > 0$  is the premium rate,  $\theta$  is the safety loading,  $\{N_t\}$  is a homogeneous Poisson process with intensity  $\lambda > 0$ , and  $\{X_i\}$  is an i.i.d. sequence of strictly positive random variables with distribution function  $F$ .  $S_t = \sum_{i=1}^{N_t} X_i$  is a compound Poisson process representing the cumulative amount of claims paid in the time interval  $[0, t]$ . The claim arrival process  $\{N_t\}$  and claim sizes  $\{X_i\}$  are assumed to be independent. Here  $\{W_t\}$  is a standard one-dimensional Brownian motion independent of the compound Poisson process  $S_t$ . We assume that  $\mathbb{E}[X_i] = \mu < \infty$  and  $F(0) = 0$ . The diffusion term  $\sigma W_t$  denotes the fluctuations associated with the surplus of the insurance company at time  $t$ . Without volatility in the surplus and claim amounts, (1) becomes the well-known Cramér-Lundberg model or the classical risk process.

We proceed as in [12] where the insurer took a combination of quota-share and excess-of-loss reinsurance arrangements. Most of the actuarial literature dealing with

reinsurance as a risk control mechanism only considers pure quota-share or excess-of-loss reinsurance. However, in reality the insurer has the choice of a combination of the two and hence the use of a combination of quota-share and XL reinsurance in this paper. We assume that the reinsurance is *cheap*, meaning that the reinsurer uses the same safety loading as the insurer. Let the quota-share retention level be  $k \in [0, 1]$ . Then the insurer's aggregate claims, net of quota-share reinsurance, are  $kX$ . If the company also buys excess-of-loss reinsurance with a retention level  $a \in [0, \infty)$ , then the insurer's aggregate claims, net of quota-share and excess-of-loss reinsurance, are given by  $kX \wedge a$ . Given that  $\bar{R}$  is a reinsurance strategy combining quota-share and excess-of-loss reinsurance, the insurer's controlled surplus process becomes

$$U_t^{\bar{R}} = u + c^{\bar{R}}t + \sigma W_t - \sum_{i=1}^{N_t} kX_i \wedge a, \quad (2)$$

where the insurance premium  $c^{\bar{R}} = c - (1 + \theta)\lambda E[(kX_i - a)^+]$ . The controlled surplus process (2) has dynamics

$$dU_t^{\bar{R}} = c^{\bar{R}}dt + \sigma dW_t - d\left(\sum_{i=1}^{N_t} kX_i \wedge a\right). \quad (3)$$

The time of ruin is defined as  $\tau^{\bar{R}} = \inf\{t \geq 0 \mid U_t^{\bar{R}} < 0\}$  and the probability of ultimate ruin is defined as  $\psi^{\bar{R}} = \mathbb{P}(U_t^{\bar{R}} < 0 \text{ for some } t > 0)$ . A reinsurance strategy  $\bar{R}$  is said to be *admissible* if  $k \in [0, 1]$  and  $a \in [0, \infty)$ . The objective is to find the quota-share level  $k$  and the excess-of-loss retention limit  $a$  to minimize the insurer's risk or to maximize the insurer's survival probability. It should be noted that when the retention limit  $a$  of the excess-of-loss reinsurance is infinite, then the treaty becomes a *pure quota-share* reinsurance, while when the quota-share level  $k = 1$ , it becomes a *pure excess-of-loss* reinsurance treaty. The premium income of the insurance company is nonnegative if  $c \geq (1 + \theta)\lambda E[(kX - a)^+]$ . Therefore, we will let  $\underline{a}$  be the XL retention level at which equality  $c = (1 + \theta)\lambda E[(kX - \underline{a})^+]$  holds.

Define the value function of this problem as

$$\begin{aligned} \psi^{\bar{R}}(u) &= \mathbb{P}\left(U_t \leq 0 \text{ for some } t \geq 0 \mid U_0^{\bar{R}} = u\right) \\ &= \mathbb{P}\left(\tau^{\bar{R}} < \infty \mid U_0^{\bar{R}} = u\right), \end{aligned} \quad (4)$$

where  $\psi^{\bar{R}}(u)$  is the probability of ultimate ruin under the policy  $\bar{R}$  when the initial surplus is  $u$ . Then the objective is to find the optimal value function, that is, the minimal ruin probability

$$\psi(u) = \inf_{(k,a) \in \mathcal{R}} \psi^{\bar{R}}(u) \quad (5)$$

and optimal policy  $(\bar{R})^* = (k^*, a^*)$  s.t.  $\psi^{\bar{R}}(u) = \psi(u)$ . Alternatively, we can find the values of  $k^*$  and  $a^*$  which maximize

the probability of ultimate survival  $\phi(u) = 1 - \psi(u)$ , so that the optimal value function becomes

$$\phi(u) = \sup_{(k,a) \in \mathcal{R}} \phi^{\bar{R}}(u), \quad (6)$$

where  $\mathcal{R}$  is the set of all reinsurance policies.

### 3. HJB, Integrodifferential, and Integral Equations

**Lemma 1.** Assume that the survival probability  $\phi(u)$  defined by (6) is twice continuously differentiable on  $(0, \infty)$ . Then  $\phi(u)$  satisfies the HJB equation

$$\begin{aligned} \sup_{(k,a) \in \mathcal{R}} \left\{ \frac{1}{2} \sigma^2 \phi''(u) + c^{\bar{R}} \phi'(u) \right. \\ \left. + \lambda \int_0^u [\phi(u - kx \wedge a) - \phi(u)] dF(x) \right\} = 0, \end{aligned} \quad (7)$$

$u > 0,$

where  $\mathcal{R}$  is the set of all reinsurance policies.

*Proof.* See [13]. □

We now present the verification theorem which is essential for solving the associated stochastic control problem.

**Theorem 2.** Suppose  $\Phi \in C^2$  is an increasing strictly concave function satisfying HJB equation (7) subject to the boundary conditions

$$\begin{aligned} \Phi(u) &= 0 \quad \text{on } u < 0 \\ \Phi(0) &= 0 \quad \text{if } \sigma^2 > 0 \\ \lim_{u \rightarrow \infty} \Phi(u) &= 1 \end{aligned} \quad (8)$$

for  $0 \leq u < \infty$ . Then the maximal survival probability  $\phi(u)$  given by (6) coincides with  $\Phi$ . Furthermore, if  $(\bar{R})^* = (k^*, a^*)$  satisfies

$$\begin{aligned} \frac{1}{2} \sigma^2 \Phi''(u) + c^{\bar{R}^*} \Phi'(u) \\ + \lambda \int_0^u [\Phi(u - k^*x \wedge a^*) - \Phi(u)] dF(x) = 0 \end{aligned} \quad (9)$$

when  $0 \leq u < \infty$

then the policy  $(\bar{R})^*$  is an optimal policy; that is,  $\Phi(u) = \phi(u) = \phi^{\bar{R}^*}(u)$ .

*Proof.* Let  $\bar{R}$  be an arbitrary reinsurance strategy and let  $U^*$  be the surplus process when  $\bar{R} = \bar{R}^*$ . Choose  $n > u$  and define

$T = \mathcal{T}_n = \inf\{t \mid U_t \notin [0, n]\}$ . Note that  $U_{T \wedge t} \in (-\infty, n]$  because the jumps are downwards. The process

$$\begin{aligned} M_t^1 &= \sum_{i=1}^{N_{T \wedge t}} [\Phi(U_{T_i}) - \Phi(U_{T_{i-}})] \\ &\quad - \lambda \int_0^{T \wedge t} \left[ \int_0^{U_s} \Phi(U_s - kx \wedge a) dF(x) \right. \\ &\quad \left. - \Phi(U_s) \right] ds \end{aligned} \quad (10)$$

is a martingale. We write

$$\begin{aligned} \Phi(U_{T \wedge t}) &= \Phi(u) + \Phi(U_{T \wedge t}) - \Phi(U_{T_{N_{T \wedge t}}}) \\ &\quad + \sum_{i=1}^{N_{T \wedge t}} [\Phi(U_{T_i}) - \Phi(U_{T_{i-}})] + M_t^1 \\ &\quad + \lambda \int_0^{T \wedge t} \left[ \int_0^{U_s} \Phi(U_s - kx \wedge a) dF(x) \right. \\ &\quad \left. - \Phi(U_s) \right] ds. \end{aligned} \quad (11)$$

By Itô's formula,

$$\begin{aligned} \Phi(U_{T_i}) - \Phi(U_{T_{i-}}) &= \int_{T_{i-}}^{T_i} \left[ \frac{1}{2} \sigma^2 \Phi''(U_s) + c^{\bar{R}} \Phi'(U_s) \right] ds \\ &\quad + \int_{T_{i-}}^{T_i} \sigma \Phi'(U_s) dW_s. \end{aligned} \quad (12)$$

The corresponding result holds for  $\Phi(U_{T \wedge t}) - \Phi(U_{T_{N_{T \wedge t}}})$ . Thus,

$$\begin{aligned} \Phi(U_{T \wedge t}) &= \Phi(u) + \int_0^{T \wedge t} \left[ \frac{1}{2} \sigma^2 \Phi''(U_s) + c^{\bar{R}} \Phi'(U_s) \right. \\ &\quad \left. + \lambda \left( \int_0^{U_s} \Phi(U_s - kx \wedge a) dF(x) - \Phi(U_s) \right) \right] ds \\ &\quad + \int_0^{T \wedge t} \sigma \Phi'(U_s) dW_s + M_t^1. \end{aligned} \quad (13)$$

Using HJB equation (7), we find that

$$\Phi(U_{T \wedge t}) \leq \Phi(u) + \int_0^{T \wedge t} \sigma \Phi'(U_s) dW_s + M_t^1 \quad (14)$$

and equality holds for  $U^*$ . Let  $\{\mathcal{S}_m\}$  be a localization sequence of the stochastic integral, and set  $\mathcal{T}_n^m = \mathcal{T}_n \wedge \mathcal{S}_m$ . Taking expectations yields

$$\mathbb{E} [\Phi(U_{\mathcal{T}_n^m \wedge t})] \leq \Phi(u). \quad (15)$$

By bounded convergence, letting  $m \rightarrow \infty$  and then  $t \rightarrow \infty$ , we have  $\mathbb{E}[\Phi(U_{\mathcal{T}_n})] \leq \Phi(u)$ . It turns out that, for  $\Phi(0) = 0$ ,

$$\begin{aligned} \mathbb{P}(\tau < \mathcal{T}_n, U_\tau = 0) + \Phi(n) \mathbb{P}(\mathcal{T}_n < \tau) \\ = \mathbb{E} [\Phi(U_{\mathcal{T}_n})] \leq \Phi(u). \end{aligned} \quad (16)$$

Note that  $\mathbb{P}(\mathcal{T}_n < \tau) \geq \phi^{\bar{R}}(u)$ . Because there is a strategy with  $\phi^{\bar{R}}(u) > 0$ , it follows that  $\Phi(u)$  is bounded. We therefore let  $n \rightarrow \infty$ , yielding  $\mathbb{E}[\Phi(U_\tau)] \leq \Phi(u)$ . In particular, we obtain

$$\begin{aligned} \phi^{\bar{R}}(u) \Phi(\infty) &\leq \phi^{\bar{R}}(u) \Phi(\infty) + \mathbb{P}(\tau < \infty, U_\tau = 0) \\ &\leq \Phi(u) \end{aligned} \quad (17)$$

which simplifies to

$$\phi^{\bar{R}}(u) \leq \phi^{\bar{R}}(u) + \mathbb{P}(\tau < \infty, U_\tau = 0) \leq \Phi(u) \quad (18)$$

since  $\Phi(\infty) = 1$ . For  $U^*$  we obtain an equality. In particular,  $\{\Phi(U_t^*)\}$  is a martingale. It remains to show that  $\mathbb{P}(U_\tau^* \neq 0) = 1$ . Note first from HJB equation (7) that  $F(x)$  must be continuous; if not, the integral in (7) is not continuous. Choose  $\varepsilon > 0$  and consider the strategy  $\bar{R} = \bar{R}^* \mathbf{1}_{u \geq \varepsilon}$ . Let  $T_\varepsilon = \inf\{t \mid U_t^* < \varepsilon\}$ . By the martingale property,  $\Phi(u) = \Phi(\infty) \mathbb{P}(T_\varepsilon = \infty) + \mathbb{E}[\Phi(T_\varepsilon), T_\varepsilon < \tau < \infty]$  which reduces to

$$\Phi(u) = \mathbb{P}(T_\varepsilon = \infty) + \mathbb{E}[\Phi(T_\varepsilon), T_\varepsilon < \tau < \infty] \quad (19)$$

the last term of which is bounded by  $\Phi(\varepsilon) \mathbb{P}(T_\varepsilon < \tau < \infty)$ . Since  $F(x)$  is continuous, it must converge to zero as  $\varepsilon \rightarrow 0$ . Because  $\mathbb{P}(T_\varepsilon = \infty) \rightarrow \phi^*(u)$ , it follows that  $\Phi(u) = \phi^*(u) \Phi(\infty)$  or  $\Phi(u) = \phi^*(u) = \phi(u)$ . That is,  $\Phi(u)$  is the optimal value function and  $\bar{R}^* = (\bar{R})^*$  is an optimal policy.  $\square$

The integrodifferential equation corresponding to optimization problem (6) immediately follows from Theorem 2 as

$$\begin{aligned} \frac{1}{2} \sigma^2 \phi''(u) + c^{\bar{R}} \phi'(u) \\ + \lambda \int_0^u [\phi(u - kx \wedge a) - \phi(u)] dF(x) = 0 \end{aligned} \quad (20)$$

for  $0 \leq u < \infty$ .

This is an integrodifferential equation of Volterra type (VIDE). Solution of this equation will require that it is transformed into a Volterra integral equation (VIE) of the second kind using successive integration by parts. Hence the following theorem is obtained.

**Theorem 3.** *Integrodifferential equation (20) can be represented as a Volterra integral equation of the second kind:*

$$\phi(u) + \int_0^u K(u, x) \phi(x) dx = h(u), \quad (21)$$

where

(1) *If  $u \leq \underline{a} < a$ , one has*

$$K(u, x) = -\frac{\lambda \bar{F}(u - kx)}{c^{\bar{R}}} \quad (22)$$

$$h(u) = \phi(0)$$

with  $\bar{F}(x) = 1 - F(x)$ , when there is no diffusion (i.e., when  $\sigma^2 = 0$ ), and

$$K(u, x) = \frac{2(c^{\bar{R}} + \lambda G(u - kx) - \lambda(u - kx))}{\sigma^2} \tag{23}$$

$$h(u) = u\phi'(0) \quad \text{if } \sigma^2 > 0$$

when there is diffusion.

(2) If  $\underline{a} < a < u$ , one has

$$K(u, x) = -\frac{\lambda H_1(x, u)}{c^{\bar{R}}} \tag{24}$$

$$h(u) = \phi(0)$$

with

$$H_1(x, u) = \begin{cases} \bar{F}(u - kx) & kx < a \\ 1 - (F(kx + a) - F(a)) & kx \geq a \end{cases} \tag{25}$$

when there is no diffusion, and

$$K(u, x) = \frac{2(c^{\bar{R}} + \lambda H_2(x, u) - \lambda(u - kx))}{\sigma^2} \tag{26}$$

$$h(u) = u\phi'(0) \quad \text{if } \sigma^2 > 0$$

with

$$H_2(x, u) = \begin{cases} G(u - kx) & kx < a \\ (F(kx + a) - F(a))(u - kx) & kx \geq a \end{cases} \tag{27}$$

and  $G(x) = \int_0^x F(v)dv$  when there is diffusion.

*Proof.* The proof for the case  $u \leq \underline{a} < a$  is similar to the proof of Theorem 2.2 in [14] but with  $r = \sigma_R^2 = 0$ ,  $k = 1$ , and  $p = c^{\bar{R}}$ . Here, we present the proof for the case  $\underline{a} < a < u$ .

Integrating (20) on  $[0, z]$  with respect to  $u$  gives

$$\begin{aligned} 0 &= \frac{1}{2}\sigma^2 [\phi'(z) - \phi'(0)] + c^{\bar{R}} [\phi(z) - \phi(0)] \\ &\quad - \lambda \int_0^z \phi(u) du \\ &\quad + \lambda \int_0^z \int_0^u \phi(u - kx \wedge a) f(x) dx du. \end{aligned} \tag{28}$$

To simplify the double integral in (28), we again use integration by parts and Fubini's Theorem (see [13]) to switch the order of integration and change the properties of the convolution integral. Thus,

$$\begin{aligned} &\int_0^z \int_0^u \phi(u - kx \wedge a) f(x) dx du \\ &= \int_0^a F(z - kx) \phi(x) dx \\ &\quad + \int_a^z \phi(v) [F(v + a) - F(a)] dv, \end{aligned} \tag{29}$$

where  $v = u - kx$ . Substituting into (28) gives

$$\begin{aligned} &\frac{1}{2}\sigma^2 \phi'(z) - \frac{1}{2}\sigma^2 \phi'(0) + c^{\bar{R}} \phi(z) - c^{\bar{R}} \phi(0) \\ &\quad - \lambda \int_0^z \phi(u) du + \lambda \left[ \int_0^a F(z - kx) \phi(x) dx \right. \\ &\quad \left. + \int_a^z \phi(v) [F(v + a) - F(a)] dv \right] = 0. \end{aligned} \tag{30}$$

Replacing  $z$  with  $u$ ,  $v$  and  $u$  with  $x$ , and  $F(v + a)$  with  $F(kx + a)$  gives

$$\begin{aligned} &\frac{1}{2}\sigma^2 \phi'(u) - \frac{1}{2}\sigma^2 \phi'(0) + c^{\bar{R}} \phi(u) - c^{\bar{R}} \phi(0) \\ &\quad - \lambda \int_0^u \phi(x) dx + \lambda \int_0^a F(u - kx) \phi(x) dx \\ &\quad + \lambda \int_a^u [F(kx + a) - F(a)] \phi(x) dx = 0. \end{aligned} \tag{31}$$

Setting  $\sigma^2 = 0$  in (31) yields the case without diffusion

$$\begin{aligned} &\phi(u) - \frac{\lambda}{c^{\bar{R}}} \int_0^a \bar{F}(u - kx) \phi(x) dx \\ &\quad - \frac{\lambda}{c^{\bar{R}}} \int_a^u [1 - (F(kx + a) - F(a))] \phi(x) dx \\ &= \phi(0) \end{aligned} \tag{32}$$

from which the kernel is clearly  $K(u, x) = -\lambda H_1(x, u)/c^{\bar{R}}$  with

$$H_1(x, u) = \begin{cases} \bar{F}(u - kx) & kx < a \\ 1 - (F(kx + a) - F(a)) & kx \geq a \end{cases} \tag{33}$$

and the forcing function is  $h(u) = \phi(0)$  as given by (24).

For the case with diffusion, repeated integration by parts of (30) on  $[0, u]$  with respect to  $z$  yields the desired result.

$$\begin{aligned} &\phi(u) + \frac{2}{\sigma^2} \int_0^u (c^{\bar{R}} - \lambda(u - kx)) \phi(x) dx \\ &\quad + \frac{2\lambda}{\sigma^2} \left[ \int_0^a G(u - kx) \phi(x) dx \right. \\ &\quad \left. + \int_a^u [F(kx + a) - F(a)] (u - kx) \phi(x) dx \right] \\ &= \frac{\sigma^2 (\phi(0) + u\phi'(0)) + 2c^{\bar{R}} u \phi(0)}{\sigma^2} \end{aligned} \tag{34}$$

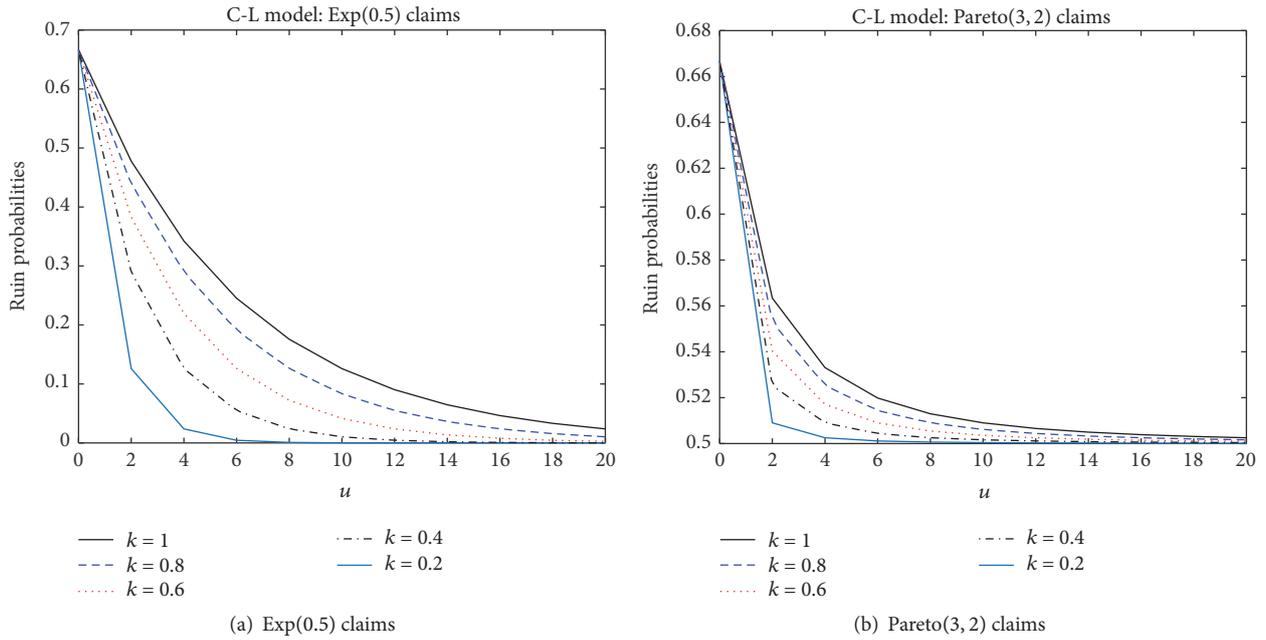


FIGURE 1: Ultimate ruin probabilities at different proportional retention levels in the Cramér-Lundberg model:  $\lambda = 2$ ,  $c = 6$ .

which is a linear VIE of the second kind with  $K(u, x)$  and  $h(u)$  as given in (26).  $\square$

#### 4. Numerical Results

We solved (21) using the fourth-order block-by-block method, a full description of which can be found in [11, 14, 15].  $\text{Exp}(\beta)$  refers to the exponential density  $f(x) = \beta e^{-\beta x}$ , so that the distribution function for the exponential distribution is  $F(x) = 1 - e^{-\beta x}$  and its tail distribution is  $\bar{F}(x) = 1 - F(x) = e^{-\beta x}$ . The mean excess function for the exponential distribution is  $e_F(x) = 1/\beta$  and  $G(x) = x - (1/\beta)F(x)$ . The Pareto( $\alpha, \kappa$ ) distribution, which is a special case of the three-parameter Burr( $\alpha, \kappa, \tau$ ) distribution, has density  $f(x) = \alpha \kappa^\alpha / (\kappa + x)^{\alpha+1}$  for  $\alpha > 0$  and  $\kappa = \alpha - 1 > 0$ , and its distribution function is  $F(x) = 1 - (\kappa / (\kappa + x))^\alpha$ . The tail distribution of the Pareto distribution is  $\bar{F}(x) = (\kappa / (\kappa + x))^\alpha$  and its mean excess function is  $e_F(x) = 1 + x/\kappa$ , so that  $G(x) = x - (1 + x/\kappa)F(x)$ . A grid size of  $h = 0.01$  was used throughout. The data simulations were performed using a Samsung Series 3 PC with an Intel Celeron 847 processor at 1.10 GHz and 6.0 GB RAM. To reduce computing time, the numerical method was implemented using the FORTRAN programming language, taking advantage of its DOUBLE PRECISION feature which gives a high degree of accuracy. The figures were constructed using MATLAB R2016a.

*4.1. Ultimate Ruin Probability in the Cramér-Lundberg Model Compounded by Proportional Reinsurance.* Here, the surplus process takes the form

$$U_t^{\bar{R}} = u + kct - \sum_{i=1}^{N_t} kX_i, \quad (35)$$

So, the survival probability  $\phi(u)$  satisfies (21) and (22) with  $a = \infty$  and  $c^{\bar{R}} = kc$ ; that is, it satisfies a VIE of the second kind with kernel and forcing function given by

$$K(u, x) = -\frac{\lambda \bar{F}(u - kx)}{kc} \quad (36)$$

$$h(u) = \phi(0).$$

Figure 1 shows the ultimate ruin probabilities in the Cramér-Lundberg model for different proportional reinsurance retention levels  $k$  and provides validity for the assertion that reinsurance does in fact reduce the ruin probability, thus increasing the insurance company's chances of survival. The results for the case  $k = 1$  (no reinsurance) are the same as those obtained in [14].

*4.2. Ultimate Ruin Probability in the Cramér-Lundberg Model Compounded by Excess-of-Loss Reinsurance.* This is the case of  $k = 1$  and  $\sigma = 0$ , so the surplus process is

$$U_t^{\bar{R}} = u + c^{\bar{R}}t - \sum_{i=1}^{N_t} X_i \wedge a, \quad (37)$$

where  $c^{\bar{R}} = c - (1 + \theta)\lambda E[(X_i - a)^+]$ . Here, for the case  $a < u$ , the kernel and forcing function are given by

$$K(u, x) = -\frac{\lambda H(x, u)}{c^{\bar{R}}} \quad (38)$$

$$h(u) = \phi(0)$$

TABLE 1: Ruin probabilities for XL reins. in CLM: Exp(0.5) claims ( $\lambda = 2, c = 6$ ).

| $u$ | $\Psi_{\infty}(u)$ | $\Psi_{35}(u)$ | $\Psi_{30}(u)$ | $\Psi_{25}(u)$ | $\Psi_{20}(u)$ |
|-----|--------------------|----------------|----------------|----------------|----------------|
| 0   | 0.6667             | 0.6667         | 0.6667         | 0.6667         | 0.6667         |
| 2   | 0.4777             | 0.4777         | 0.4777         | 0.4777         | 0.4777         |
| 4   | 0.3423             | 0.3423         | 0.3423         | 0.3423         | 0.3422         |
| 6   | 0.2453             | 0.2453         | 0.2453         | 0.2453         | 0.2453         |
| 8   | 0.1757             | 0.1757         | 0.1757         | 0.1757         | 0.1757         |
| 10  | 0.1259             | 0.1259         | 0.1259         | 0.1259         | 0.1258         |
| 12  | 0.0902             | 0.0902         | 0.0902         | 0.0902         | 0.0901         |
| 14  | 0.0646             | 0.0646         | 0.0646         | 0.0646         | 0.0646         |
| 16  | 0.0463             | 0.0463         | 0.0463         | 0.0463         | 0.0462         |
| 18  | 0.0332             | 0.0332         | 0.0332         | 0.0332         | 0.0331         |
| 20  | 0.0238             | 0.0238         | 0.0238         | 0.0238         | 0.0237         |

TABLE 2: Ruin probabilities for XL reins. in CLM: Par(3, 2) claims ( $\lambda = 2, c = 6$ ).

| $u$ | $\Psi_{\infty}(u)$ | $\Psi_{35}(u)$ | $\Psi_{30}(u)$ | $\Psi_{25}(u)$ | $\Psi_{20}(u)$ |
|-----|--------------------|----------------|----------------|----------------|----------------|
| 0   | 0.6667             | 0.6667         | 0.6667         | 0.6667         | 0.6667         |
| 2   | 0.5634             | 0.5636         | 0.5637         | 0.5639         | 0.5641         |
| 4   | 0.5331             | 0.5335         | 0.5336         | 0.5338         | 0.5341         |
| 6   | 0.5198             | 0.5202         | 0.5204         | 0.5206         | 0.5210         |
| 8   | 0.5130             | 0.5134         | 0.5135         | 0.5138         | 0.5142         |
| 10  | 0.5090             | 0.5095         | 0.5096         | 0.5099         | 0.5103         |
| 12  | 0.5066             | 0.5070         | 0.5072         | 0.5074         | 0.5079         |
| 14  | 0.5050             | 0.5054         | 0.5056         | 0.5058         | 0.5063         |
| 16  | 0.5039             | 0.5043         | 0.5045         | 0.5048         | 0.5052         |
| 18  | 0.5031             | 0.5036         | 0.5037         | 0.5040         | 0.5044         |
| 20  | 0.5025             | 0.5030         | 0.5032         | 0.5034         | 0.5039         |

with

$$H(x, u) = \begin{cases} \bar{F}(u - x) & x < a \\ 1 - (F(x + a) - F(a)) & x \geq a. \end{cases} \quad (39)$$

This is simply (22) and (24) with  $k = 1$  and  $c\bar{R} = c - (1 + \theta)\lambda E[(X_i - a)^+]$ .

Ruin probabilities for the Cramér-Lundberg model compounded by excess-of-loss (XL) reinsurance are given in Table 1 for different values of the XL retention level  $a$  ranging from 20 to infinity. Clearly, for Exp(0.5) claims, the ruin probabilities for the different retention levels reduce only very slightly as the retention level reduces. For Pareto(3, 2) claims, the ruin probabilities increase slightly as the retention level reduces (as shown in Table 2), meaning that it is optimal not to reinsure. But comparing these probabilities with Figure 1 leads to the conclusion that proportional reinsurance results in much lower ruin probabilities for the CLM as well as the perturbed model.

4.3. Ultimate Ruin Probability in the Perturbed Classical Risk Process Compounded by Proportional Reinsurance. The

survival probability  $\phi(u)$  satisfies (21) and (26) with  $a = \infty$ ; that is,

$$\begin{aligned} \phi(u) &+ \frac{2}{\sigma^2} \int_0^u [kc - \lambda(u - kx) + \lambda G(u - kx)] \phi(x) dx \quad (40) \\ &= \frac{\sigma^2 (\phi(0) + u\phi'(0)) - 2kcu\phi(0)}{\sigma^2} \end{aligned}$$

which is a VIE of the second kind with kernel and forcing function given, respectively, by

$$\begin{aligned} K(u, x) &= \frac{2[kc - \lambda(u - kx) + \lambda G(u - kx)]}{\sigma^2} \quad (41) \\ h(u) &= u\phi'(0) \quad \text{if } \sigma^2 > 0. \end{aligned}$$

Figure 2 depicts the ruin probabilities for the diffusion-perturbed model compounded by proportional reinsurance for different retention levels ranging from  $k = 1$  (no reinsurance) to  $k = 0.2$  (80% reinsurance). In the case of both Exp(0.5) claims and Pareto(3, 2) claims, applying proportional reinsurance significantly reduces the ultimate ruin probability of an insurance company.

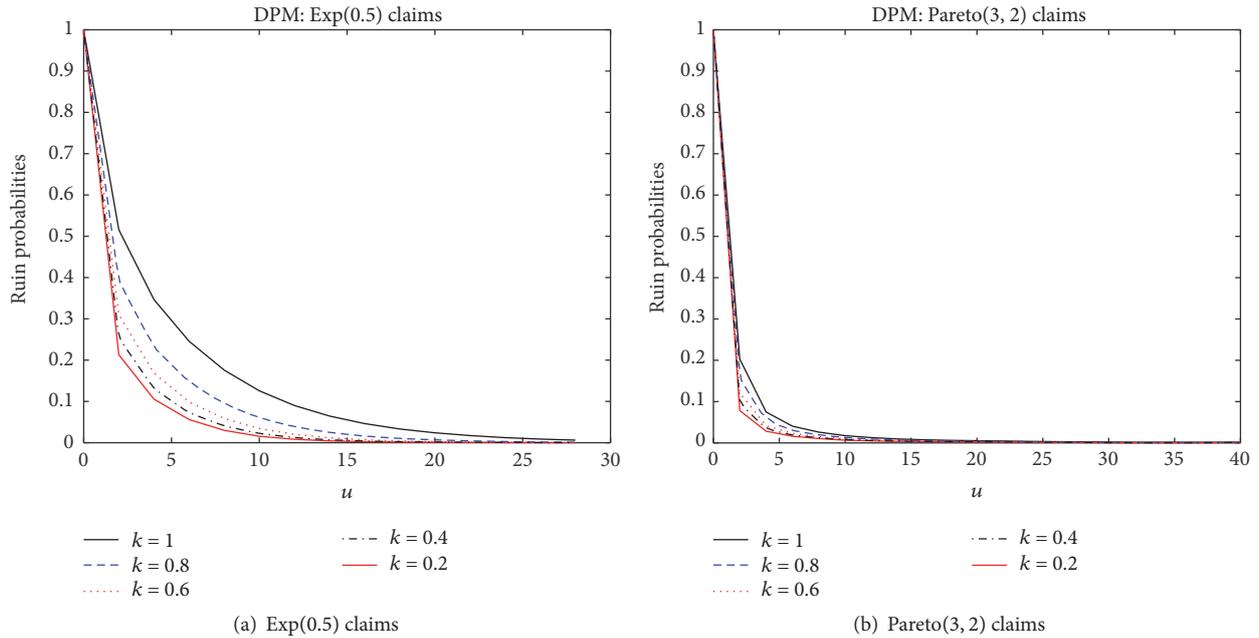


FIGURE 2: Ultimate ruin probabilities at different proportional retention levels in the diffusion-perturbed model:  $\lambda = 2, c = 6, \sigma = 0.02$ .

TABLE 3: Ruin probabilities for XL reins. in DPM: Exp(0.5) claims ( $\lambda = 2, c = 6, \sigma = 0.02$ ).

| $u$ | $\Psi_{\infty}(u)$ | $\Psi_{35}(u)$ | $\Psi_{30}(u)$ | $\Psi_{25}(u)$ | $\Psi_{20}(u)$ |
|-----|--------------------|----------------|----------------|----------------|----------------|
| 0   | 1.0000             | 1.0000         | 1.0000         | 1.0000         | 1.0000         |
| 2   | 0.5159             | 0.5159         | 0.5159         | 0.5155         | 0.5109         |
| 4   | 0.3467             | 0.3467         | 0.3466         | 0.3461         | 0.3399         |
| 6   | 0.2458             | 0.2458         | 0.2457         | 0.2451         | 0.2380         |
| 8   | 0.1759             | 0.1759         | 0.1758         | 0.1752         | 0.1674         |
| 10  | 0.1257             | 0.1258         | 0.1257         | 0.1250         | 0.1167         |
| 12  | 0.0901             | 0.0901         | 0.0901         | 0.0893         | 0.0807         |
| 14  | 0.0646             | 0.0646         | 0.0645         | 0.0638         | 0.0550         |
| 16  | 0.0463             | 0.0463         | 0.0463         | 0.0455         | 0.0365         |
| 18  | 0.0333             | 0.0333         | 0.0332         | 0.0324         | 0.0233         |
| 20  | 0.0240             | 0.0241         | 0.0240         | 0.0232         | 0.0140         |

4.4. *Ultimate Ruin Probability in the Perturbed Classical Risk Process Compounded by Excess-of-Loss Reinsurance.* The survival probability  $\phi(u)$  satisfies a VIE of the second kind with kernel  $K(u, x)$  as given in (23) (for the case  $u \leq \underline{a} < a$ ) and (26) (for the case  $\underline{a} < a < u$ ), with  $k = 1$ , and forcing function  $h(u) = u\phi'(0)$  in both cases. That is,

$$\text{for } u \leq \underline{a} < a, K(u, x) = 2[c^{\bar{R}} + \lambda G(u-x) - \lambda(u-x)]/\sigma^2;$$

$$\text{for } \underline{a} < a < u, K(u, x) = 2[c^{\bar{R}} + \lambda H_2(x, u) - \lambda(u-x)]/\sigma^2$$

with

$$H_2(x, u) = \begin{cases} G(u-x) & x < a \\ (F(x+a) - F(a))(u-x) & x \geq a. \end{cases} \quad (42)$$

The impact of XL reinsurance on the ruin probabilities in a diffusion-perturbed model is evident from Table 3 which shows a reduction in the ruin probabilities for XL retentions not exceeding  $a = 30$  for small claims. However, as can be seen from Table 4, the ruin probabilities for large claims are higher for values of  $a$  exceeding 150 but reduce significantly for values of  $a$  below 150. But again, if we compare these results with Figure 2 we see that the ruin probabilities are much lower for proportional reinsurance.

4.5. *Optimal Reinsurance Strategy: Asymptotic Ruin Probabilities.* It is known that the optimal quota-share retention  $k^*$  tends to the asymptotically optimal  $k^p$  that maximizes the adjustment coefficient  $\rho$  [13]. Therefore, since it was not possible to determine the optimal retention  $k^*$  from the results discussed in Sections 4.1–4.4, we will use asymptotic ruin probabilities. For illustrative purposes, we will now find

TABLE 4: Ruin probabilities for XL reins. in DPM: Par(3, 2) claims ( $\lambda = 2, c = 6, \sigma = 0.02$ ).

| $u$ | $\psi_{\infty}(u)$ | $\psi_{200}(u)$ | $\psi_{150}(u)$ | $\psi_{100}(u)$ | $\psi_{50}(u)$ |
|-----|--------------------|-----------------|-----------------|-----------------|----------------|
| 0   | 1.0000             | 1.0000          | 1.0000          | 1.0000          | 1.0000         |
| 2   | 0.2026             | 0.2029          | 0.2027          | 0.2022          | 0.1973         |
| 4   | 0.0744             | 0.0747          | 0.0745          | 0.0740          | 0.0683         |
| 6   | 0.0401             | 0.0405          | 0.0403          | 0.0397          | 0.0338         |
| 8   | 0.0257             | 0.0260          | 0.0258          | 0.0252          | 0.0192         |
| 10  | 0.0171             | 0.0174          | 0.0172          | 0.0167          | 0.0106         |
| 12  | 0.0124             | 0.0127          | 0.0125          | 0.0119          | 0.0058         |
| 14  | 0.0093             | 0.0096          | 0.0094          | 0.0088          | 0.0027         |
| 16  | 0.0072             | 0.0075          | 0.0073          | 0.0067          | 0.0006         |
| 18  | 0.0058             | 0.0061          | 0.0059          | 0.0054          | 0.0008         |
| 20  | 0.0050             | 0.0054          | 0.0052          | 0.0042          | 0.0015         |

TABLE 5: Asympt. ruin prob. for CLM with proportional reins. (Pareto claims) ( $c = 6, \lambda = 2, \theta = \eta = 1$ ).

| $u$ | $\psi_1(u)$ | $\psi_{0.6}(u)$ | $\psi_{0.2}(u)$ | $\psi_{0.05}(u)$ | $\psi_{0.0125}(u)$ | $\psi_{0.003125}(u)$ |
|-----|-------------|-----------------|-----------------|------------------|--------------------|----------------------|
| 0   | 1.0000      | 1.0000          | 1.0000          | 1.0000           | 1.0000             | 1.0000               |
| 2   | 0.3333      | 0.2308          | 0.0909          | 0.0244           | 0.0062             | 0.0016               |
| 4   | 0.2000      | 0.1304          | 0.0476          | 0.0123           | 0.0031             | 0.0008               |
| 6   | 0.1429      | 0.0909          | 0.0323          | 0.0083           | 0.0021             | 0.0005               |
| 8   | 0.1111      | 0.0698          | 0.0244          | 0.0062           | 0.0016             | 0.0004               |
| 10  | 0.0909      | 0.0566          | 0.0196          | 0.0050           | 0.0012             | 0.0003               |
| 12  | 0.0769      | 0.0476          | 0.0164          | 0.0041           | 0.0010             | 0.0003               |
| 14  | 0.0667      | 0.0411          | 0.0141          | 0.0036           | 0.0009             | 0.0002               |
| 16  | 0.0588      | 0.0361          | 0.0123          | 0.0031           | 0.0008             | 0.0002               |
| 18  | 0.0526      | 0.0323          | 0.0110          | 0.0028           | 0.0007             | 0.0002               |
| 20  | 0.0476      | 0.0291          | 0.0099          | 0.0025           | 0.0006             | 0.0002               |

the optimal strategies only in the CLM for both the small and large claim cases.

4.5.1. *Exponential Claims.* We note, as in [13], that for exponential claims the optimal choice of the quota-share retention  $k$  that maximizes the adjustment coefficient  $\rho(k)$  is given by

$$k^{\rho} = \min \left\{ \left( 1 - \frac{\eta}{\theta} \right) \left( 1 + \frac{1}{\sqrt{1 + \theta}} \right), 1 \right\}, \quad (43)$$

where  $\theta$  and  $\eta$  are, respectively, the safety loadings of the reinsurer and insurer. Because maximizing the adjustment coefficient yields the asymptotically best strategy, we expect that the optimal retention  $k^*$  will tend to  $k^{\rho}$ . Since this study assumes cheap reinsurance (i.e.,  $\theta = \eta$ ), we have the fact that  $k^{\rho} = 0$ . That is, it is optimal for the insurance company to reinsure the entire portfolio or to take full proportional reinsurance.

4.5.2. *Pareto Claims.* For a given initial surplus  $u$  and a retention level  $k \in [0, 1]$ , let the calculated ruin probability be

given by  $\psi_k(u)$ . Then, for large claims, the asymptotic values of the ruin probability are given by

$$\psi_k(u) = \frac{1}{k\theta - (\theta - \eta)} \frac{k}{1 + u/k}. \quad (44)$$

This ruin probability is minimized when  $k^{\rho} = 2(\theta - \eta)u / (\theta u - (\theta - \eta))$ . Thus, for Pareto-distributed claims, assuming  $\theta = \eta = 1$ , we find that  $\psi_k(u) = k / (k + u)$  and that  $k^{\rho} = 0$  as well. The insurance company should reinsure the entire portfolio of risks. The results for different values of  $k$  are summarized in Table 5 and shown in Figure 3.

It is clear from Figure 3 that the ruin probabilities become smaller as  $k \rightarrow 0$ , meaning that the asymptotically optimal retention must be  $k^{\rho} = 0$ . This confirms the results shown in Figure 1. And since the optimal retention  $k^*$  tends to the asymptotically optimal  $k^{\rho}$  that maximizes the adjustment coefficient, it follows that  $k^* = 0$ . This means that the insurance company must cede the entire portfolio of risks to a reinsurer. We can therefore conclude that the optimal combinational quota-share and XL reinsurance strategy is  $(k^*, a^*) = (0, \infty)$ .

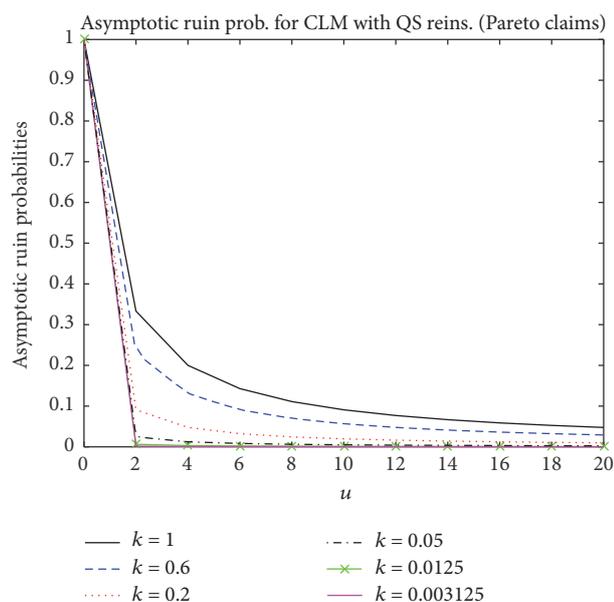


FIGURE 3: Asymptotic ruin probabilities for large claims in the CLM with proportional reinsurance ( $c = 6$ ,  $\lambda = 2$ ,  $\theta = \eta = 1$ ).

## 5. Conclusion

While the results presented in the previous section show that proportional and XL reinsurance both result in a reduction in the ruin probabilities, the reduction is more drastic for Pareto than for exponential claims in both the Cramér-Lundberg and diffusion-perturbed models. On the one hand, a comparison of the figures presented in the foregoing shows that proportional reinsurance results in lower ruin probabilities than XL reinsurance and is therefore optimal. The optimal quota-share retention was found as  $k^* = 0$ , meaning that in both the small and large claim cases in the Cramér-Lundberg model, it is optimal for the insurance company to reinsure the whole portfolio using proportional reinsurance. Going by the results in Figure 3, the same conclusion can be drawn about the diffusion-perturbed model. Thus, the optimal combinational quota-share and XL reinsurance strategy is a pure quota-share reinsurance with  $k^* = 0$ ; that is,  $(k^*, a^*) = (0, \infty)$ . It should be noted that full reinsurance is not ideal from the reinsurer's standpoint and this provides a strong argument for the use of noncheap reinsurance.

On the other hand, the literature shows that the optimal reinsurance strategy is a pure XL, that is,  $(1, a^*)$  (see, e.g., [7, 8, 16]). Possible extensions to the work are the inclusion of investments and dividend payouts as well as considering noncheap reinsurance, whereby, for a given risk, the reinsurer requires more premium and therefore uses a higher safety loading, than the insurer.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## Acknowledgments

This work was supported by Mulungushi University, the Nelson Mandela African Institution of Science and Technology, and the Zambian Ministry of Higher Education through the Support to Science and Technology Education Project (SSTEP) funded by the African Development Bank Group. The authors wish to also thank Christian Kasumo whose M.S. thesis [15] provided some theoretical background on proportional reinsurance, ultimate ruin probabilities, and the numerical solution of Volterra integral equations.

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**Output 2: Paper on dividend maximization in a diffusion-perturbed model compounded by proportional and excess-of-loss reinsurance**

## Dividend maximization in a diffusion-perturbed classical risk process compounded by proportional and excess-of-loss reinsurance

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### ABSTRACT

*We study an optimal dividend problem for an insurance company whose surplus is modelled by a diffusion-perturbed classical risk process. The company chooses to enter into reinsurance treaties involving a combination of proportional and excess-of-loss reinsurance arrangements, and is allowed to pay dividends to the shareholders. Our main objective is to find an optimal dividend and reinsurance policy that maximizes the total expected discounted dividend payouts. We derive the Hamilton-Jacobi-Bellman equation and transform the resulting Volterra integrodifferential equation into a Volterra integral equation of the second kind. This integral equation is then solved numerically using the block-by-block method to determine the dividend and reinsurance strategies that optimize the dividend payouts to the shareholders. Numerical examples with both light- and heavy-tailed distributions in the diffusion case are given. We have obtained the optimal dividend barriers that maximize the total expected discounted dividend payouts. For the diffusion-perturbed model, the results show that the optimal reinsurance policy is not to reinsure.*

**Keywords:** dividends, optimal barrier, diffusion-perturbed model, HJB equation, Volterra equation, block-by-block method, proportional reinsurance, excess-of-loss reinsurance.

**2000 Mathematics Subject Classification:** 65C20, 49L20, 45D05, 62P05.

## 1 Introduction

The problem of maximizing the total expected discounted dividend payouts can be traced back to the seminal work of (De Finetti, 1957) who proposed the need to measure the performance of an insurance portfolio by considering the maximum possible dividend paid during its lifetime

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rather than merely focusing on the safety aspect measured by the firm's ruin probability. Since then, the general dividend maximization problem based on controlled diffusion models has drawn much attention in the actuarial literature, sometimes in combination with such issues as risk control, solvency constraints, investments, capital injections or refinancing, transaction costs and reinsurance arrangements.

Several studies have investigated the optimal dividend problem for maximizing equity value by applying stochastic control techniques. (Paulsen, 2003) solved the dividend optimization problem of a firm under solvency constraints and showed that the optimal policy was of barrier type. (Højgaard and Taksar, 1999) attempted to find a policy that maximizes the expected discounted dividends paid until ruin for an insurance company that is allowed to invest part of its surplus in a risky asset and to control its risk exposure through the choice of business activity. They showed that the optimal value function is concave and that a barrier strategy is optimal. (Belhaj, 2010) studied the optimal dividend problem in a jump-diffusion model for a firm faced with two types of liquidity risks, viz., a Brownian risk and a Poisson risk, and showed that the optimal dividend policy was of barrier type. (Kasozi, Mayambala and Mahera, 2011) used the homotopy analysis method (HAM) to maximize dividend payments in the Cramér-Lundberg model under a barrier strategy but found that the HAM was not convergent when applied to a model with stochastic return on investments.

More recently, (Nansubuga, Mayambala, Mahera and Kasozi, 2016) considered maximization of dividend payouts under infinite ruin probability constraints. They derived Volterra integral equations which they solved using the block-by-block method and established the optimal barrier to use for paying dividends provided the ruin probability is no larger than the predetermined tolerance. (Hernández and Junca, 2015) studied the classical optimal dividend problem in the Cramér-Lundberg model with exponential claim sizes subject to a constraint on the ruin time and obtained the value function as a point-wise infimum of auxiliary value functions indexed by Lagrange multipliers. (Yang and Jin, 2013) considered a jump-diffusion asset pricing model paying dividends continuously and driven by a more general counting process. By conventional stochastic analysis techniques, they obtained explicit expressions for the optimal portfolio, the value function and the optimal wealth process in some particular utility function. The aforementioned studies, however, did not include any reinsurance considerations.

One study that considered reinsurance was (Kasozi, Mahera and Mayambala, 2013) which dealt with the problem of controlling the ultimate ruin probability by quota-share reinsurance arrangements. Under the assumption that the insurer could invest part of the surplus in a risk-free and risky asset, (Kasozi et al., 2013) found that quota-share reinsurance and investments play an important role in helping an insurance company to reduce its probability of ruin. But the study also established that higher volatility in the stock prices results in higher ruin probabilities. However, (Kasozi et al., 2013) used only proportional reinsurance, included investments and did not consider dividend payments.

In this paper, we consider the dividend maximization problem in a jump-diffusion model compounded by quota-share (QS) and excess-of-loss (XL) reinsurance. In addition to managing

the company's risk through reinsurance, management is allowed to pay dividends to the shareholders. It is known that under some reasonable assumptions, optimality in the jump-diffusion setting is achieved by using a barrier strategy (see, e.g., (Yin and Wang, 2009); (Yuen and Yin, 2011)). For this reason, throughout this paper, as in most papers in the literature, we work with a barrier strategy. We assume that dividends are paid out continuously to the shareholders according to a barrier strategy with level  $b > 0$  and only until ruin. Whenever the surplus exceeds the barrier level  $b$ , the excess or 'overflow' is immediately paid out as dividends. No dividends are paid out when the surplus falls below the barrier level  $b$ . Once the surplus is negative, the company experiences ruin and the process therefore stops. Thus, mathematically, we denote by  $\bar{D} = (D_t^b)_{t \in \mathbb{R}^+} \in \Pi_u^D$  the barrier strategy at level  $b$  which is defined by  $D_0^b = 0$  and

$$D_t^b = \left( \sup_{0 \leq s < t} U_s - b \right) \vee 0 \text{ for } t > 0.$$

While the term 'dividend' may be understood in the broad sense of including any amount taken from the reserves for such purposes as investment and share repurchases, in this study it is used in the narrower and more ordinary sense of taxable payments declared by the insurer's board of directors and given to shareholders out of the company's current or retained earnings (Kasozzi and Paulsen, 2005).

The outline for the rest of the paper is as follows. In Section 2, we formulate the problem and describe the model to be considered. In Section 3, we use dynamic programming to derive the Hamilton-Jacobi-Bellman (HJB) equation as well as Volterra integrodifferential and integral equations corresponding to the problem. Section 4 gives numerical results and examples for validation of the numerical method, followed, in Section 5, by some concluding remarks and possible extensions of the work.

## 2 Model formulation

To lay ground for the study, we assume that all random variables and stochastic quantities are defined on a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}^+}, \mathbb{P})$  satisfying the usual conditions, i.e., the filtration  $\{\mathcal{F}_t\}_{t \in \mathbb{R}^+}$  is right-continuous and  $\mathbb{P}$ -complete. Here,  $\Omega$  is an abstract sample space whose elements are denoted by  $\omega$ ;  $\mathcal{F}$  is a  $\sigma$ -algebra on  $\Omega$  and is a collection of subsets of  $\Omega$ ;  $\mathbb{P}$  is a probability measure and  $\{\mathcal{F}_t\}_{t \in \mathbb{R}^+}$  is a filtration. A filtration is simply an increasing and right-continuous family of sub  $\sigma$ -algebras of  $\mathcal{F}$  satisfying  $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$  whenever  $0 \leq s \leq t$  and  $\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s > t} \mathcal{F}_s$  for all  $t \geq 0$  (see, e.g., (Grimmett and Stirzaker, 2001)). In the absence of dividend payouts and reinsurance, the surplus of an insurance company is governed by the diffusion-perturbed classical risk process:

$$U_t = u + ct + \sigma W_t - \sum_{i=1}^{N_t} X_i, \quad t \geq 0 \tag{2.1}$$

where  $u = U_0 \geq 0$  is the initial reserve,  $c = (1 + \theta)\lambda\mu > 0$  is the premium rate (where  $\theta > 0$  is a safety loading),  $\{N_t\}$  is a homogeneous Poisson process with intensity  $\lambda > 0$  and  $\{X_i\}$

is an i.i.d. sequence of strictly positive random variables with common distribution function  $F$ . This paper assumes the mutual independence of the claim arrival process  $\{N_t\}$  and claim sizes  $\{X_i\}$ . Here  $\{W_t\}$  is a standard Brownian motion independent of the compound Poisson process. We assume that  $\mathbb{E}[X_i] = \mu < \infty$  and  $F(0) = 0$ . The Brownian term  $\sigma W_t$ , where  $\sigma \geq 0$  is a diffusion or volatility coefficient, represents random fluctuations in the surplus process; without volatility in the surplus process, (2.1) reduces to the well-known Cramér-Lundberg model (CLM) or classical risk process.

We proceed as in (Centeno, 1985) where the insurer took a combination of quota-share and XL reinsurance arrangements. Most of the actuarial literature dealing with reinsurance as a risk control mechanism only considers pure quota-share or excess-of-loss reinsurance. But because in reality the insurer has the choice of a combination of the two, this paper assumes a combination of QS and XL reinsurance. Let the quota-share retention level be  $k \in [0, 1]$ . Then the insurer's aggregate claims, net of QS reinsurance, are  $kX$ . Also, let the XL reinsurance retention level be  $a \in [0, \infty)$ . Then the insurer's aggregate claims, net of QS and XL reinsurance, are  $kX \wedge a$ . When the retention limit  $a$  of the XL reinsurance is infinite, then the treaty becomes a *pure quota-share* reinsurance, while a QS level  $k = 1$  makes it a *pure excess-of-loss* reinsurance treaty. Though these two scenarios are somewhat extreme, they are still real possibilities for an insurance company. The premium income of the insurance company is non-negative if  $c \geq (1 + \theta)\lambda\mathbb{E}[(kX - a)^+]$ . Therefore, we will let  $\underline{a}$  be the XL retention level at which equality  $c = (1 + \theta)\lambda\mathbb{E}[(kX - \underline{a})^+]$  holds.

If we also let the cumulative amount of dividend payouts distributed to shareholders up to time  $t$  be  $D_t^b$ , where  $b$  is the dividend barrier level, then the dividend and reinsurance strategy is a pair of  $\mathcal{F}_t$ -adapted measurable processes  $(\bar{D}, \bar{R})$ . Here,  $\bar{D} = (D_t^b)_{t \in \mathbb{R}^+}$  is a dividend strategy and  $\bar{R} = (R_t)_{t \in \mathbb{R}^+}$  is a reinsurance strategy combining QS and XL reinsurance. Thus, given a dividend and reinsurance strategy  $(\bar{D}, \bar{R})$ , the insurer's controlled surplus process now becomes

$$U_t^{\bar{D}, \bar{R}} = U_t^{\bar{R}} - D_t^b \tag{2.2}$$

where  $U_t^{\bar{R}} = u + c^{\bar{R}}t + \sigma W_t - \sum_{i=1}^{N_t} kX_i \wedge a$  is the insurer's surplus in the presence of reinsurance and  $c^{\bar{R}} = c - (1 + \theta)\lambda\mathbb{E}[(kX_i - a)^+]$  is the insurance premium rate under these conditions. The controlled surplus process (2.2) has dynamics

$$dU_t^{\bar{D}, \bar{R}} = c^{\bar{R}}dt + \sigma dW_t - d\left(\sum_{i=1}^{N_t} kX_i \wedge a\right) - dD_t^b. \tag{2.3}$$

The controlled process  $U_t^{\bar{D}, \bar{R}}$  is observed until the time of ruin, defined as  $\tau^{\bar{D}, \bar{R}} = \inf\{t \geq 0 | U_t^{\bar{D}, \bar{R}} < 0\}$ . The probability of ultimate ruin is defined as  $\psi^{\bar{D}, \bar{R}} = \mathbb{P}(U_t^{\bar{D}, \bar{R}} < 0 \text{ for some } t > 0)$ . A dividend and reinsurance strategy  $(\bar{D}, \bar{R})$  is said to be *admissible* if  $k \in [0, 1]$ ,  $a \in [0, \infty)$ , and if at any time prior to ruin a lump sum dividend payment is smaller than the size of the available liquidity reserves, i.e., in the time interval  $[t, t+]$

$$D_{t+}^b - D_t^b \leq \max\{0, U_t^{\bar{D}, \bar{R}}\} \text{ for } t \leq \tau^{\bar{D}, \bar{R}}.$$

In addition,  $D_t^b$  must be non-negative, non-decreasing and right-continuous with left limits (or càdlàg). Given an admissible dividend and reinsurance strategy  $(\bar{D}, \bar{R}) \in \Pi_u^{D,R}$ , where  $\Pi_u^{D,R}$  is the set of all admissible dividend and reinsurance strategies, the total expected discounted dividends paid out to the shareholders until ruin when the initial capital is  $u \geq 0$  are given by

$$V_u^{\bar{D}, \bar{R}} = \mathbb{E}_u \left[ \int_0^{\tau^{\bar{D}, \bar{R}}} e^{-\delta t} dD_t^b \right]$$

where  $\delta > 0$  is the constant rate at which dividends are discounted and  $\mathbb{E}_u$  denotes expectation with respect to  $\mathbb{P}_u(\cdot) = \mathbb{P}(\cdot | U_0^{\bar{D}, \bar{R}} = u)$ , the probability measure conditioned on the initial capital  $U_0^{\bar{D}, \bar{R}} = u$ . Define the optimal value function of this problem as

$$V_u^* = \sup_{(\bar{D}, \bar{R}) \in \Pi_u^{D,R}} V_u^{\bar{D}, \bar{R}} \tag{2.4}$$

where the supremum is taken over all admissible strategies  $\Pi_u^{D,R}$ . The problem is formulated as an optimal control problem in which the controls are the retention levels for quota-share and XL reinsurance (i.e.,  $k$  and  $a$ , respectively) as well as the dividend barrier level  $b > 0$ . Solution of the problem will lead to finding the optimal dividend and reinsurance strategy  $(\bar{D}, \bar{R})^* = (D_t^*, R_t^*)$ , where  $D_t^* = (b^*)$  and  $R_t^* = (k^*, a^*)$ , with  $b^*$  denoting the optimal dividend barrier level and  $k^*$  and  $a^*$  representing, respectively, the optimal retention levels for QS and XL reinsurance, so that  $V_u^{(\bar{D}, \bar{R})^*} = V_u^*$ .

### 3 HJB, integrodifferential and integral equations

In this section, we derive the Hamilton-Jacobi-Bellman (HJB) equation and the corresponding integrodifferential equation. We begin by showing that the value function  $V$  satisfies the *dynamic programming principle* (DPP).

*Lemma 3.1.* (Dynamic programming principle) For any initial surplus  $u > 0$  and any stopping time  $\bar{\tau}$ , the value function  $V$  fulfills the dynamic programming principle

$$V(u) = \sup_{(\bar{D}, \bar{R}) \in \Pi_u^{D,R}} \mathbb{E}_u \left[ \int_0^{\bar{\tau} \wedge \tau^{\bar{D}, \bar{R}}} e^{-\delta s} dD_s^b + e^{-\delta(\bar{\tau} \wedge \tau^{\bar{D}, \bar{R}})} V \left( U_{\bar{\tau} \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}} \right) \right] \tag{3.1}$$

*Proof.* See (Azcue and Muler, 2014). □

We now derive the HJB equation for the optimization problem (2.4).

*Theorem 3.2.* (HJB equation) The HJB equation corresponding to the optimization problem under consideration is

$$\max\{1 - V'(u), \sup_{R \in \mathcal{R}} \tilde{\mathcal{L}}(V)(u)\} = 0 \tag{3.2}$$

with  $V(0) = 0$ . The infinitesimal generator  $\tilde{\mathcal{L}}(V)(u)$  is defined by

$$\tilde{\mathcal{L}}(V)(u) = \frac{1}{2} \sigma^2 V''(u) + c^{\bar{R}} V'(u) - (\lambda + \delta) V(u) + \lambda \int_0^u V(u - kx \wedge a) dF(x)$$

and  $\mathcal{R}$  is the set of all reinsurance policies.

*Proof.* Given any dividend payment rate  $l \geq 0$  and reinsurance function  $R$ , we consider the admissible strategy  $(\bar{D}, \bar{R}) = ((lt), (R))_{t \geq 0}$  which pays dividends at a constant rate  $l$  and takes reinsurance with constant retained function  $R(k, a, x) = kX_i \wedge a$ . Let the corresponding controlled surplus process be denoted by  $U_t^{\bar{D}, \bar{R}} = U_t^{\bar{R}} - lt$  and the corresponding ruin time by  $\tau^{\bar{D}, \bar{R}}$ . The surplus process  $U_{t \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}}$  stopped at the ruin time is a Markov process, so that, as in  $\mathcal{G} \left( (U_{t \wedge \tau^{\bar{D}, \bar{R}}})_{t \geq 0}, f \right) (u) = \frac{1}{2} \sigma^2 f''(u) + cf'(u) - (\lambda + \delta)f(u) + \lambda \mathcal{I}(f)(u)$  and Remark 1.7 in (Azcue and Muler, 2014), if the company pays dividends at constant rate  $l \geq 0$ , we have

$$\tilde{\mathcal{G}} \left( U_{t \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}}, V \right) (u) = \begin{cases} (c^{\bar{R}} - l)V'(u) - (\lambda + \delta)V(u) + \lambda \mathcal{I}(V)(u) & l \leq c^{\bar{R}} \\ (c^{\bar{R}} - l)V'(u) - (\lambda + \delta)V(u) + \lambda \mathcal{I}(V)(u^-) & l > c^{\bar{R}} \end{cases}$$

where  $\tilde{\mathcal{G}}$  is the discounted infinitesimal generator defined by

$$\tilde{\mathcal{G}} \left( (U_{t \wedge \tau^{\bar{D}, \bar{R}}})_{t \geq 0}, V \right) := \lim_{t \rightarrow 0} \frac{\mathbb{E}_u \left[ e^{-\delta t} V \left( U_{t \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}} \right) \right] - V(u)}{t}$$

and  $\mathcal{I}(V)$  is the integral operator

$$\mathcal{I}(V)(u) = \int_0^\infty V(u - R(k, a, x))dF(x) = \int_0^u V(u - kx \wedge a)dF(x) \tag{3.3}$$

As in  $\sup_{l \geq 0} \left\{ l + \tilde{\mathcal{G}} \left( (U_{t \wedge \tau^{\bar{D}, \bar{R}}})_{t \geq 0}, V \right) (u) \right\} \leq 0$  but using Lemma 3.1, we obtain the inequality

$$\sup_{l \geq 0, R \in \mathcal{R}} \left\{ l + \tilde{\mathcal{G}} \left( U_{t \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}}, V \right) (u) \right\} \leq 0$$

so the HJB equation of this problem is

$$\sup_{l \geq 0, R \in \mathcal{R}} \left\{ l + \tilde{\mathcal{G}} \left( U_{t \wedge \tau^{\bar{D}, \bar{R}}}^{\bar{D}, \bar{R}}, V \right) (u) \right\} = 0$$

which can be rewritten as

$$\max \left\{ 1 - V'(u), \sup_{R \in \mathcal{R}} \tilde{\mathcal{L}}(V)(u) \right\} = 0$$

with  $V(0) = 0$ , where  $\tilde{\mathcal{L}}(V)(u) = \frac{1}{2} \sigma^2 V''(u) + c^{\bar{R}} V'(u) - (\lambda + \delta)V(u) + \lambda \mathcal{I}(V)(u)$  with the operator  $\mathcal{I}(V)$  as defined in (3.3). □

**Theorem 3.3.** Let  $\bar{D} = (D_t^b)$  denote the dividend policy corresponding to the barrier level  $b$  and let  $V_b(u) = V_b^{\bar{D}, \bar{R}}(u) = V^{\bar{D}, \bar{R}}(u; b)$ . If  $V_b(u)$  solves

$$\tilde{\mathcal{L}}(V_b)(u) = \delta V_b(u) \tag{3.4}$$

on  $[0, b]$  for some dividend barrier  $b$ , together with the conditions  $V_b(u) = 0$  on  $u < 0$ ,  $V_b(0) = 0$  if  $\sigma^2 > 0$ ,  $V_b'(b) = 1$  and  $V_b(u) = u - b + V_b(b)$  on  $u > b$ , then, for  $0 < u \leq \infty$ , the HJB equation (3.2) takes the form

$$\sup \tilde{\mathcal{L}}(V_b)(u) = 0 \tag{3.5}$$

*Proof.* The equation  $\sup \tilde{\mathcal{L}}(V_b)(u) = 0$  is derived using Itô's formula and as motivated by (Schmidli, 2008) which can be consulted for details. □

It follows from Theorem 3.3 that

$$\frac{1}{2}\sigma^2 V_b''(u) + c\bar{R}V_b'(u) - (\lambda + \delta)V_b(u) + \lambda \int_0^u V_b(u - kx \wedge a) dF(x) = 0 \quad (3.6)$$

with boundary condition  $V_b(u) = 0$  on  $u < 0$ . This is the Volterra integrodifferential equation (VIDE) corresponding to the optimization problem being considered in this paper. By successive integration by parts, the VIDE (3.6) is transformed into a Volterra integral equation (VIE) of the second kind. The following theorem is pertinent in this regard.

*Theorem 3.4.* The integrodifferential equation (3.6) can be represented as a Volterra integral equation of the second kind

$$V_b(u) + \int_0^u K(u, x)V_b(x)dx = h(u) \quad (3.7)$$

where

1. If  $u \leq \underline{a} < a$ , we have

$$K(u, x) = -\frac{\delta + \lambda\bar{F}(u - kx)}{c\bar{R}}$$

$$h(u) = V_b(0)$$

with  $\bar{F}(x) = 1 - F(x)$ , when there is no diffusion (i.e., when  $\sigma^2 = 0$ ), and

$$K(u, x) = \frac{2\left(c\bar{R} + \lambda G(x, u) - (\lambda + \delta)(u - kx)\right)}{\sigma^2}$$

$$h(u) = uV_b'(0) \text{ if } \sigma^2 > 0$$

when there is diffusion.

2. If  $\underline{a} < a < u$ , we have

$$K(u, x) = -\frac{\delta + \lambda H_1(x, u)}{c\bar{R}}$$

$$h(u) = V_b(0) \quad (3.8)$$

with

$$H_1(x, u) = \begin{cases} \bar{F}(u - kx) & kx < a \\ 1 - (F(kx + a) - F(a)) & kx \geq a \end{cases}$$

and  $V_b(u) = u - b + V_b(b)$ , for  $u > b$  when there is no diffusion, and

$$K(u, x) = \frac{2\left(c\bar{R} + \lambda H_2(x, u) - (\lambda + \delta)(u - kx)\right)}{\sigma^2}$$

$$h(u) = uV_b'(0) \text{ if } \sigma^2 > 0 \quad (3.9)$$

with

$$H_2(x, u) = \begin{cases} G(u - kx) & kx < a \\ (F(kx + a) - F(a))(u - kx) & kx \geq a \end{cases}$$

and  $G(x) = \int_0^x F(v)dv$  when there is diffusion.

*Proof.* The proof for the case  $u \leq \underline{a} < a$  is similar to the proof of Theorem 2.2 in (Paulsen, Kasozi and Steigen, 2005) but with  $r = \sigma_R^2 = 0$ ,  $k = 1$  and  $p = c^{\bar{R}}$ . Here, we present the proof for the case  $\underline{a} < a < u$ .

Integrating (3.6) on  $[0, z]$  w.r.t.  $u$  gives

$$0 = \frac{1}{2}\sigma^2 V_b'(z) - \frac{1}{2}\sigma^2 V_b'(0) + c^{\bar{R}}V_b(z) - c^{\bar{R}}V_b(0) - (\lambda + \delta) \int_0^z V_b(u)du + \lambda \int_0^z \int_0^u V_b(u - kx \wedge a) f(x) dx du \tag{3.10}$$

To simplify the double integral in (3.10), we again use integration by parts and Fubini's theorem (Schmidli, 2008) to switch the order of integration and change the properties of the convolution integral. Thus, with  $\nu = u - kx$ ,

$$\int_0^z \int_0^u V_b(u - kx \wedge a) f(x) dx du = \int_0^a F(z - kx) V_b(x) dx + \int_a^z V_b(\nu) [F(\nu + a) - F(a)] d\nu$$

Substituting into (3.10) gives

$$\frac{1}{2}\sigma^2 V_b'(z) - \frac{1}{2}\sigma^2 V_b'(0) + c^{\bar{R}}V_b(z) - c^{\bar{R}}V_b(0) - (\lambda + \delta) \int_0^z V_b(u)du + \lambda \left[ \int_0^a F(z - kx) V_b(x) dx + \int_a^z V_b(\nu) [F(\nu + a) - F(a)] d\nu \right] = 0 \tag{3.11}$$

Replacing  $z$  with  $u$ ,  $\nu$  and  $u$  with  $x$  and  $F(\nu + a)$  with  $F(kx + a)$  gives

$$\frac{1}{2}\sigma^2 V_b'(u) - \frac{1}{2}\sigma^2 V_b'(0) + c^{\bar{R}}V_b(u) - c^{\bar{R}}V_b(0) - (\lambda + \delta) \int_0^u V_b(x) dx + \lambda \int_0^a F(u - kx) V_b(x) dx + \lambda \int_a^u [F(kx + a) - F(a)] V_b(x) dx = 0 \tag{3.12}$$

The case without diffusion

$$V_b(u) - \frac{\delta}{c^{\bar{R}}} \int_0^u V_b(x) dx - \frac{\lambda}{c^{\bar{R}}} \int_0^a \bar{F}(u - kx) V_b(x) dx - \frac{\lambda}{c^{\bar{R}}} \int_a^u [1 - (F(kx + a) - F(a))] V_b(x) dx = V_b(0) \tag{3.13}$$

follows by setting  $\sigma^2 = 0$  in (3.12) and making some algebraic rearrangements. From (3.13), the kernel is  $K(u, x) = -\frac{\delta + \lambda H_1(x, u)}{c^{\bar{R}}}$  with

$$H_1(x, u) = \begin{cases} \bar{F}(u - kx) & kx < a \\ 1 - (F(kx + a) - F(a)) & kx \geq a \end{cases}$$

and the forcing function is  $h(u) = V_b(0)$  as given by (3.8).

For the case with diffusion, repeated integration by parts of equation (3.11) yields

$$V_b(u) + \frac{2}{\sigma^2} \int_0^u (c^{\bar{R}} - (\lambda + \delta)(u - kx)) V_b(x) dx + \frac{2\lambda}{\sigma^2} \left[ \int_0^a G(u - kx) V_b(x) dx + \int_a^u [F(kx + a) - F(a)] (u - kx) V_b(x) dx \right] = \frac{\sigma^2 (V_b(0) + u V_b'(0)) + 2c^{\bar{R}} u V_b(0)}{\sigma^2} \tag{3.14}$$

which is a linear VIE of the second kind with  $K(u, x)$  and  $h(u)$  as given in (3.7) and (3.9).  $\square$

## 4 Numerical results

We employed the fourth-order block-by-block method and Simpson's rule to (3.7). A detailed description of this method is found, e.g., in (Paulsen et al., 2005) and (Linz, 1969; Linz, 1985). In the numerical examples that follow, two distributions are used for modelling the claim sizes  $X_i$ . These are the exponential and Pareto distributions. The exponential distribution has the advantage of mathematical tractability, while the Pareto distribution is very versatile and has wide application in economics, finance, actuarial science, survival analysis and telecommunications due to its heavy tail properties. Furthermore, the computationally simple form of the Pareto distribution function enhances its desirability for modelling claim sizes in a variety of actuarial applications.

The  $\text{Exp}(\beta)$  distribution, a special case of the Weibull( $\alpha, \beta$ ) distribution, has density  $f(x) = \beta e^{-\beta x}$ , with corresponding distribution and tail functions  $F(x) = 1 - e^{-\beta x}$  and  $\bar{F}(x) = 1 - F(x) = e^{-\beta x}$ , respectively. The mean excess function for the exponential distribution is  $e_F(x) = \frac{1}{\beta}$ ; thus,  $G(x) = x - \frac{1}{\beta}F(x)$ . The Pareto( $\alpha, \kappa$ ) distribution, which is a special case of the three-parameter Burr( $\alpha, \kappa, \tau$ ) distribution, has density  $f(x) = \frac{\alpha \kappa^\alpha}{(\kappa+x)^{\alpha+1}}$  for  $\alpha > 0$  and  $\kappa = \alpha - 1 > 0$ , and its distribution function is  $F(x) = 1 - \left(\frac{\kappa}{\kappa+x}\right)^\alpha$ . The Pareto tail distribution is  $\bar{F}(x) = \left(\frac{\kappa}{\kappa+x}\right)^\alpha$  and its mean excess function is  $e_F(x) = 1 + \frac{x}{\kappa}$ , so that  $G(x) = x - \left(1 + \frac{x}{\kappa}\right)F(x)$ . Alternatively, we can use the formula  $G(x) = x - 1 + \left(\frac{\kappa}{\kappa+x}\right)^\kappa$  given in (Paulsen et al., 2005).

A grid size of  $h = 0.01$  was used throughout. The data simulations were performed using a Samsung Series 3 PC with an Intel Celeron 847 processor at 1.10GHz and 6.0GB RAM. To reduce computing time, the numerical method was implemented using the FORTRAN programming language, taking advantage of its DOUBLE PRECISION feature which gives satisfactory accuracy. All the figures in this paper were constructed using MATLAB R2016a.

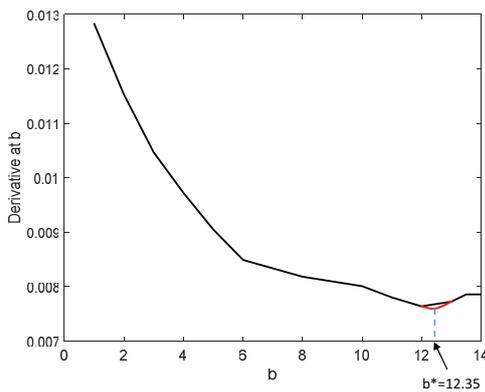
The various cases of dividend payouts with reinsurance in the diffusion-perturbed model (DPM) can be derived from Theorem 3.4), with appropriate values of  $k$ ,  $a$  and  $\sigma$ . Since results based on the CLM have been widely published (e.g., (Kasozzi and Paulsen, 2005),(Nansubuga et al., 2016)), in the following sections we only present results of dividends for the DPM with and without QS and XL reinsurance.

### 4.1 Dividends in the diffusion-perturbed model without reinsurance: optimal dividend barriers

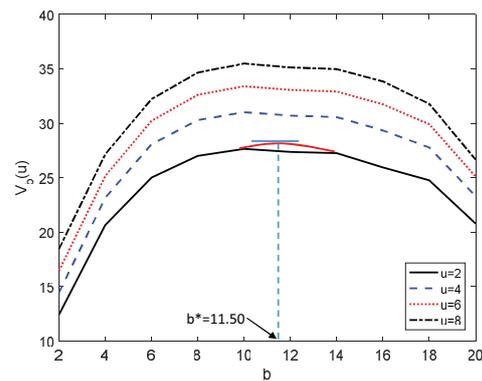
Before considering the diffusion-perturbed model compounded by reinsurance, it is important to obtain the optimal dividend barriers for small and large claims. The dividend values for  $\text{Exp}(0.5)$  claims are shown in Table 1. In this case, the optimal barrier  $b$  lies in the interval  $[12, 14]$  for small claims. The actual optimal barrier was obtained as  $b^* = 12.35$  by plotting the values of  $f'(b)$  at different values of  $b$  and approximating the value of  $b$  at which the derivative was a minimum, as shown in Figure 1(a).

Table 1: Dividends in the DPM without reinsurance: Exp(0.5) claims  
 ( $c = 6, \lambda = 2, \delta = 0.1, \sigma = 1$ )

| $b$           | $u = 2$ | $u = 4$ | $u = 6$ | $u = 8$ | $u = 10$ | $u = 12$ | $u = 14$ |
|---------------|---------|---------|---------|---------|----------|----------|----------|
| 2             | 5.6462  | 7.6462  | 9.6462  | 11.6462 | 13.6462  | 15.6462  | 17.6462  |
| 4             | 6.6968  | 8.8651  | 10.8651 | 12.8651 | 14.8651  | 16.8651  | 18.8651  |
| 6             | 7.6659  | 10.1478 | 12.2855 | 14.2855 | 16.2855  | 18.2855  | 20.2855  |
| 8             | 7.9578  | 10.5342 | 12.7533 | 14.7682 | 16.7682  | 18.7682  | 20.7682  |
| 10            | 8.1250  | 10.7556 | 13.0213 | 15.0786 | 17.0385  | 19.0385  | 21.0385  |
| 12            | 8.5155  | 11.2725 | 13.6471 | 15.8032 | 17.8573  | 19.8924  | 21.8924  |
| 14            | 8.2853  | 10.9678 | 13.2782 | 15.3760 | 17.3746  | 19.3547  | 21.3774  |
| 16            | 7.2210  | 9.5590  | 11.5726 | 13.4010 | 15.1428  | 16.8686  | 18.6314  |
| 18            | 6.8983  | 9.1318  | 11.0554 | 12.8020 | 14.4660  | 16.1146  | 17.7987  |
| 20            | 6.7561  | 8.8729  | 10.6974 | 12.3579 | 13.9448  | 15.5232  | 17.1408  |
| $b^* = 12.35$ | 8.6865  | 11.4989 | 13.9212 | 16.1206 | 18.2160  | 20.2919  | 22.6480  |



(a) Optimal barrier: Exp(0.5) claims



(b) Optimal barrier: Pareto (3,2) claims

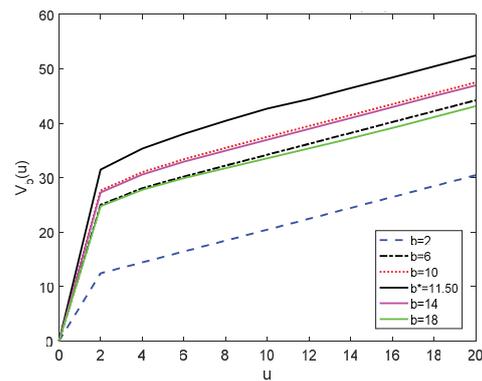
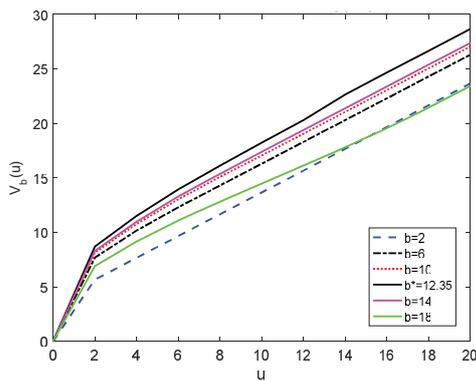
Figure 1: Optimal dividend barriers for the DPM for small and large claims

For Pareto(3,2) claims, the dividend values are given in Table 2. Here, the optimal barrier  $b^* \in [10, 12]$ , with the actual  $b^*$  found as 11.50. This is shown in Figure 1(b). The dividend values for various barrier levels  $b$  (including the optimal barrier) are given in Figure 2 for small and large claim sizes. Now, fixing  $b = b^*$ , we computed the dividend values for small and large claims in the DPM. The results are presented in the following sections for varying retention levels for quota-share and XL reinsurance.

It was established that for exponential claims the value functions for all  $u$  start reducing between  $b = 12$  and  $b = 14$  (i.e.,  $V_{12}(u) > V_{14}(u)$ ), it follows that the maximum value functions  $V^*$  are obtained when  $b^* \in [12, 14]$ . The optimal dividend barrier is found to be  $b^* = 12.35$ . Note that all other value functions are less than  $V^*$  despite  $b > b^*$  (see Figure 2(a)). The same can be said about Pareto claims (see Figure 2(b)).

Table 2: Dividends in the DPM without reinsurance: Pareto(3,2) claims  
 ( $c = 6, \lambda = 2, \lambda = 2, \delta = 0.1, \sigma = 1$ )

| $b$           | $u = 2$ | $u = 4$ | $u = 6$ | $u = 8$ | $u = 10$ | $u = 12$ | $u = 14$ |
|---------------|---------|---------|---------|---------|----------|----------|----------|
| 2             | 12.4488 | 14.4488 | 16.4488 | 18.4488 | 20.4488  | 22.4488  | 24.4488  |
| 4             | 20.6442 | 23.1525 | 25.1525 | 27.1525 | 29.1525  | 31.1525  | 33.1525  |
| 6             | 25.0237 | 28.0642 | 30.2186 | 32.2186 | 34.2186  | 36.2186  | 38.2186  |
| 8             | 27.0098 | 30.2917 | 32.6171 | 34.6541 | 36.6541  | 38.6541  | 40.6541  |
| 10            | 27.6548 | 31.0150 | 33.3960 | 35.4816 | 37.4898  | 39.4898  | 41.4898  |
| 12            | 27.3761 | 30.7024 | 33.0594 | 35.1240 | 37.1120  | 39.1035  | 41.1035  |
| 14            | 27.2562 | 30.5680 | 32.9146 | 34.9702 | 36.9495  | 38.9322  | 40.9698  |
| 16            | 25.9341 | 29.3282 | 31.7332 | 33.8399 | 35.8684  | 37.9004  | 39.9887  |
| 18            | 24.7624 | 27.7711 | 29.9030 | 31.7705 | 33.5687  | 35.3701  | 37.2212  |
| 20            | 20.7777 | 23.3022 | 25.0911 | 26.6581 | 28.1669  | 29.6784  | 31.2317  |
| $b^* = 11.50$ | 31.5039 | 35.3317 | 38.0441 | 40.4200 | 42.7077  | 44.4236  | 46.4236  |



(a) DP model: Dividends for Exp(0.5) claims

(b) DP model: Dividends for Pareto(3,2) claims

Figure 2: Dividends in the DPM (no reinsurance),  $c = 6, \lambda = 2, \delta = 0.1, \sigma = 1$

### 4.2 Dividends in the diffusion-perturbed model with proportional reinsurance

Here,  $\sigma > 0, a = \infty$  and so the model is

$$U_t^{\overline{D}, \overline{R}} = u + kct + \sigma W_t - \sum_{i=1}^{N_t} kX_i - D_t^b \tag{4.1}$$

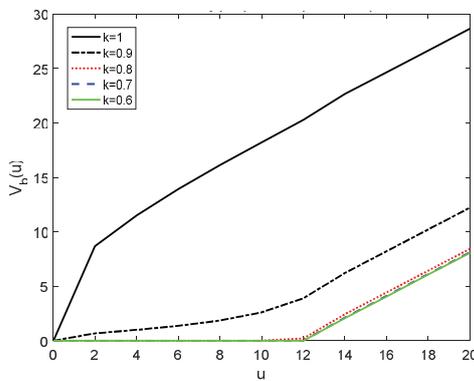
By Itô's formula, the IDE for this model is

$$\frac{1}{2}\sigma^2 V_b''(u) + kcV_b'(u) - (\lambda + \delta)V_b(u) + \lambda \int_0^u V_b(u - kx)dF(x) = 0 \tag{4.2}$$

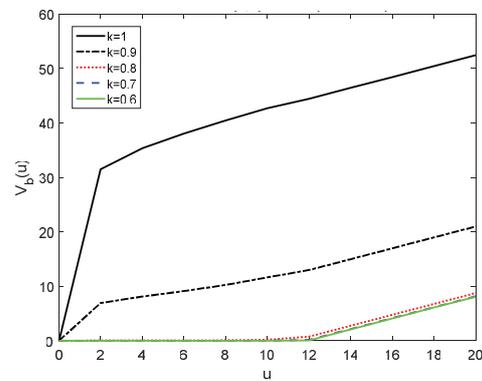
Successive integration by parts transforms this VIDE into a VIE of the second kind with kernel  $K(u, x) = \frac{2[kc - (\lambda + \delta)(u - kx) + \lambda G(u - kx)]}{\sigma^2}$  and forcing function  $h(u) = uV_b'(0)$ .

1. Exponential claims

For the small claim case in the DPM, the optimal barrier was found as  $b^* = 12.35$ . Figure 3(a) gives the dividend values for small claims in the diffusion-perturbed model. The optimal retention level for quota-share reinsurance is  $k^* = 1$  because this is the retention value giving the highest dividend values. This is confirmed by Figure 3(a). Thus, if the claim sizes are exponentially distributed it is optimal not to take proportional reinsurance in the diffusion-perturbed classical risk process. We also observe from Figure 3 that in the small and large claim cases involving the DPM and for values of the retention  $k$  below 0.8, the company can only begin to pay dividends at higher values of the initial surplus  $u$ , in particular, for  $u \geq 12$ . This is because as the retention level reduces, the cession level increases, meaning that the company pays more to reinsurers for risk-sharing. Consequently, the funds available for dividend distribution are reduced, necessitating a higher initial capital to guarantee survival of the insurance company.



(a) DPM: Dividends for Exp(0.5) claims



(b) DPM: Dividends for Pareto(3,2) claims

Figure 3: Dividend value functions in the DPM with proportional reinsurance,  $c = 6, \lambda = 2, \delta = 0.1, \sigma = 1$

2. Pareto claims

The optimal barrier for large claims in the DPM was found to be  $b^* = 11.50$  and the dividend values for the DPM compounded by XL reinsurance are given in Figure 3(b). In the large claim case involving the diffusion-perturbed classical risk model, the optimal retention level for quota-share reinsurance is clearly  $k^* = 1$  since this is the retention value yielding the highest dividend values as shown in Figure 3(b).

4.3 Dividends in the diffusion-perturbed model with excess-of-loss reinsurance

This is a diffusion model compounded only by XL reinsurance (i.e.,  $\sigma > 0, k = 1$ ), so we have

$$U_t^{\overline{D}, \overline{R}} = u + c\overline{R}t + \sigma W_t - \sum_{i=1}^{N_t} X_i \wedge a - D_t^b \tag{4.3}$$

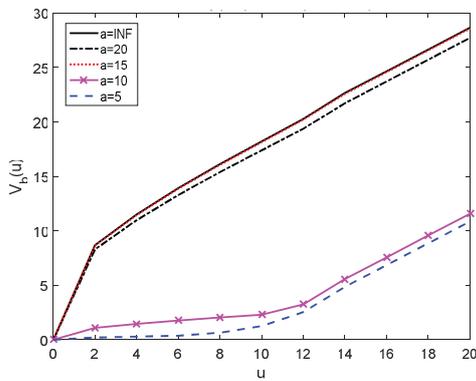
where  $c^{\bar{R}} = c - (1 + \theta)\lambda\mathbb{E}[(X_i - a)^+]$ . The corresponding IDE follows from Itô's formula as

$$\frac{1}{2}\sigma^2V_b''(u) + c^{\bar{R}}V_b'(u) - (\lambda + \delta)V_b(u) + \lambda \int_0^u V_b(u - x \wedge a)dF(x) = 0 \quad (4.4)$$

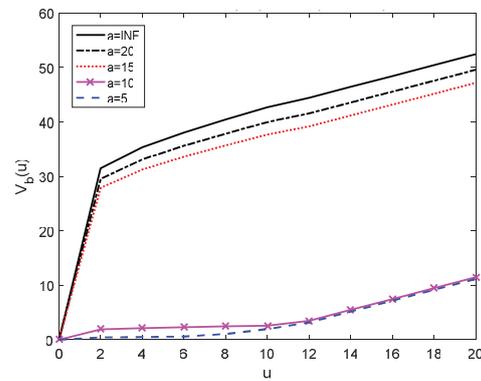
which transforms into a VIE of the second kind with  $K(u, x)$  and  $h(u)$  as given in (3.7) and (3.9) but with  $k = 1$ .

1. Exponential claims

As already noted, the optimal barrier for small claims in the DPM was found to be  $b^* = 12.35$  and the dividend values when XL reinsurance is used as a risk measure are given in Figure 4(a) which shows that it is optimal not to take XL reinsurance, i.e.,  $a^* = \infty$ . This means that in the DPM for small claims, reinsurance does not serve the cause of dividend maximization. In other words, the insurance company should not reinsure as doing so reduces the funds available for distribution as dividends.



(a) DPM: Dividends for Exp(0.5) claims



(b) DPM: Dividends for Pareto(3,2) claims

Figure 4: Dividend value functions for large claims for DPM with XL reinsurance,  $c = 6, \lambda = 2, \delta = 0.1, \sigma = 1.0$

2. Pareto claims

Here, the optimal dividend barrier is  $b^* = 11.5$  and the dividend values for the DPM compounded by XL reinsurance for large claims is given in Figure 4(b). As in the cases previously discussed, Figure 4 shows that it is optimal not to take XL reinsurance. But Figure 4 also shows that in both the small and large claim cases in the DPM, for values of the retention  $a$  below 15, the company will only be able to pay more in dividends at higher values of the initial surplus  $u$ .

## 5 Conclusion

From the results presented in the previous section, it can be concluded that at the optimal dividend barrier  $b^*$  for the DPM it is optimal not to take reinsurance. This applies both in the light- and heavy-tailed cases and the optimal retentions do not vary according to the claim size distribution used. Thus, for small claims the optimal policy is  $(\bar{D}, \bar{R}) = (12.35, (1, \infty))$ , while for large claims it is  $(\bar{D}, \bar{R}) = (11.50, (1, \infty))$ . For the chosen parameters, dividends are maximized when no reinsurance is taken. Thus, though the literature (e.g., (Li, Li and Young, 2017; Zhang and Liang, 2016)) shows that in a combinational proportional and XL reinsurance scenario the optimal strategy is a pure XL, i.e.,  $(1, a^*)$ , this study has established that it is optimal that the company does not reinsure (or that it retains all business), i.e., the optimal reinsurance policy is  $(k^*, a^*) = (1, \infty)$ . This means that in the DPM neither QS nor XL reinsurance has any advantage over the other as far as dividend maximization is concerned. Therefore, for purposes of dividend maximization the company should consider using other risk measures such as investment, capital injections or refinancing, portfolio selection and premium control. The study has also established that the company should use a higher optimal barrier for small claims than for large ones.

The work can be extended by: (1) determining ruin probabilities in the context of a combination of proportional and excess-of-loss reinsurance; (2) including solvency constraints in the form of ruin probability targets; (3) including investments; (4) incorporating transaction costs when paying dividends; (5) exploring optimality of other dividend strategies (e.g., threshold or band); and (6) replacing the claim number process  $N_t$  by a general renewal process so that the surplus process becomes a Sparre-Andersen model. Results on (1) can be found in (Kasumo, Kasozi and Kuznetsov, 2018) and some work towards (2) is underway.

## Acknowledgements

This work was supported by Mulungushi University, the Nelson Mandela African Institution of Science and Technology and the Zambian Ministry of Higher Education through the Support to Science and Technology Education Project (SSTEP) funded by the African Development Bank Group. The authors thank the anonymous referees for their valuable comments which significantly improved the paper.

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**Output 3: Draft manuscript on dividend maximization under ruin probability targets**

# Dividend maximization under a set ruin probability target in the presence of proportional and excess-of-loss reinsurance

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## Abstract

We study dividend maximization with set ruin probability targets for an insurance company whose surplus is modelled by a diffusion perturbed classical risk process. The company is permitted to enter into proportional or excess-of-loss reinsurance arrangements. By applying stochastic control theory, we derive Volterra integral equations and solve them numerically using block-by-block methods. In each of the models, we have established the optimal barrier to use for paying dividends provided the ruin probability does not exceed a predetermined target. Numerical examples involving the use of both light- and heavy-tailed distributions have been given. The results show that ruin probability targets result in an improvement in the optimal barrier to be used for dividend payouts. This is the case for light- and heavy-tailed distributions and applies regardless of the risk model used.

**Keywords:** *HJB equation, Volterra equation, Block-by-block method, Reinsurance, Dividends, Ruin probability, Ruin probability target*

## 1 Introduction

The dividend maximization problem has preoccupied researchers for several decades now. Several studies have investigated the optimal dividend problem for maximizing equity value by applying stochastic control techniques, among them Choulli *et al.* (2003), Kasozi and Mahera (2013) and Nansubuga *et al.* (2016). Kasozi *et al.* (2011) used homotopy analysis method (HAM) to maximize dividend payments in the Cramér-Lundberg model under a barrier strategy but found that the HAM was not convergent when applied to a model with stochastic return on investments. Kasozi and Paulsen (2005) studied the problem of dividend maximization in the classical risk model for a company that has invested some of the surplus in a risk-free asset. They obtained the optimal barrier strategy that maximizes

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<sup>2</sup>Corresponding author

the dividends to be paid out prior to ruin. Marciniak and Palmowski (2016) focused on the optimal dividend problem for insurance risk models with surplus-dependent premiums, using as their basic model a piece-wise deterministic Markov process (PDMP).

But many other studies have emerged in the actuarial literature focusing on dividend maximization under solvency constraints. Paulsen (2003) solved the dividend optimization problem of a firm under solvency constraints and showed that the optimal policy is of barrier type. Dickson and Dreik (2006) considered dividend optimization under a ruin probability constraint but for models different from ours. He *et al.* (2008) studied the optimal control problem for an insurance company that adopts a proportional reinsurance policy under solvency constraints. They gave a rigorous probability proof on the bankrupt probability decreasing with respect to some dividend barrier. Nansubuga *et al.* (2016) considered maximization of dividend payouts under infinite ruin probability constraints. They derived Volterra integral equations which they solved using the block-by-block method and established the optimal barrier to use to pay dividends provided the ruin probability is no larger than the predetermined tolerance.

Hernández and Junca (2015) studied the classical optimal dividend problem in the Cramér-Lundberg model with exponential claim sizes subject to a constraint on the ruin time and obtained the value function as a point-wise infimum of auxiliary value functions indexed by Lagrange multipliers. Using the fundamental tool of scale functions and fluctuation theory, Hernández, Junca and Moreno-Franco (2018) extended the results of Hernández and Junca (2015) for spectrally one-sided Lévy risk processes by introducing a longevity feature in the classical dividend problem through addition of a constraint on the time of ruin of the firm. Hipp (2016) studied control for minimizing ruin probability as well as maximizing dividend payments. In particular, he considered an optimal control problem concerned with maximizing the total expected discounted dividend payments with a ruin constraint and found that a ruin constraint is cheaper when an appropriate reinsurance cover is available.

In this paper, we consider dividend maximization under a set ruin probability target in a jump-diffusion model compounded by proportional and excess-of-loss (XL) reinsurance. In addition to managing the company's risk through reinsurance, management is allowed to pay dividends to the shareholders provided they are paid continuously according to a barrier level  $b$  and only until ruin. No dividends are paid when the surplus falls below  $b$ , and the ruin probability target is taken into account. The term 'dividend' refers to taxable payments declared by the insurer's board of directors and given to shareholders out of the company's current or retained earnings (Kasozzi & Paulsen, 2005). The 'ruin probability' is given by  $\psi(u) = \mathbb{P}(\tau_u < \infty)$ , where  $\tau_u = \{t > 0 | U_t < 0\}$ , called the 'ruin time', is the first time the surplus process  $U$  becomes negative, with  $\tau_u = \infty$  if  $U$  always stays positive. At an initial surplus  $u$ , the probability of ruin occurring before time horizon  $T$  is  $\psi(t, u) = \mathbb{P}(\tau_u < T)$ . If the surplus process is  $U_t^{\bar{D}, \bar{R}}$ , where  $(\bar{D}, \bar{R}) \in \Pi_u^{D, R}$  is an admissible dividend and reinsurance strategy, and if  $\bar{D}$  incorporates a dividend barrier  $b$ , then the ruin probability *at barrier level*  $b$  is defined as  $\psi(u) = \mathbb{P}(\tau_b^{\bar{D}, \bar{R}} < T)$ . At time horizon  $T$  and a ruin tolerance  $\epsilon > 0$ , the ruin

probability at barrier level  $b$  is defined as  $\psi_b(T, b) := \psi_b(T, u)|_{u=b} = \mathbb{P}(\tau_b^{\overline{D}, \overline{R}} < T) = \epsilon$ . The ‘ultimate ruin probability’ is the probability that the surplus process ever falls below zero, represented mathematically as  $\psi(u) = \mathbb{P}(\tau_u^{\overline{D}, \overline{R}} < \infty | U_0 = u) = 1 - \phi(u)$ , where  $\phi(u)$  is the survival probability.

The main contribution of this paper is that it seeks to extend the work of Nansubuga *et al.* (2016) by allowing the company to enter into reinsurance arrangements while at the same time distributing a portion of the surplus in the form of dividends to the shareholders. But there has to be a delicate trade-off between stability and profitability. Maximizing dividend payments leads to certain ruin (which is unacceptable for the policyholders), while maximizing survival probability results in a reduction in solvency capital, thus making dividend distribution impossible (which is unacceptable for the shareholders). To strike a balance, we seek to maximize dividend payments under a ruin probability constraint or target (Hipp, 2003). The idea in this paper is to find the optimal reinsurance strategies and then use them to determine the dividend value functions under a set ruin probability target. A key objective will be to establish which form of reinsurance is better between quota-share (QS) and excess-of-loss (XL) reinsurance.

The outline for the rest of the paper is as follows. In Section 2, we formulate the model and derive the relevant HJB, integrodifferential and integral equations corresponding to the problem. Section 3 presents numerical results and examples for validation of the numerical method. In Section 4, we present some concluding remarks and suggest possible extensions to this work.

## 2 Model formulation and analysis

### 2.1 Model

To make a rigorous mathematical formulation of the problem, we assume throughout this paper that all random variables and stochastic quantities are defined on a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}^+}, \mathbb{P})$  satisfying the usual conditions, that is, the filtration  $\{\mathcal{F}_t\}_{t \in \mathbb{R}^+}$  is right-continuous and  $\mathbb{P}$ -complete. In the absence of dividend payouts and reinsurance, the surplus of an insurance company is governed by the diffusion-perturbed classical risk process:

$$U_t = u + ct + \sigma W_t - \sum_{i=1}^{N_t} X_i, \quad t \geq 0 \quad (2.1)$$

where  $u = U_0 \geq 0$  is the initial reserve,  $c = (1 + \theta)\lambda\mu > 0$  is the premium rate assumed to be computed by the expected value premium principle,  $\{N_t\}$  is a homogeneous Poisson process with intensity  $\lambda > 0$  and  $\{X_i\}$  is an i.i.d. sequence of strictly positive random variables with distribution function  $F(x)$  and probability density function  $f(x)$ .  $\theta$  is called the *safety loading*

and represents the additional premium received by the insurer on account of uncertainty. The claim arrival process  $\{N_t\}$  and claim sizes  $\{X_i\}$  are assumed to be independent. Here  $\{W_t\}$  is a standard Brownian motion independent of the compound Poisson process  $S_t = \sum_{i=1}^{N_t} X_i$ . We assume that  $\mathbb{E}[X_i] = \mu < \infty$  and  $F(0) = 0$ . The Brownian term  $\sigma W_t$  represents random variations or fluctuations in the surplus process. However, when there is no volatility in the surplus and claim amounts (that is, when  $\sigma = 0$ ), (2.1) reduces to the Cramér-Lundberg model (CLM) or the classical risk process.

We assume that the insurer takes a combination of quota-share and XL reinsurance as proposed by Centeno (1985). Let the quota-share retention level be  $k \in [0, 1]$ . Then the insurer's aggregate claims, net of quota-share reinsurance, are  $kX$ . Also, let the XL reinsurance retention level be  $a \in [0, \infty)$ . Then the insurer's aggregate claims, net of quota-share and XL reinsurance, are  $kX \wedge a$ . When the retention limit  $a$  of the XL reinsurance is infinite, then the treaty becomes a *pure quota-share* reinsurance, while a quota-share level  $k = 1$  makes it a *pure excess-of-loss* reinsurance treaty. Though these two scenarios are somewhat extreme, they are still real possibilities for an insurance company. The premium income of the insurance company is non-negative if  $c \geq (1 + \theta)\mathbb{E}[(kX - a)^+]$ . Therefore, we will let  $\underline{a}$  be the XL retention level at which equality  $c = (1 + \theta)\mathbb{E}[(kX - \underline{a})^+]$  holds.

Thus, given a reinsurance strategy  $\bar{R} = (R_t)_{t \in \mathbb{R}^+}$  combining quota-share and XL reinsurance, the controlled surplus process becomes

$$U_t^{\bar{R}} = u + c^{\bar{R}}t + \sigma W_t - \sum_{i=1}^{N_t} (kX_i \wedge a), \quad t \geq 0 \quad (2.2)$$

where  $U_0^{\bar{R}} = u$  is the initial surplus of the company and  $c^{\bar{R}}$  is the premium rate net of the reinsurance premium. By Itô's formula, the generator for the process  $U_t^{\bar{R}}$  is given by

$$\mathcal{L}g(u) = \frac{1}{2}\sigma^2 g''(u) + c^{\bar{R}}g'(u) + \lambda \int_0^\infty [g(u - kx \wedge a) - g(u)]dF(x) \quad (2.3)$$

Paulsen and Gjessing (1997) have shown that if the equation  $\mathcal{L}(\psi)(u) = 0$ , where  $\mathcal{L}$  is the infinitesimal generator defined in (2.3), has a solution satisfying the boundary conditions

$$\begin{aligned} \psi(u) &= 1 \text{ on } u < 0 \\ \psi(0) &= 1 \text{ if } \sigma^2 > 0 \\ \lim_{u \rightarrow \infty} \psi(u) &= 0 \end{aligned} \quad (2.4)$$

then that solution is the probability of ruin. Minimizing the ultimate ruin probability  $\psi(u)$  is the same as maximizing the ultimate survival probability  $\phi(u)$  s.t.  $\mathcal{L}(\phi)(u) = \mathcal{L}(1 - \psi(u)) = 0$

with the boundary conditions

$$\begin{aligned}\phi(u) &= 0 \text{ on } u < 0 \\ \phi(0) &= 0 \text{ if } \sigma^2 > 0 \\ \lim_{u \rightarrow \infty} \phi(u) &= 1\end{aligned}\tag{2.5}$$

## 2.2 HJB, integrodifferential and integral equations

The following theorem presents the Hamilton-Jacobi-Bellman (HJB) equation for the survival probability maximization problem:

**Theorem 2.1.** *Assume that the survival probability  $\phi(u)$  is twice continuously differentiable on  $(0, \infty)$ . Then  $\phi(u)$  satisfies the HJB equation*

$$\sup_{R \in \mathcal{R}} \mathcal{L}\phi(u) = 0, \quad u > 0\tag{2.6}$$

where  $\mathcal{R}$  is the set of all reinsurance policies and  $\mathcal{L}$  is the infinitesimal generator defined in (2.3) for  $0 < u \leq \infty$ .

*Proof.* The proof of this theorem is standard and can be found in Schmidli (2008).  $\square$

It follows from Theorem 2.1 that

$$\frac{1}{2}\sigma^2\phi''(u) + c\bar{R}\phi'(u) + \lambda \int_0^u \phi(u - kx \wedge a) dF(x) - \lambda\phi(u) = 0\tag{2.7}$$

which is a second-order Volterra integrodifferential equation (VIDE). This equation is transformed into an ordinary Volterra integral equation (VIE) using successive integration by parts as stated in the following theorem.

**Theorem 2.2.** *The Volterra integrodifferential equation (2.7) can be represented as a Volterra integral equation of the second kind*

$$\phi(u) + \int_0^u K(u, x)\phi(x)dx = h(u)\tag{2.8}$$

where

1. For  $u \leq \underline{a} < a$ , we have

$$\begin{aligned}K(u, x) &= -\frac{\lambda\bar{F}(u - kx)}{c\bar{R}} \\ h(u) &= \phi(0)\end{aligned}\tag{2.9}$$

with  $\bar{F}(x) = 1 - F(x)$ , when there is no diffusion (that is, when  $\sigma^2 = 0$ ), and

$$K(u, x) = \frac{2 \left( c^{\bar{R}} + \lambda G(x, u) - \lambda(u - kx) \right)}{\sigma^2} \quad (2.10)$$

$$h(u) = u\phi'(0) \text{ if } \sigma^2 > 0$$

when there is diffusion.

2. For  $\underline{a} < a < u$ , we have

$$K(u, x) = -\frac{\lambda H_1(x, u)}{c^{\bar{R}}} \quad (2.11)$$

$$h(u) = \phi(0)$$

with

$$H_1(x, u) = \begin{cases} \bar{F}(u - kx) & kx < a \\ 1 - (F(kx + a) - F(a)) & kx \geq a \end{cases}$$

when there is no diffusion, and

$$K(u, x) = \frac{2 \left( c^{\bar{R}} + \lambda H_2(x, u) - \lambda(u - kx) \right)}{\sigma^2} \quad (2.12)$$

$$h(u) = u\phi'(0) \text{ if } \sigma^2 > 0$$

with

$$H_2(x, u) = \begin{cases} G(u - kx) & kx < a \\ (F(kx + a) - F(a))(u - kx) & kx \geq a \end{cases}$$

and  $G(x) = \int_0^x F(v)dv$  when there is diffusion.

*Proof.* The proof for the case  $u \leq \underline{a} < a$  is similar to the proof of Theorem 2.2 in Paulsen *et al.* (2005) but with  $r = \sigma_R^2 = 0$ ,  $k = 1$  and  $p = c^{\bar{R}}$ . The case  $\underline{a} < a < u$  has been proved in Kasumo *et al.* (2018a).  $\square$

Assume that the company pays out dividends  $D_t^b$  up to time  $t$  under a ruin probability target, say  $\psi(u) = \epsilon$ . The dividend process  $\bar{D} = (D_t^b)_{t \in \mathbb{R}^+}$  is non-negative, non-decreasing, right-continuous with left limits (or càdlàg) and  $\mathcal{F}_t$ -adapted. The dynamics of the company's wealth is therefore given by

$$dU_t^{\bar{D}, \bar{R}} = dU_t^{\bar{R}} - dD_t^b \quad (2.13)$$

where  $dU_t^{\bar{R}} = c^{\bar{R}}dt + \sigma dW_t - d \left( \sum_{i=1}^{N_t} kX_i \wedge a \right)$  and the superscript  $b$  is a dividend barrier level. The insurance premium rate under these conditions is  $c^{\bar{R}} = c - c^R$ , where  $c^R = (1 + \theta)\lambda\mathbb{E}[(kX_i - a)^+]$  is the reinsurance premium, and  $dW_t = \xi_t dt$ ,  $\xi_t$  being a white noise process. The time of ruin, when dividends and reinsurance are taken into account, is defined

as  $\tau_b^{\bar{D}, \bar{R}} = \inf\{t \geq 0 | U_t^{\bar{D}, \bar{R}} < 0\}$  and the probability of ultimate ruin is defined as  $\psi^{\bar{D}, \bar{R}} = \mathbb{P}(U_t^{\bar{D}, \bar{R}} < 0 \text{ for some } t > 0)$ .

The objective is to maximize the total expected discounted dividends paid out to the shareholders until ruin

$$V^{\bar{D}, \bar{R}}(u) = \mathbb{E}_u \left[ \int_0^{\tau_b^{\bar{D}, \bar{R}}} e^{-\delta t} dD_t^b \right] \quad (2.14)$$

under a set ruin probability target

$$\psi^{\bar{D}, \bar{R}}(u) = \mathbb{P}(\tau_b^{\bar{D}, \bar{R}} < \infty) \leq \epsilon \quad (2.15)$$

The quantity  $\delta > 0$  is the constant rate at which dividends are discounted and  $\mathbb{E}_u$  denotes expectation with respect to  $\mathbb{P}_u$ , the probability measure conditioned on the initial capital  $U_0^{\bar{D}, \bar{R}} = u$ . Thus the optimal value function of this problem becomes

$$V_b(u) = V^{\bar{D}, \bar{R}}(u, \epsilon) = \sup_{(\bar{D}, \bar{R}) \in \Pi_u^{\bar{D}, \bar{R}}} \left\{ \mathbb{E}_u \left[ \int_0^{\tau_b^{\bar{D}, \bar{R}}} e^{-\delta t} dD_t^b \right] : \psi^{\bar{D}, \bar{R}}(u) \leq \epsilon \right\} \quad (2.16)$$

where  $0 < \epsilon \leq 1$  is the permitted ruin probability and  $\psi^{\bar{D}, \bar{R}}(u)$  is the with-dividend-and-reinsurance ruin probability.

Paulsen and Gjessing (1997) have showed that if  $V_b(u)$  solves  $\mathcal{L}V_b(u) = \delta V_b(u)$  on  $0 < u < b$ , subject to the conditions

$$\begin{aligned} V_b(u) &= 0 \text{ on } u < 0 \\ V_b(0) &= 0 \text{ if } \sigma^2 > 0 \\ V_b'(b) &= 1 \\ V_b(u) &= V_b(b) + u - b \text{ on } u > b \end{aligned} \quad (2.17)$$

then  $V_b(u)$  is given by (2.16). For  $0 \leq u \leq b$ , the integrodifferential equation for  $V_b$  is

$$\frac{1}{2}\sigma^2 V_b''(u) + c^{\bar{R}} V_b'(u) + \lambda \int_0^u V_b(u - kx \wedge a) dF(x) - (\lambda + \delta)V_b(u) = 0 \quad (2.18)$$

Equation (2.18) is a Volterra integrodifferential equation (VIDE) which can be transformed into a Volterra integral equation of the second kind using successive integration by parts, as shown by the following theorem.

**Theorem 2.3.** *The Volterra integrodifferential equation (2.18) can be represented as a Volterra integral equation of the second kind*

$$V_b(u) + \int_0^u K(u, x)V_b(x)dx = h(u) \quad (2.19)$$

where

1. For  $u \leq \underline{a} < a$ , we have

$$\begin{aligned} K(u, x) &= -\frac{\delta + \lambda \bar{F}(u - kx)}{c^{\bar{R}}} \\ h(u) &= V_b(0) \end{aligned} \quad (2.20)$$

with  $\bar{F}(x) = 1 - F(x)$ , when there is no diffusion (that is, when  $\sigma^2 = 0$ ), and

$$\begin{aligned} K(u, x) &= \frac{2 \left( c^{\bar{R}} + \lambda G(x, u) - (\lambda + \delta)(u - kx) \right)}{\sigma^2} \\ h(u) &= uV_b'(0) \text{ if } \sigma^2 > 0 \end{aligned} \quad (2.21)$$

when there is diffusion.

2. For  $\underline{a} < a < u$ , we have

$$\begin{aligned} K(u, x) &= -\frac{\delta + \lambda H_1(x, u)}{c^{\bar{R}}} \\ h(u) &= V_b(0) \end{aligned} \quad (2.22)$$

with

$$H_1(x, u) = \begin{cases} \bar{F}(u - kx) & kx < a \\ 1 - (F(kx + a) - F(a)) & kx \geq a \end{cases}$$

and  $V_b(u) = V_b(b) + u - b$ , for  $u > b$  when there is no diffusion, and

$$\begin{aligned} K(u, x) &= \frac{2 \left( c^{\bar{R}} + \lambda H_2(x, u) - (\lambda + \delta)(u - kx) \right)}{\sigma^2} \\ h(u) &= uV_b'(0) \text{ if } \sigma^2 > 0 \end{aligned} \quad (2.23)$$

with

$$H_2(x, u) = \begin{cases} G(u - kx) & kx < a \\ (F(kx + a) - F(a))(u - kx) & kx \geq a \end{cases}$$

and  $G(x) = \int_0^x F(v)dv$  when there is diffusion.

*Proof.* The proof for the case  $u \leq \underline{a} < a$  is similar to the proof of Theorem 2.2 in Paulsen *et al.* (2005) but with  $r = \sigma_R^2 = 0$ ,  $k = 1$  and  $p = c^{\bar{R}}$ , while the proof for the case  $\underline{a} < a < u$  can be found in Kasumo *et al.* (2018b).  $\square$

The following results are relevant to the case involving ruin probability targets.

**Theorem 2.4.** *At every dividend barrier level  $b$ , there exists a unique probability  $\epsilon_b$  such that  $\mathbb{P}(\tau_b^{\bar{D}, \bar{R}} < T) = \epsilon_b$  and if  $b_1 < b_2$ , then  $\epsilon_{b_2} < \epsilon_{b_1}$ .*

*Proof.* We note that  $\mathbb{P}(\tau_u^{\bar{D}, \bar{R}} < T)$  is defined  $\forall u > 0$ . This implies that  $\mathbb{P}(\tau_b^{\bar{D}, \bar{R}} < T)$  follows by setting  $u = b$  and since  $\mathbb{P}(\tau_b^{\bar{D}, \bar{R}} < T)$  is a decreasing function of  $u$ ,  $b_1 < b_2$  implies that  $\epsilon_{b_2} < \epsilon_{b_1}$ .  $\square$

**Theorem 2.5.** *Suppose that the barrier that solves the VIDE (2.18) is  $b^*$  and that a ruin target  $\mathbb{P}(\tau_b^{\overline{D}, \overline{R}} < T) = \epsilon_b$  is enforced by the insurance company. Then*

- (i) *If  $b \leq b^*$ , the optimal strategy is to pay dividends using barrier level  $b^*$ .*
- (ii) *If  $b > b^*$ , the optimal strategy is to pay dividends using barrier level  $b$ .*

*Proof.* The proof of this theorem is similar to that of Theorem 2.5 in Nansubuga *et al.* (2016). □

### 3 Numerical results

To solve for the survival probability  $\phi(u)$  (from which we obtain the ruin probability  $\psi(u) = 1 - \phi(u)$ ) and for the dividend value function  $V_b(u)$ , we use the fourth-order block-by-block method a comprehensive description of which can be found in Linz (1969, 1985) and Paulsen *et al.* (2005). To maximize dividends under a ruin probability target for a model with initial capital  $u$  and ruin probability tolerance  $\epsilon$ , the following calculations have been performed:

- (a) Using  $u$ , we solve the dividend maximization problem to determine the optimal barrier  $b^*$ .
- (b) For each optimal dividend barrier  $b^*$ , we incorporate proportional and XL reinsurance into the CLM and the DPM.
- (c) In the ultimate ruin problem, we choose  $b_0$  so that  $\psi(b_0) = \epsilon$ , which is the ultimate ruin probability at  $b_0$ . This means that dividends cannot be paid unless the survival probability  $1 - \epsilon$  is greater than  $\epsilon$ .

With both  $b^*$  and  $b_0$ , the decision is based on Theorem 2.5. Some results are now presented based on the exponential and Pareto distributions. The  $\text{Exp}(\beta)$  distribution, a special case of the Weibull( $\alpha, \beta$ ) distribution, has density  $f(x) = \beta e^{-\beta x}$ , with corresponding distribution and tail functions  $F(x) = 1 - e^{-\beta x}$  and  $\overline{F}(x) = 1 - F(x) = e^{-\beta x}$ , respectively. The mean excess function for the exponential distribution is  $e_F(x) = \frac{1}{\beta}$  and  $G(x) = x - \frac{1}{\beta} F(x)$ . The Pareto( $\alpha, \kappa$ ) distribution, which is a special case of the three-parameter Burr( $\alpha, \kappa, \tau$ ) distribution, has density  $f(x) = \frac{\alpha \kappa^\alpha}{(\kappa+x)^{\alpha+1}}$  for  $\alpha > 0$  and  $\kappa = \alpha - 1 > 0$ , and its distribution function is  $F(x) = 1 - \left(\frac{\kappa}{\kappa+x}\right)^\alpha$ . The Pareto tail distribution is  $\overline{F}(x) = \left(\frac{\kappa}{\kappa+x}\right)^\alpha$  and its mean excess function is  $e_F = 1 + \frac{x}{\kappa}$ , so that  $G(x) = x - \left(1 + \frac{x}{\kappa}\right) F(x)$ . Alternatively, for the Pareto distribution  $G(x)$  can be written as  $x - 1 + \left(\frac{\kappa}{\kappa+x}\right)^\kappa$ . A gridsize of  $h = 0.01$  was used throughout. The data simulations in this section were performed on a Samsung Series 3 PC with an Intel Celeron 847 processor at 1.10GHz and 6.0GB RAM. To reduce computing time, the numerical method was implemented using the FORTRAN programming language, taking

advantage of its DOUBLE PRECISION feature which gives a high degree of accuracy. The graphs were constructed using MATLAB R2016a.

The various cases of dividend payouts with reinsurance can be derived from equations (2.8)-(2.12) (Theorem 2.3), which represent dividend models compounded by proportional and XL reinsurance, with appropriate values of  $k$ ,  $a$  and  $\sigma$ . The results based on the Cramér-Lundberg model (CLM) and diffusion-perturbed model(DPM) are given in the following sections.

### 3.1 Ruin probability targets for the Cramér-Lundberg model: exponential claims

Here the kernel and forcing function are given by

$$\begin{aligned} K(u, x) &= -\frac{\lambda \bar{F}(u-x)}{c} \\ h(u) &= \phi(0) \end{aligned} \tag{3.1}$$

with  $\bar{F}(x) = 1 - F(x)$ . Table 1 gives the ultimate ruin probabilities in the CLM with no reinsurance and dividends.

Table 1: Ultimate ruin probabilities in the CLM for Exp(0.5) claims,  $c = 6$ ,  $\lambda = 2$

| $u$       | 0      | 4      | 8      | 10     | 12     | 14     | 16     | 18     | 20     |
|-----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\psi(u)$ | 0.6667 | 0.3423 | 0.1757 | 0.1259 | 0.0902 | 0.0646 | 0.0463 | 0.0332 | 0.0238 |

As expected, increasing the initial capital  $u$  reduces the ruin probability  $\psi(u)$ . We now set ruin probabilities to obtain different values of initial capital to be used as ruin probability target values in the dividend model for the CLM without reinsurance, that is, with  $k = 1$  and  $a = \infty$ . Let  $\epsilon_i \equiv i$ -th ruin probability used. Then, choosing arbitrarily from Table 1,

$$\psi(b_0^1) = \epsilon_1 = 0.1259 \text{ gives } b_0^1 = 10.00$$

$$\psi(b_0^2) = \epsilon_2 = 0.0902 \text{ gives } b_0^2 = 12.00$$

$$\psi(b_0^3) = \epsilon_3 = 0.0646 \text{ gives } b_0^3 = 14.00$$

$$\psi(b_0^4) = \epsilon_4 = 0.0332 \text{ gives } b_0^4 = 18.00$$

### 3.2 Dividends for the Cramér-Lundberg model: exponential claims

The VIE in this case has kernel and forcing function

$$\begin{aligned} K(u, x) &= -\frac{\delta + \lambda \bar{F}(u-x)}{c} \\ h(u) &= V_b(0) \end{aligned} \quad (3.2)$$

The exact solution can be found in Kasozi and Paulsen (2005). The total expected present value of dividends is given by the value function

$$V_b(u) = \begin{cases} \frac{f(u)}{f'(b)} & u \leq b \\ \frac{f(b)}{f'(b)} + u - b & u > b \end{cases} \quad (3.3)$$

where  $f(u) = (\beta + r_1)e^{r_1 u} - (\beta + r_2)e^{r_2 u}$  (Kasozi *et al.*, 2011), where  $r_1$  and  $r_2$  are given by

$$r_{1,2} = \frac{-(c\beta - \lambda - \delta) \pm \sqrt{(c\beta - \lambda - \delta)^2 + 4c\beta\delta}}{2c} \quad (3.4)$$

The optimal barrier  $b^*$  is obtained by solving the equation  $f''(b^*) = 0$ , that is,  $(\beta + r_1)r_1^2 e^{r_1 b^*} - (\beta + r_2)r_2^2 e^{r_2 b^*} = 0$ . For any arbitrary starting point  $f(0)$ ,  $f(u)$  is the  $O(h^4)$  numerical solution obtained using the block-by-block method. To find  $f'(b)$ , we use the approximation  $f'(b) \approx \lim_{h \rightarrow 0} \frac{f(b+h) - f(b-h)}{2h}$ , where  $h$  is the grid size. We have solved (3.3) for several values of  $b$ . Using a FORTRAN program which, at each run, gives the  $O(h^4)$  solution to the Volterra equations of the second kind (Theorem 2.3), we have computed the values of  $f'(b)$ . The results indicate that for any two barriers  $b_1$  and  $b_2$ , with  $0 < b_1 < b_2 < \infty$ ,  $f'(b_1) > f'(b_2)$ . Eventually, some interval  $[b_1, b_2]$  gives  $f'(b_1) < f'(b_2)$  for the first time. This interval contains the optimal  $b^*$  which gives the optimal value function  $V_b(b^*)$ . The results are given in Table 2 and Figure 1.

Using Theorem 2.5, we obtain the optimal barriers under ruin probability targets. For example, for initial capital  $u = 2$  the optimal dividend barrier is  $b^* = 8.5923$  and  $b_0^1 = 10.0000$ . Since  $b_0^1 > b^*$ , we take 10.0000 as the optimal barrier. For  $u = 6$ ,  $b^* = 13.6047$  and  $b_0^1 = 10.0000$ . Since  $b^* > b_0^1$ , we take 13.6047 as the optimal barrier. The optimal barriers for other values of  $u$  can be obtained in a similar manner. The results are presented in Table 2 and Figure 1. The company pays out dividends to the shareholders whenever  $b^* > b_0^1$  because of the ruin probability target. Figure 1 shows that as the ruin probability reduces, the optimal dividend barrier increases and this is precisely the goal of dividend maximization.

It should be noted that for Pareto claim sizes,  $b^* > b_0^i$  ( $i = 1, 2, 3, 4$ ) for all  $u$ . Thus, the company can pay dividends at all barrier levels.

Table 2: Dividends in the CLM with Exp(0.5) claims,  $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$

| $b$             | $u = 2$ | $u = 4$ | $u = 6$ | $u = 8$ | $u = 10$ | $u = 12$ | $u = 14$ |
|-----------------|---------|---------|---------|---------|----------|----------|----------|
| 2               | 5.8278  | 7.8278  | 9.8278  | 11.8278 | 13.8278  | 15.8278  | 17.8278  |
| 4               | 6.9460  | 9.1224  | 11.1224 | 13.1224 | 15.1224  | 17.1224  | 19.1224  |
| 6               | 7.8320  | 10.2860 | 12.4009 | 14.4009 | 16.4009  | 18.4009  | 20.4009  |
| 8               | 8.3886  | 11.0169 | 13.2822 | 15.3439 | 17.3439  | 19.3439  | 21.3439  |
| 10              | 8.5897  | 11.2810 | 13.6006 | 15.7117 | 17.7294  | 19.7294  | 21.7294  |
| 12              | 8.4894  | 11.1494 | 13.4419 | 15.5284 | 17.5225  | 19.5058  | 21.5058  |
| 14              | 8.1872  | 10.7525 | 12.9634 | 14.9757 | 16.8988  | 18.8115  | 20.7718  |
| 16              | 7.7567  | 10.1871 | 12.2817 | 14.1881 | 16.0101  | 17.8223  | 19.6794  |
| 18              | 7.2728  | 9.5516  | 11.5155 | 13.3030 | 15.0114  | 16.7105  | 18.4517  |
| 20              | 6.7561  | 8.8729  | 10.6974 | 12.3579 | 13.9448  | 15.5232  | 17.1408  |
| $b^* = 10.27$   | 8.5923  | 11.2844 | 13.6047 | 15.7165 | 17.7348  | 19.9749  | 21.9749  |
| $b_0^1 = 10.00$ | 10.0000 | 11.2844 | 13.6047 | 15.7165 | 17.7348  | 19.9749  | 21.9749  |
| $b_0^2 = 12.00$ | 12.0000 | 12.0000 | 13.6047 | 15.7165 | 17.7348  | 19.9749  | 21.9749  |
| $b_0^3 = 14.00$ | 14.0000 | 14.0000 | 14.0000 | 15.7165 | 17.7348  | 19.9749  | 21.9749  |
| $b_0^4 = 18.00$ | 18.0000 | 18.0000 | 18.0000 | 18.0000 | 18.0000  | 19.9749  | 21.9749  |

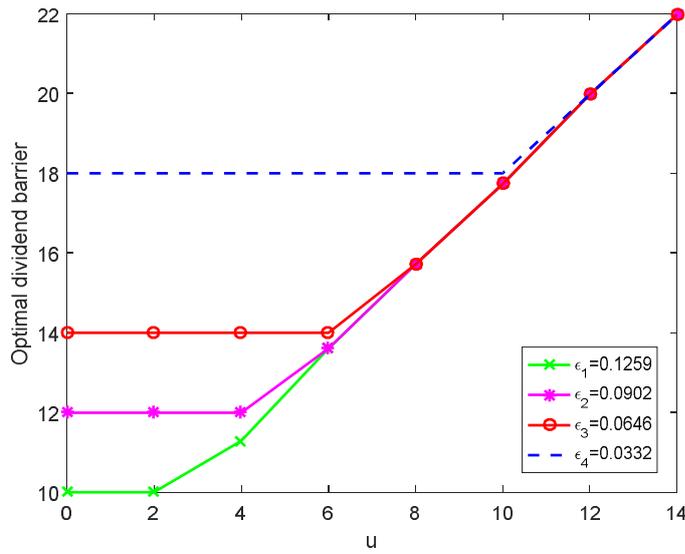


Figure 1: Numerical optimal barriers in CLM for Exp(0.5) claims,  $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$

### 3.3 Dividends for the Cramér-Lundberg model compounded by proportional reinsurance: exponential claims

The VIE for the CLM compounded by proportional reinsurance has kernel and forcing function

$$\begin{aligned}
K(u, x) &= -\frac{\delta + \lambda \bar{F}(u - kx)}{kc} \\
h(u) &= V_b(0)
\end{aligned}
\tag{3.5}$$

where  $k \in [0, 1]$  is the retention level for quota-share reinsurance. It has been shown in Kasumo *et al.* (2018b) that for dividend maximization it is optimal not to take proportional reinsurance in the small claim case involving the CLM. Therefore, the ruin probability targets and optimal barriers under proportional reinsurance are the same as shown in the immediately preceding sections (see Tables 1 and 2).

### 3.4 Dividends for the Cramér-Lundberg model compounded by excess-of-loss reinsurance: exponential claims

The kernel and forcing function for this case are given by

$$\begin{aligned}
K(u, x) &= -\frac{\delta + \lambda H_1(x, u)}{c\bar{R}} \\
h(u) &= V_b(0)
\end{aligned}
\tag{3.6}$$

with

$$H_1(x, u) = \begin{cases} \bar{F}(u - x) & x < a \\ 1 - (F(x + a) - F(a)) & x \geq a \end{cases}$$

where  $c\bar{R} = c - (1 + \theta)\lambda\mathbb{E}[(X_i - a)^+]$  is the insurance premium rate. Kasumo *et al.* (2018b) have shown that it is optimal not to take XL reinsurance in the CLM for exponential claims. This means that the ruin probability targets and optimal dividend barriers for XL reinsurance are the same as those for QS reinsurance as shown in Tables 1 and 2.

### 3.5 Ruin probability targets for the Cramér-Lundberg model compounded by excess-of-loss reinsurance: Pareto claims

Since in the large claim case involving the CLM the optimal policy is to take XL reinsurance with  $a^* = 10$  (see Kasumo *et al.* (2018b)), we have to compute ruin probabilities for  $a = 10$ . These are shown in Table 3.

Table 3: Ultimate ruin probabilities in the CLM for Par(3,2) claims,  $c = 6$ ,  $\lambda = 2$

| $u$       | 0      | 4      | 8      | 10     | 10.5   | 11     | 11.5   | 12     | 12.5   |
|-----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\psi(u)$ | 0.6667 | 0.5363 | 0.5169 | 0.5132 | 0.4888 | 0.4318 | 0.3512 | 0.2492 | 0.1251 |

The ruin probability  $\psi(u)$  reduces quite slowly as the initial capital  $u$  increases. We now set

ruin probabilities to obtain different values of initial capital to be used as ruin probability target values in the dividend model for the CLM with optimal XL reinsurance retention  $a^* = 10$ . Let  $\epsilon_i \equiv i$ -th ruin probability used. Then, choosing arbitrarily from Table 3,

$$\begin{aligned} \psi(b_0^1) = \epsilon_1 = 0.5169 \text{ gives } b_0^1 &= 8.00 \\ \psi(b_0^2) = \epsilon_2 = 0.5132 \text{ gives } b_0^2 &= 10.00 \\ \psi(b_0^3) = \epsilon_3 = 0.4318 \text{ gives } b_0^3 &= 11.00 \\ \psi(b_0^4) = \epsilon_4 = 0.2492 \text{ gives } b_0^4 &= 12.00 \end{aligned}$$

### 3.6 Dividends for the Cramér-Lundberg model compounded by excess-of-loss reinsurance: Pareto claims

Table 4: Dividends in the CLM with XL reinsurance: Par(3,2) claims  
( $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$ )

| $b$          | $u = 2$ | $u = 4$ | $u = 6$ | $u = 8$ | $u = 10$ | $u = 12$ | $u = 14$ |
|--------------|---------|---------|---------|---------|----------|----------|----------|
| 2            | 13.7872 | 15.7872 | 17.7872 | 19.7872 | 21.7872  | 23.7872  | 25.7872  |
| 4            | 22.2092 | 24.6831 | 26.6831 | 28.6831 | 30.6831  | 32.6831  | 34.6831  |
| 6            | 26.8729 | 29.8662 | 32.0364 | 34.0364 | 36.0364  | 38.0364  | 40.0364  |
| 8            | 28.6866 | 31.8819 | 34.1985 | 36.2515 | 38.2515  | 40.2515  | 42.2515  |
| 10           | 27.6877 | 30.7718 | 33.0077 | 34.9893 | 36.9052  | 38.9052  | 40.9052  |
| $b^* = 9.75$ | 28.9684 | 32.1951 | 34.5345 | 36.6077 | 38.3522  | 40.3522  | 42.3522  |

The optimal dividend barrier for this model is found to be  $b^* \approx 9.75$  and the results are given in Table 4. It turns out that for Pareto(3,2) claims in the CLM compounded by XL reinsurance  $b^* > b_0^i$  ( $i = 1, 2, 3, 4$ )  $\forall u > 0$ . Therefore, the company can pay dividends at all dividend barrier levels.

### 3.7 Ruin probability targets for the diffusion-perturbed model: exponential claims

The ruin probabilities for the DPM with  $\sigma = 1.0$  are given in Table 5.

Table 5: Ultimate ruin probabilities in the DPM for Exp(0.5) claims,  $c = 6$ ,  $\lambda = 2$

| $u$       | 0      | 4      | 8      | 10     | 12     | 14     | 16     | 18     | 20     |
|-----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\psi(u)$ | 1.0000 | 0.3581 | 0.1872 | 0.1354 | 0.0979 | 0.0708 | 0.0512 | 0.0370 | 0.0268 |

Since the optimal reinsurance policy for the dividend maximization problem is  $(k^*, a^*) = (1, \infty)$ , that is, do not reinsure, we have used only the ruin probabilities  $\psi_{k=1}(u)$  which are

the same as  $\psi_{a=\infty}(u)$  to choose ruin probability targets and set optimal dividend barriers under set ruin probability targets.

Thus, from Table 5, we arbitrarily choose

$$\begin{aligned}\psi(b_0^1) &= \epsilon_1 = 0.1354, \text{ giving } b_0^1 = 10.00 \\ \psi(b_0^2) &= \epsilon_2 = 0.0708, \text{ giving } b_0^2 = 14.00 \\ \psi(b_0^3) &= \epsilon_3 = 0.0512, \text{ giving } b_0^3 = 16.00 \\ \psi(b_0^4) &= \epsilon_4 = 0.0268, \text{ giving } b_0^4 = 20.00\end{aligned}$$

### 3.8 Dividends for the diffusion-perturbed model compounded by proportional reinsurance: exponential claims

The kernel and forcing function for this case are, respectively,

$$\begin{aligned}K(u, x) &= \frac{2(kc + \lambda G(u - kx) - (\lambda + \delta)(u - kx))}{\sigma^2} \\ h(u) &= uV_b'(0) \text{ if } \sigma^2 > 0\end{aligned}\tag{3.7}$$

with  $G(x) = \int_0^x F(v)dv$ . The optimal barriers for varying initial surplus values are shown in Figure 2. The optimal barrier without a ruin probability target in this case was found as  $b^* = 12.35$ .

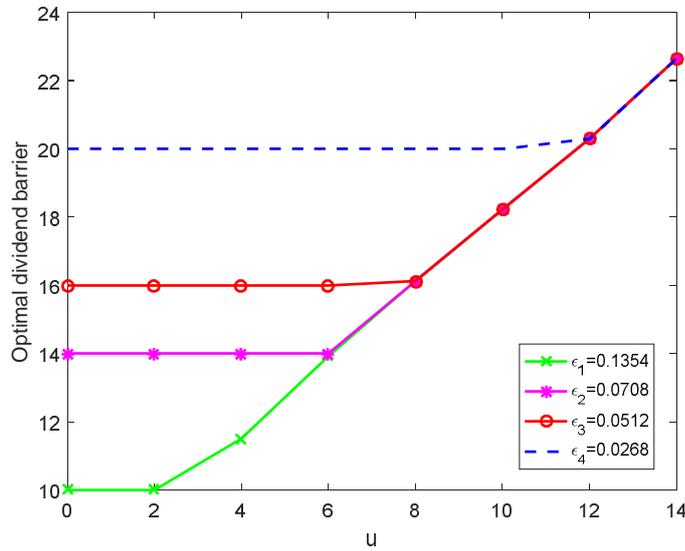


Figure 2: Numerical optimal barriers in DPM for Exp(0.5) claims,  $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$ ,  $\sigma = 1.0$

At  $u = 2$ , for example, the optimal barrier without a ruin probability target is 12.35. Thus,

the insurer only pays out dividends to the shareholders when the surplus exceeds 12.35. However, with a ruin probability target, say  $\epsilon_3 = 0.0512$ , the optimal barrier is 16.00 instead of 12.35. This increases the optimal barrier, thereby reducing the chances of the company undergoing ruin. This applies to all other models considered in this section (see Figures 1 and 2).

### 3.9 Ruin probability targets for the diffusion-perturbed model compounded by excess-of-loss reinsurance: Pareto claims

The kernel and forcing function for this case are given by (2.12) (Theorem 2.2) when  $k = 1$  and  $c^{\bar{R}}$  is as defined in Section 3.4. The infinite ruin probabilities for the DPM for Pareto(3,2) claim sizes are given in Table 6.

Table 6: Ultimate ruin probabilities in the DPM for Par(3,2) claims,  
 $c = 6, \lambda = 2, \sigma = 1.0$

| $u$       | 0      | 8      | 16     | 24     | 32     | 34     | 36     | 38     | 40     |
|-----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\psi(u)$ | 1.0000 | 0.0273 | 0.0080 | 0.0036 | 0.0020 | 0.0018 | 0.0016 | 0.0014 | 0.0012 |

Choosing arbitrarily from Table 6, we have

$$\begin{aligned} \psi(b_0^1) &= \epsilon_1 = 0.0020, \text{ giving } b_0^1 = 32.00 \\ \psi(b_0^2) &= \epsilon_2 = 0.0016, \text{ giving } b_0^2 = 36.00 \\ \psi(b_0^3) &= \epsilon_3 = 0.0014, \text{ giving } b_0^3 = 38.00 \\ \psi(b_0^4) &= \epsilon_4 = 0.0012, \text{ giving } b_0^4 = 40.00 \end{aligned}$$

### 3.10 Dividends for the diffusion-perturbed model compounded by excess-of-loss reinsurance: Pareto claims

By Theorem 2.3, the kernel and forcing function are given by (2.23) for  $k = 1$  and  $c^{\bar{R}}$  as defined in Section 3.4. We use the same analysis and discussion of results as in Sections 3.1 and 3.2. We find the optimal barrier without a ruin probability target to be  $b^* = 11.50$ . However, under ruin probability targets, the optimal barriers for Pareto claims are obtained using Theorem 2.5 and given in Figure 3 for selected ruin probabilities.

## 4 Conclusion

The study has shown that as the ruin probability reduces the optimal dividend barrier to use for payment of dividends increases, and vice versa. Therefore, the use of ruin probability

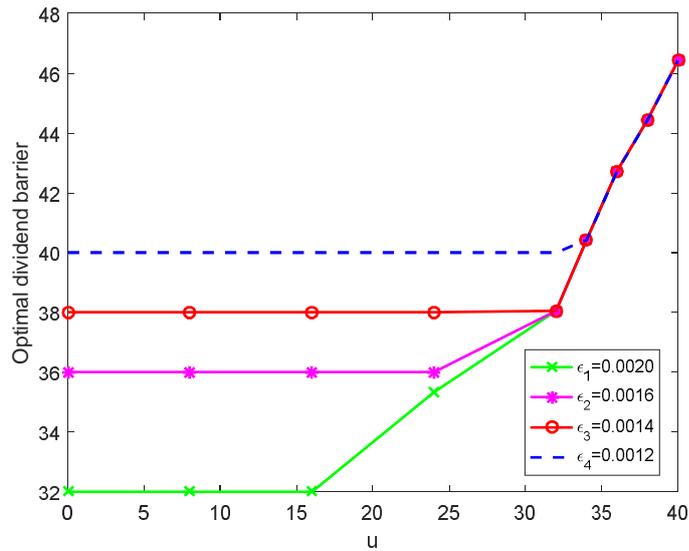


Figure 3: Numerical optimal barriers in DPM for Par(3,2) claims,  
 $c = 6$ ,  $\lambda = 2$ ,  $\delta = 0.1$ ,  $\sigma = 1.0$

targets by the insurance company is highly desirable from the shareholders' point of view. This is because a lower ruin probability makes it possible for the insurance company to pay more in dividends to the shareholders (that is, to use a higher dividend value function or optimal barrier).

The study has established that the reinsurance strategies are no different than before imposing ruin probability targets but that the optimal dividend barriers to use for dividend maximization increase as the ruin probability reduces. Insurance companies should therefore work towards reducing their ruin probabilities using some risk measures as this has a desirable effect on the optimal dividend barriers to be used for dividend payouts.

This work could be extended by: (1) including investments; (2) incorporating transaction costs when paying dividends; (3) exploring optimality of other dividend strategies (e.g., threshold or band); and (4) replacing the claim number process  $N$  by a general renewal process so that the surplus process becomes a Sparre-Andersen model.

## Acknowledgements

This work was supported by Mulungushi University, the Nelson Mandela African Institution of Science and Technology and the Zambian Ministry of Higher Education through the Support to Science and Technology Education Project (SSTEP) funded by the African Development Bank Group. The authors thank the anonymous referees for their valuable comments.

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**Output 4: Poster presentation on minimizing ruin probabilities by reinsurance**

# Minimizing the Ruin Probability in a Diffusion-Perturbed Model with Proportional and Excess-of-loss Reinsurance

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## Abstract

We consider a diffusion-perturbed risk model compounded by proportional and excess-of-loss (XL) reinsurance. Using the HJB approach, we derive a second-order Volterra integro-differential equation (VIDE) which we transform into a linear Volterra integral equation (VIE) of the second kind. We then proceed to solve this linear VIE numerically using the block-by-block method for the optimal reinsurance policy that minimizes the ultimate ruin probability for the chosen parameters which are commonly used in the insurance literature. Numerical examples with exponential and Pareto claims are given. Results show that reinsurance causes ruin probabilities to reduce significantly for both exponential and Pareto claims but the reduction is faster for Pareto claims.

## Model

### 1.1 Diffusion-perturbed classical risk process

$$U_t = u + ct + \sigma W_t - \sum_{i=1}^{N_t} X_i, t \geq 0$$

where:  
 $u$ : initial capital  
 $c$ : premium rate  
 $X_i$ : iid claims  
 $N_t$ : Poisson process with intensity  $\lambda$ , independent of  $X_i$   
 $W_t$ : 1-dimensional standard Brownian motion  
 $\sigma$ : volatility or diffusion coefficient

### 1.2 Reinsurance

**Definition:** Insurance of insurance companies

- Decision of how much reinsurance to buy depends on:
- Retention level  $k \in [0, 1]$  for proportional reinsurance and  $\alpha \in [0, \infty)$  for XL reinsurance
  - Premium rate  $c^R$  retained by the insurer if proportional and XL reinsurance are applied

### 1.3 Risk process with reinsurance

$$U_t^{\bar{R}} = u + c^R t + \sigma W_t - \sum_{i=1}^{N_t} (kX_i \wedge a)$$

where  $(\bar{R}) = (k, a)$  is the reinsurance strategy

## Analysis

### 2.1 Return and value functions

**Return function** – The ruin (or survival) probability  
**Value function** – The infimum (or supremum) of the return function

$$\psi(u) = \inf_{R \in \bar{R}} \psi^R(u)$$

Value function
Return function

### 2.2 HJB and integro-differential equations

**HJB equation:**  $\sup_R L(\phi)(u) = 0$

where  $L(\phi)(u) = \frac{1}{2}\sigma^2\phi''(u) + c^R\phi'(u) - \lambda\phi(u) + \lambda \int_0^u \phi(u - kx \wedge a) dF(x)$

**Integro-differential equation:**

$$\frac{1}{2}\sigma^2\phi''(u) + c^R\phi'(u) - \lambda\phi(u) + \lambda \int_0^u \phi(u - kx \wedge a) dF(x) = 0$$

where  $\phi(u) = 1 - \psi(u)$  is the survival probability

### 2.3 Integral equation

$$\phi(u) + \int_0^u K(u, x)\phi(x)dx = h(u)$$

where:

$$K(u, x) = -\frac{\lambda H_1(x, u)}{c^R}, h(u) = 0 \text{ (when } \sigma^2 = 0)$$

and

$$K(u, x) = -2[c^R - \lambda(u - kx) + \lambda H_2(x, u)]$$

$$h(u) = u\phi'(0) \text{ (when } \sigma^2 > 0)$$

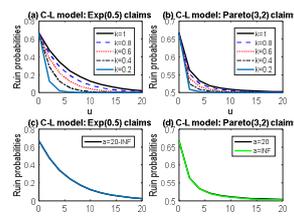
$$H_1(x, u) = \begin{cases} F(u - kx), & x < a \\ 1 - [F(kx + a) - F(a)], & x \geq a \end{cases}$$

$$H_2(x, u) = \begin{cases} G(u - kx), & x < a \\ [F(kx + a) - F(a)](u - kx), & x \geq a \end{cases}$$

Solution of this integral equation is obtained using the block-by-block numerical method

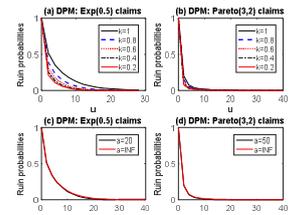
## Results

### 3.1 Cramér-Lundberg model



**Figure 1:** Ruin probabilities with proportional reinsurance ((a) and (b)) and XL reinsurance ((c) and (d)) in the C-L model

### 3.2 Diffusion-perturbed model



**Figure 2:** Ruin probabilities with proportional reinsurance ((a) and (b)) and XL reinsurance ((c) and (d)) in the DP model

## Discussion

From the results obtained, it was evident that reinsurance does result in a reduction in the probability of ultimate ruin, the reduction being more drastic for Pareto than for exponential claims in both the C-L and DP models.

## Conclusions

1. Reinsurance is important to insurance companies as it generally reduces their ruin probabilities.
2. Proportional reinsurance always leads to lower ruin probabilities than XL reinsurance for both the C-L and DP models. Therefore, a pure proportional reinsurance is optimal.

## Acknowledgements

Special gratitude is due to the Zambian Government and the Ministry of Higher Education for financial support through the AIDB-supported SSTEP project, as well as to NM-AIST and Mulungushi University.

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