

**MODELLING THE IMPACTS OF ANTHROPOGENIC ACTIVITIES  
ON FOREST BIOMASS AND FOREST DEPENDENT WILDLIFE  
POPULATION**

**Ibrahim M. Fanuel**

**A Thesis Submitted in Fulfilment of the Requirements for the Degree of PhD in  
Mathematical and Computer Sciences and Engineering of the Nelson Mandela African  
Institution of Science and Technology**

**Arusha, Tanzania**

**May, 2024**

## ABSTRACT

The depletion of forest biomass and declining of forest-dependent wildlife populations are urgent ecological and societal issues resulting from human activities such as deforestation and land-use changes. This study aims to comprehend the influence of anthropogenic activities on forest biomass and the population of wildlife dependent on forests, and to formulate appropriate management measures. Specific objectives include forecasting forest land loss in Tanzania, analysing mathematical model describing the impact of human activities on forests and wildlife, examining the influence of fuzzy parameters on model dynamics, and evaluating the effects of economic measures and technological efforts on conservation. The study considered Tanzania where local communities heavily rely on forest resources for their livelihoods. This region also supports a rich biodiversity of wildlife species, where the forest provides essential habitats and resources for their survival. Furthermore, the study also acknowledged the existence of various human activities carried out in the area which may have a significant impact on the forest ecosystem and the wildlife populations that depend on it. Four models are presented: a time series model and three dynamical system models. The key findings include: (i) The univariate time series model accurately predicts an increase in forest land loss in Tanzania with a 96.2% accuracy rate ( $MAPE = 0.0377$ ), highlighting the urgency for sustainable forest management practices and conservation policies. (ii) Depletion of forest biomass due to human activities has severe implications for wildlife survival and ecological balance. Achieving this balance requires ensuring the growth rate of forest biomass and wildlife populations exceeds their rates of utilisation and depletion, respectively. (iii) Incorporating fuzzy parameters improves model reliability by accounting for uncertainties in climate, geography, and human activities, enhancing decision-making processes. (iv) Economic measures and technological efforts have the potential to conserve forest biomass and wildlife populations. However, careful implementation and comprehensive understanding of forest ecosystem dynamics are crucial to prevent destabilisation. The study emphasises the need for interdisciplinary collaboration and stakeholders engagement to ensure sustainable forest use and conservation, considering the complexities and uncertainties of natural systems. Balancing forest conservation with socio-economic needs is key for the well-being of local communities and future generations.

## DECLARATION

I, **Ibrahim M. Fanuel**, hereby declare to the Senate of Nelson Mandela African Institution of Science and Technology that this thesis is my own original work. I confirm that it has neither been submitted nor is currently being concurrently submitted for a degree award in any other institution.

Ibrahim M. Fanuel,



**Name and signature of candidate**

24.05.2024

**Date**

The above declaration is confirmed

Dr. Francis Moyo,



**Name and signature of supervisor 1**

24.05.2024

**Date**

Dr. Silas Mirau,

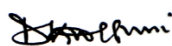


**Name and signature of supervisor 2**

24.05.2024

**Date**

Dr. Damian Kajunguri,



**Name and signature of supervisor 3**

24.05.2024

**Date**

## **COPYRIGHT**

This thesis is copyright material protected under the Berne Convention, the Copyright Act of 1999 and other international and national enactments, in that behalf, on intellectual property. It must not be reproduced by any means, in full or in part, except for short extracts in fair dealing; for researcher private study, critical scholarly review or discourse with an acknowledgement, without the written permission of the office of Deputy Vice-Chancellor for Academic, Research and Innovation on behalf of both the author and the NM-AIST.

## CERTIFICATION

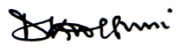
The undersigned hereby certify that they have thoroughly read the thesis and find it to be in a form that meets the requirements for examination and evaluation.

Dr. Francis Moyo,   
\_\_\_\_\_  
**Name and signature of supervisor 1**

24.05.2024  
\_\_\_\_\_  
**Date**

Dr. Silas Mirau,   
\_\_\_\_\_  
**Name and signature of supervisor 2**

24.05.2024  
\_\_\_\_\_  
**Date**

Dr. Damian Kajunguri,   
\_\_\_\_\_  
**Name and signature of supervisor 3**

24.05.2024  
\_\_\_\_\_  
**Date**

## **ACKNOWLEDGEMENTS**

I would like to express my heartfelt gratitude to all those who have contributed to the successful completion of my doctoral journey and the writing of this thesis.

First and foremost, I would like to express my heartfelt gratitude to God Almighty for His blessings, guidance, and unwavering presence throughout this entire journey. His grace and divine wisdom have been the source of strength and inspiration that sustained me during challenging times and moments of doubt. I am deeply grateful for His countless blessings and for guiding me towards this achievement. Without His divine guidance and support, this accomplishment would not have been possible. I humbly acknowledge and attribute all my successes to His boundless grace and mercy.

I am deeply indebted to my supervisors Dr. Francis Moyo, Dr. Silas Mirau and Dr. Damian Kajunguri, whose unwavering support, guidance, and encouragement have been invaluable throughout this research endeavour. Your expertise, patience, and insightful feedback have played a pivotal role in shaping the direction of this work.

I am also grateful to my family for their love, encouragement, and unwavering belief in my abilities. Their continuous support and understanding have been a source of strength and motivation, especially during challenging times.

I would like to extend my special thanks to College of Business Education for granting me study leave to pursue my PhD. Their unwavering support and belief in the value of higher education allowed me the time and resources to dedicate myself fully to this academic pursuit. Their support has been instrumental in making this journey possible and has greatly contributed to the successful completion of my PhD thesis. The financial support provided enabled me to focus on my research without financial worries.

In conclusion, the completion of this thesis was made possible through the unwavering support, guidance, and inspiration provided by the individuals and organizations acknowledged earlier. However, I hereby affirm that I am solely responsible and accountable for any deficiencies in this thesis.

## TABLE OF CONTENTS

|                                          |              |
|------------------------------------------|--------------|
| <b>ABSTRACT</b>                          | <b>i</b>     |
| <b>DECLARATION</b>                       | <b>ii</b>    |
| <b>COPYRIGHT</b>                         | <b>iii</b>   |
| <b>CERTIFICATION</b>                     | <b>iv</b>    |
| <b>ACKNOWLEDGEMENTS</b>                  | <b>v</b>     |
| <b>TABLE OF CONTENTS</b>                 | <b>vi</b>    |
| <b>LIST OF TABLES</b>                    | <b>xi</b>    |
| <b>LIST OF FIGURES</b>                   | <b>xii</b>   |
| <b>LIST OF APPENDICES</b>                | <b>xvii</b>  |
| <b>LIST OF ABBREVIATIONS AND SYMBOLS</b> | <b>xviii</b> |
| <b>CHAPTER ONE</b>                       | <b>1</b>     |
| <b>INTRODUCTION</b>                      | <b>1</b>     |
| 1.1 Background of the Problem            | 1            |
| 1.2 Statement of the Problem             | 5            |
| 1.3 Rationale of the Study               | 6            |
| 1.4 Research Objective                   | 7            |
| 1.4.1 General objective                  | 7            |
| 1.4.2 Specific objectives                | 7            |
| 1.5 Research Questions                   | 7            |
| 1.6 Significance of the Study            | 8            |
| 1.7 Delineation of the Study             | 9            |
| <b>CHAPTER TWO</b>                       | <b>10</b>    |

|                                                                                                                                       |           |
|---------------------------------------------------------------------------------------------------------------------------------------|-----------|
| <b>LITERATURE REVIEW . . . . .</b>                                                                                                    | <b>10</b> |
| 2.1 Introduction . . . . .                                                                                                            | 10        |
| 2.2 Human-Forest Interactions Dynamics . . . . .                                                                                      | 10        |
| 2.3 Human-Environment Systems Theory (HEST) . . . . .                                                                                 | 11        |
| 2.4 Mathematical Modelling . . . . .                                                                                                  | 12        |
| 2.5 Conclusion and Research Agenda . . . . .                                                                                          | 17        |
| <b>CHAPTER THREE . . . . .</b>                                                                                                        | <b>18</b> |
| <b>MATERIALS AND METHODS . . . . .</b>                                                                                                | <b>18</b> |
| 3.1 Introduction . . . . .                                                                                                            | 18        |
| 3.2 Univariate Time Series Model . . . . .                                                                                            | 18        |
| 3.2.1 Model identification . . . . .                                                                                                  | 19        |
| 3.2.2 Parameter estimation . . . . .                                                                                                  | 22        |
| 3.2.3 Model validation . . . . .                                                                                                      | 24        |
| 3.3 Dynamical System Models . . . . .                                                                                                 | 24        |
| 3.3.1 Modelling Depletion of Forest Biomass and Wildlife Population in a<br>Crisp Environment . . . . .                               | 25        |
| 3.3.2 Formulation of fuzzy parameter based model on depletion of forest<br>biomass and forest-dependent wildlife population . . . . . | 27        |
| 3.3.3 Formulation of forest biomass and forest-dependent wildlife population<br>conservation model . . . . .                          | 30        |
| 3.3.4 Equilibrium and stability analysis . . . . .                                                                                    | 32        |
| 3.3.5 The Volterra-Lyapunov matrix theory for global stability analysis . . . . .                                                     | 33        |
| 3.3.6 System permanence . . . . .                                                                                                     | 35        |

|                                         |                                                                                                                           |           |
|-----------------------------------------|---------------------------------------------------------------------------------------------------------------------------|-----------|
| 3.4                                     | Parameter Estimation . . . . .                                                                                            | 36        |
| 3.4.1                                   | Least Squares . . . . .                                                                                                   | 36        |
| 3.4.2                                   | Markov Chain Monte Carlo Method . . . . .                                                                                 | 37        |
| 3.5                                     | Uncertainty and Sensitivity Analysis of the Input Model Parameters . . . . .                                              | 39        |
| 3.5.1                                   | Latin Hypercube Sampling and Partial Rank Correlation Coefficients . . .                                                  | 40        |
| 3.6                                     | Numerical Simulation . . . . .                                                                                            | 42        |
| <b>CHAPTER FOUR . . . . .</b>           |                                                                                                                           | <b>44</b> |
| <b>RESULTS AND DISCUSSION . . . . .</b> |                                                                                                                           | <b>44</b> |
| 4.1                                     | ARIMA model for Forest Loss Forecasting in Tanzania . . . . .                                                             | 44        |
| 4.1.1                                   | Data exploration . . . . .                                                                                                | 44        |
| 4.1.2                                   | Model identification . . . . .                                                                                            | 46        |
| 4.1.3                                   | Parameter estimation for ARIMA(1,2,0) and ARIMA(1,2,1) models . . .                                                       | 48        |
| 4.1.4                                   | Model diagnostic and forecast . . . . .                                                                                   | 50        |
| 4.2                                     | Mathematical Modelling of Forest Biomass Depletion and its impact on<br>Wildlife Dependent on Forest Ecosystems . . . . . | 52        |
| 4.2.1                                   | Basic properties of the model . . . . .                                                                                   | 55        |
| 4.2.2                                   | System permanence . . . . .                                                                                               | 57        |
| 4.2.3                                   | Equilibrium analysis . . . . .                                                                                            | 60        |
| 4.2.4                                   | Stability analysis . . . . .                                                                                              | 68        |
| 4.2.5                                   | Existence of trans-critical bifurcation . . . . .                                                                         | 76        |
| 4.2.6                                   | The Volterra-Lyapunov matrix theory for global stability analysis of in-<br>terior equilibrium point $E^*$ . . . . .      | 79        |
| 4.2.7                                   | Markov Chain Monte Carlo (MCMC) sampling for parameter identifiability                                                    | 85        |

|                                       |                                                                                                      |            |
|---------------------------------------|------------------------------------------------------------------------------------------------------|------------|
| 4.2.8                                 | Numerical simulation . . . . .                                                                       | 92         |
| 4.3                                   | Fuzzy Modelling on Depletion of Forest Biomass and Forest-Dependent<br>Wildlife Population . . . . . | 100        |
| 4.3.1                                 | Boundedness and persistence of the solutions . . . . .                                               | 104        |
| 4.3.2                                 | Equilibrium analysis . . . . .                                                                       | 105        |
| 4.3.3                                 | Stability analysis . . . . .                                                                         | 106        |
| 4.3.4                                 | Numerical simulation . . . . .                                                                       | 108        |
| 4.4                                   | Forest Biomass and Forest-Dependent Wildlife Conservation Model . . . . .                            | 111        |
| 4.4.1                                 | Economic and technological efforts for resources conservation . . . . .                              | 111        |
| 4.4.2                                 | Model formulation . . . . .                                                                          | 112        |
| 4.4.3                                 | Basic properties of the model . . . . .                                                              | 115        |
| 4.4.4                                 | Equilibrium analysis . . . . .                                                                       | 116        |
| 4.4.5                                 | Physical interpretation of the equilibrium points . . . . .                                          | 120        |
| 4.4.6                                 | Stability analysis . . . . .                                                                         | 121        |
| 4.4.7                                 | Parameter estimation and model fitting . . . . .                                                     | 127        |
| 4.4.8                                 | Uncertainty and Sensitivity (US) analysis of the input parameters . . . .                            | 128        |
| 4.4.9                                 | Numerical simulation . . . . .                                                                       | 136        |
| <b>CHAPTER FIVE</b>                   | <b>. . . . .</b>                                                                                     | <b>144</b> |
| <b>CONCLUSION AND RECOMMENDATIONS</b> | <b>. . . . .</b>                                                                                     | <b>144</b> |
| 5.1                                   | Conclusion . . . . .                                                                                 | 144        |
| 5.2                                   | Recommendations . . . . .                                                                            | 146        |
| <b>LIST OF REFERENCES</b>             | <b>. . . . .</b>                                                                                     | <b>149</b> |
| <b>APPENDICES</b>                     | <b>. . . . .</b>                                                                                     | <b>160</b> |

|                         |             |
|-------------------------|-------------|
| <b>RESEARCH OUTPUTS</b> | <b>.171</b> |
|-------------------------|-------------|

## LIST OF TABLES

|           |                                                                              |     |
|-----------|------------------------------------------------------------------------------|-----|
| Table 1:  | General inferences from theoretical ACFs and PACFs . . . . .                 | 22  |
| Table 2:  | Model variables and description . . . . .                                    | 25  |
| Table 3:  | Model parameter symbol and description . . . . .                             | 26  |
| Table 4:  | Description of fuzzy parameters . . . . .                                    | 28  |
| Table 5:  | Model Parameter and description . . . . .                                    | 31  |
| Table 6:  | General system stability inferences from eigenvalues . . . . .               | 33  |
| Table 7:  | Summary of <code>auto.arima</code> stepwise search to minimize AIC . . . . . | 47  |
| Table 8:  | ARIMA(1,2,0) model results . . . . .                                         | 48  |
| Table 9:  | ARIMA(1,2,1) model results . . . . .                                         | 49  |
| Table 10: | Model performance measures . . . . .                                         | 50  |
| Table 11: | Model parameter, values and source . . . . .                                 | 85  |
| Table 12: | Statistics of MCMC (Number of simulations = 100 000) . . . . .               | 86  |
| Table 13: | Estimated parameter values . . . . .                                         | 127 |
| Table 14: | Mean values and ranges of parameters values used in LHS . . . . .            | 132 |
| Table 15: | Pairwise PRCC comparisons (Unadjusted p-values) . . . . .                    | 135 |
| Table 16: | Pairwise PRCC comparisons (FDR Adjusted p-values) . . . . .                  | 135 |
| Table 17: | Parameters different after FDR adjustment? . . . . .                         | 136 |

## LIST OF FIGURES

|            |                                                                                                                                                                                              |    |
|------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----|
| Figure 1:  | Annual deforestation in ha (a) Europe, (b) Asia (c) Africa (d) Tanzania, from 1990 to 2020. . . . .                                                                                          | 2  |
| Figure 2:  | Global expansion of agricultural lands into forested areas (Source: Ritchie and Roser [101]) . . . . .                                                                                       | 2  |
| Figure 3:  | Human population including UN projections 1950 to 2100. The projections are based on the UN's medium-fertility scenario (Source: United Nations, World Population Prospects [116]) . . . . . | 3  |
| Figure 4:  | (a) Triangular fuzzy number (b) The $\alpha$ -cut of a Triangular fuzzy number . . .                                                                                                         | 29 |
| Figure 5:  | Algorithm for proving Volterra–Lyapunov stability of $4 \times 4$ matrix A . . . .                                                                                                           | 35 |
| Figure 6:  | Probability distributions for input parameters $\theta$ and $\vartheta$ . . . . .                                                                                                            | 42 |
| Figure 7:  | (a) Plot showing the forest loss in hectares (ha) from 1991 to 2021 with trend line. (b) Moving average plot of the forest loss data over the same interval . . . . .                        | 44 |
| Figure 8:  | Lag scatter plot showing strong correlation among the lags . . . . .                                                                                                                         | 45 |
| Figure 9:  | The order of differencing with respective ACF plots . . . . .                                                                                                                                | 46 |
| Figure 10: | Second order differencing with respective ACF and PACF plots . . . . .                                                                                                                       | 47 |
| Figure 11: | Residuals plots for ARIMA(1,2,0) with respectively probability distribution with standardised scale which shows the residual are normally distributed with mean 0 . . . . .                  | 49 |
| Figure 12: | Residuals plots for ARIMA(1,2,1) with respectively probability distribution with standardised scale which shows the residual are normally distributed with mean 0 . . . . .                  | 49 |
| Figure 13: | Forest land loss prediction: Comparison of ARIMA(1,2,0) and ARIMA(1,2,1)                                                                                                                     | 50 |
| Figure 14: | Residuals plots, from top left corner is standardise residuals, histogram and estimated density, qq-plot and autocorrelation . . . . .                                                       | 51 |

|            |                                                                                                                                                                                                                                                          |    |
|------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----|
| Figure 15: | Time series forecast plot using ARIMA(1,2,0), the shaded regions signify the confidence interval . . . . .                                                                                                                                               | 52 |
| Figure 16: | Schematic diagram for model system (15) . . . . .                                                                                                                                                                                                        | 53 |
| Figure 17: | Invariance in (a) $B - W$ , (b) $N - H$ , (c) $B - N$ and (d) $B - H$ planes . . .                                                                                                                                                                       | 56 |
| Figure 18: | The isoclines (42 and 43) plot shows an intersection at a unique point $(B^*, W^*)$ in the interior of the first quadrant . . . . .                                                                                                                      | 66 |
| Figure 19: | Geweke's diagnostics. The first row: $L$ , $M$ , $s$ and $\beta_0$ , the second row: $\beta_1$ , $\lambda$ , $\sigma$ and $K_0$ , the third row: $K_1$ , $\theta$ , $\beta_2$ and $\gamma_1$ , fourth row: $\gamma$ , $\eta$ , $\nu_1$ and $\nu_2$ . . . | 88 |
| Figure 20: | (a) Trends of MCMC samples demonstrating the quality of the mixing (b) The sampled parameters' marginal density distribution . . . . .                                                                                                                   | 89 |
| Figure 21: | The scatter plots show cross-correlations and histogram on diagonal of the combined chains for the first six parameters to visualize the 1-D and 2-D marginal posterior distributions . . . . .                                                          | 90 |
| Figure 22: | Sample autocorrelation plots for the 16 MCMC sample parameters. The ACF coefficients are represented on the $x$ axis, while lags are represented on the $y$ axis . . . . .                                                                               | 91 |
| Figure 23: | Histograms illustrating the probability density function estimation for model parameters, showcasing Gaussian distributions indicative of effective mixing . . . . .                                                                                     | 91 |
| Figure 24: | Solutions of the model system (15) using the parameter values in Table 12; (a) Forest biomass, (b) Wildlife population, (c) Human population and (d) Human activities . . . . .                                                                          | 92 |
| Figure 25: | Non-linear stability of the equilibrium point $E^*$ in (a) $B - W$ plane, (b) $B - W - N$ plane . . . . .                                                                                                                                                | 93 |
| Figure 26: | Trans-critical bifurcation between $E_3$ and $E^*$ with respect to the depletion rate of forest biomass due to human population ( $\beta_1$ ), other parameter values were kept fixed as in Table 12 and taking $s = 0.8$ . . . . .                      | 94 |

|            |                                                                                                                                                                                                                                                                   |     |
|------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| Figure 27: | Variations of $B, W, N$ , and $H$ with respect to time ( $t$ ) for different values of $\beta_1$ . . . . .                                                                                                                                                        | 95  |
| Figure 28: | Variations of $B, W, N$ , and $H$ with respect to time ( $t$ ) for different values of $\beta_2$ . . . . .                                                                                                                                                        | 96  |
| Figure 29: | Variations of $B, W, N$ , and $H$ with respect to time ( $t$ ) for different values of $\gamma$                                                                                                                                                                   | 96  |
| Figure 30: | Variations of $B, W, N$ , and $H$ with respect to time ( $t$ ) for different values of $\nu_2$                                                                                                                                                                    | 97  |
| Figure 31: | Evolution of solution trajectories of (a) forest biomass ( $B$ ), (b) forest-dependent wildlife population ( $W$ ), (c) human population ( $N$ ) and (d) human activities ( $H$ ) with different values of $\beta_2$ , other parameters were kept fixed . . . . . | 98  |
| Figure 32: | Evolution of solution trajectories of (a) forest biomass ( $B$ ), (b) forest-dependent wildlife population ( $W$ ), (c) human population ( $N$ ) and (d) human activities ( $H$ ) with different values of $\gamma$ , other parameters were kept fixed . . . . .  | 99  |
| Figure 33: | Evolution of solution trajectories of (a) forest biomass ( $B$ ), (b) forest-dependent wildlife population ( $W$ ), (c) human population ( $N$ ) and (d) human activities ( $H$ ) with different values of $\nu_2$ , other parameters were kept fixed . . . . .   | 99  |
| Figure 34: | Evolution of the dynamics of (a) forest biomass, (b) wildlife population, (c) human population and (d) human activities with time in crisp and fuzzy environments with the same initial condition . . . . .                                                       | 109 |
| Figure 35: | Non-linear stability of the interior equilibrium point $E_*$ in (a) $B - W - N$ and (b) $B - W - H$ planes . . . . .                                                                                                                                              | 110 |
| Figure 36: | Forest biomass and forest-dependent wildlife conservation model flow diagram                                                                                                                                                                                      | 112 |
| Figure 37: | Model fitting (lines) corresponding to estimated parameters for (a) Forest biomass (b) Wildlife population (c) Human population (d) population pressure (e) Human activities (f) Technological efforts and (g) Economic efforts                                   | 128 |

|            |                                                                                                                                                                                                                                                                                                                                                                                                            |      |
|------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------|
| Figure 38: | Evolution of model solutions using literature values (solid red lines) versus estimated values (dash black lines) for (a) Forest biomass (b) Wildlife population (c) Technological efforts (d) Human population (e) Population pressure, (f) Human activities and (g) Economic efforts . . . . .                                                                                                           | .129 |
| Figure 39: | The sample autocorrelation of the residuals for the model outputs arranged from the top left corner as follows: forest biomass, wildlife population, human population, population pressure, human activities, technological efforts, and economic efforts . . . . .                                                                                                                                        | .130 |
| Figure 40: | Probability distribution functions (pdfs) of the twenty one sampled input parameter values obtained from LHS using a sample size of 200 . . . . .                                                                                                                                                                                                                                                          | .131 |
| Figure 41: | PRCC of the model parameters with respect to (a) Forest biomass ( $B(t)$ ), (b) Forest-dependent wildlife population ( $W(t)$ ), (c) Population pressure ( $P(t)$ ) and (d) Human activities ( $H(t)$ ) plotted over a time course (sample size =200) . . . . .                                                                                                                                            | .132 |
| Figure 42: | PRCC of the model parameters with respect to (a) Forest biomass ( $B(t)$ ) and (b) Forest-dependent wildlife population ( $W(t)$ ) at time point $t = 8$ . .                                                                                                                                                                                                                                               | .133 |
| Figure 43: | PRCC of the model parameter with respect to (a) Forest biomass ( $B(t)$ ) and (b) Forest-dependent wildlife population ( $W(t)$ ) at time point $t = 80$ .                                                                                                                                                                                                                                                 | .134 |
| Figure 44: | PRCC of the model parameters with respect to (a) Population pressure ( $P(t)$ ) and (b) Human activities ( $H(t)$ ) at time point $t = 8$ . . . . .                                                                                                                                                                                                                                                        | .134 |
| Figure 45: | PRCC of the model parameters with respect to (a) Population pressure ( $P(t)$ ) and (b) Human activities ( $H(t)$ ) at time point $t = 80$ . . . . .                                                                                                                                                                                                                                                       | .135 |
| Figure 46: | Temporal evolution of: (a) Forest biomass, (b) Wildlife population in forest habitats, (c) Human population, (d) Population pressure, (e) Human activities, (f) Economic efforts, and (g) Technological efforts. Forest biomass, wildlife population and Technological efforts were simulated for the period of 50 years, whereas remaining variables were simulated for the period of 100 years . . . . . | .137 |

|            |                                                                                                                                                                                                                                                                                                                         |      |
|------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------|
| Figure 47: | Non-linear stability of the equilibrium point $\bar{E}$ in $B - W$ plane . . . . .                                                                                                                                                                                                                                      | .138 |
| Figure 48: | Non-linear stability of the equilibrium point $\bar{E}$ in (a) $P - H - E$ and (b) $N - P - E$ planes . . . . .                                                                                                                                                                                                         | .138 |
| Figure 49: | Variation of forest biomass ( $B$ ), forest-dependent wildlife population ( $W$ ), human population ( $N$ ) and human activities ( $H$ ) with time ( $t$ ) for different values of $\beta_1$ and $\beta_2$ . . . . .                                                                                                    | .139 |
| Figure 50: | Variation of forest biomass ( $B$ ), wildlife population ( $W$ ), population pressure ( $P$ ) and human activities ( $H$ ) with time ( $t$ ) for different values of $\phi$ and $\phi_2$ . . . . .                                                                                                                      | .140 |
| Figure 51: | Variation of forest biomass ( $B$ ), wildlife population ( $W$ ), population pressure ( $P$ ) and human activities ( $H$ ) with time ( $t$ ) for different values of $\beta_2$ and $\gamma$ . . . . .                                                                                                                   | .141 |
| Figure 52: | Variation of forest biomass ( $B$ ), wildlife population ( $W$ ), population pressure ( $P$ ) and human activities ( $H$ ) with time ( $t$ ) for different values of $\rho$ and $\omega$ . . . . .                                                                                                                      | .142 |
| Figure 53: | Evolution of (a) Forest biomass, (b) Wildlife population in forest habitats, (c) Human population, (d) Population pressure, (e) Human activities, (f) Technological efforts, and (g) Economic efforts for different values of $\rho$ and $\omega$ . The remaining parameter values were kept fixed as in Table 13 . . . | .143 |

## LIST OF APPENDICES

|                                                            |     |
|------------------------------------------------------------|-----|
| Appendix 1: Python code for Time Series Analysis . . . . . | 161 |
| Appendix 2: MATLAB code for Bifurcation Analysis .. . . .  | 169 |

## LIST OF ABBREVIATIONS AND SYMBOLS

|         |                                                                          |
|---------|--------------------------------------------------------------------------|
| ACF     | Autocorrelation Function                                                 |
| ADF     | Augmented Dickey–Fuller                                                  |
| AIC     | Akaike Information Criteria                                              |
| AMCSE   | Applied Mathematics and Computational Sciences and Engineering           |
| AR      | Autoregressive                                                           |
| ARIMA   | Autoregressive Integrated Moving Average                                 |
| BIC     | Bayesian Information Criteria                                            |
| CoCSE   | School of Computational and Communication Sciences and Engineering       |
| FAO     | Food and Agriculture Organization                                        |
| FDR     | False Discovery Rate                                                     |
| GPS     | Global Positioning System                                                |
| HEST    | Human–Environment Systems Theory                                         |
| iid     | Independent and Identically Distributed                                  |
| LHS     | Latin Hypercube Sampling                                                 |
| MA      | Moving Average                                                           |
| MAE     | Mean Absolute Error                                                      |
| MAPE    | Mean Absolute Percentage Error                                           |
| MCMC    | Markov Chain Monte Carlo                                                 |
| MLE     | Maximum Likelihood Estimation                                            |
| MSE     | Mean Square Error                                                        |
| NAFORMA | National Forest Resources Monitoring and Assessment of Tanzania Mainland |
| NM-AIST | Nelson Mandela African Institution of Science and Technology             |
| PACF    | Partial Autocorrelation Function                                         |
| pdf     | Probability Density Function                                             |
| PRCC    | Partial Rank Correlation Coefficient                                     |
| SOFO    | State of the World’s Forests                                             |
| SSE     | Error Sum of Squares                                                     |
| UNEP    | United Nations Environment Programme                                     |

# CHAPTER ONE

## INTRODUCTION

### 1.1 Background of the Problem

Forests are essential ecosystems that support a vast array of flora and fauna [26]. They provide habitats to 80-90% of the terrestrial biodiversity [79, 95]. The highest ratio of endemics per unit area of all biodiversity hotspots is found in the forests of the Eastern arc mountains of Tanzania and Kenya; the area covers 2000  $km^2$  and offer habitats to about 121 endemic vertebrate species [109, 84]. They also provide various ecosystem services, such as carbon sequestration, water regulation, and soil stabilisation [7]. To humans, forests are sources of wood, energy, medicine, and fodder [72]. The World Bank estimates that forests are the main source of medicine for about one billion individuals, with 25% percent of all cancer-fighting drugs coming from the rainforest [51, 114]. However, the resource is facing an increasing threats from anthropogenic activities such as logging, agriculture, urbanisation, mining, and other forms of land-use changes [73]. These activities are causing significant damage to the forest ecosystems and threaten the survival of many forest-dependent wildlife species.

The Forest Resources Assessment 2020 report reveals a concerning trend that primary forests worldwide have witnessed a substantial decline, decreasing by over 80 million hectares since 1990 [27]. Even more alarming is the estimated rate of deforestation on a global scale, which reached approximately 10 million hectares per year between 2015 and 2020 [27]. This alarming figure highlights the urgent need for conservation efforts to protect our valuable forest resources. The situation becomes even more critical when focusing on the African continent. As illustrated in Fig. 1, the deforestation rates in Africa are notably higher, presenting a greater threat to the region's forests and biodiversity.

The primary driver behind global deforestation and forest degradation continues to be agricultural expansion, as shown in Fig. 2. Between 2000 and 2010, a significant portion of forest loss (approximately 40%) was attributed to large-scale commercial agricultural activities, while local subsistence agriculture was responsible for 33% of the forest loss during the same period [27].

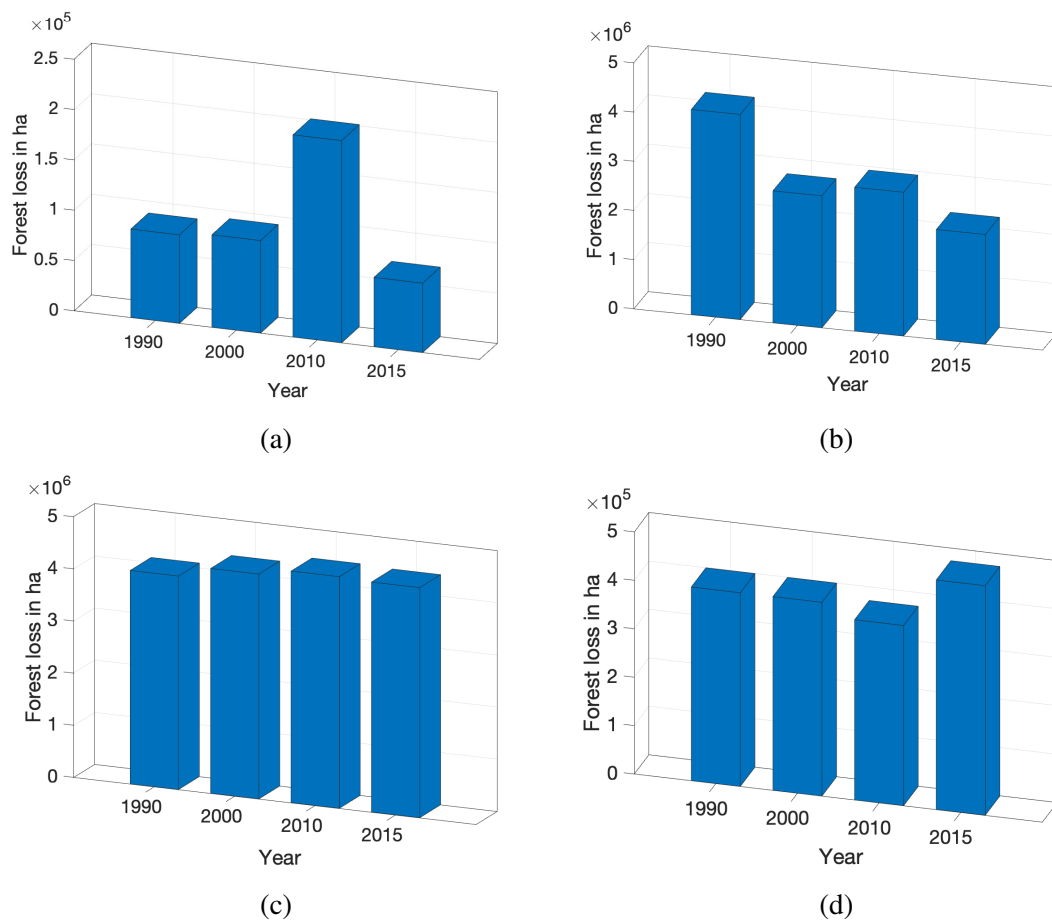


**Figure 1:** Annual deforestation in ha (a) Europe, (b) Asia (c) Africa (d) Tanzania from, 1990 to 2020 (Source: FAO-Forest Resources Assessment [26])

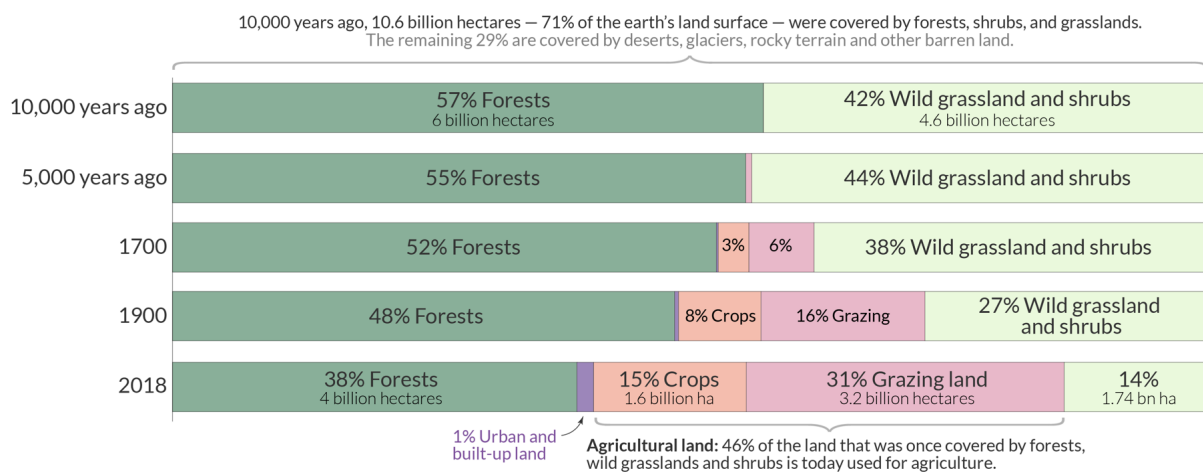


**Figure 2:** Global expansion of agricultural lands into forested areas (Source: Ritchie and Roser [101])

In developing countries, the issue of forest degradation and deforestation is often exacerbated by high levels of poverty and rapid population growth [47]. Poverty-stricken communities, struggling to meet their basic needs, frequently turn to natural resources such as forests for sustenance and livelihoods. The rapid increase in population further intensifies the demand for food, fuel, and other resources, putting additional pressure on forests. Figure 3 illustrates that population growth is not evenly distributed across the world. The latest projections indicate that Africa is expected to experience the highest rate of population growth compared to other regions [116]. This trend holds significant implications for forest conservation efforts on the continent. Without appropriate measures to address the underlying drivers of deforestation, such as poverty and population growth, the situation is likely to worsen, leading to irreversible damage to ecosystems, loss of biodiversity, and negative impacts on human well-being.

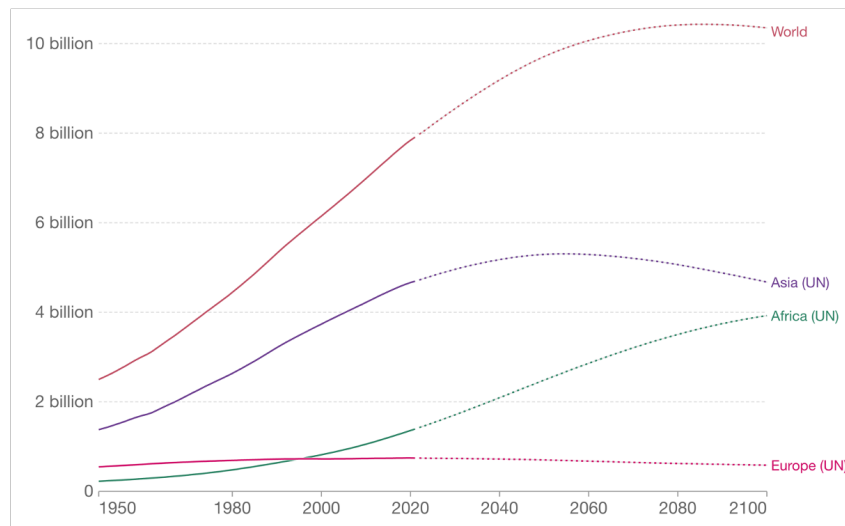


**Figure 3:** Human population including UN projections 1950 to 2100. The projections are based on the UN’s medium-fertility scenario (Source: United Nations, World Population Prospects [116])

Similarly, FAO [27] estimates that 90% of the people living in extreme poverty depend on forest for their livelihoods. Poor people often lack access to alternative sources of income and are therefore forced to rely on forest resources for their livelihoods [86]. Poverty can lead to the expansion of subsistence agriculture, which involves clearing land for crops and livestock, leading to further deforestation [27]. Furthermore, the lack of economic resources and opportunities can encourage migration to forested areas, resulting in increased demand for land and forest resources for settlement and agriculture [86]. Despite the decline of extreme poverty globally, yet, in Sub-Saharan Africa, the number of people living in extreme poverty

is rapidly increasing. The forecast shows that by 2030, 9 in 10 extremely poor people will be living in Sub-Saharan Africa [118], meaning increasing threats to Sub-Saharan Africa forests.

The economic and social burden of forest biomass loss is substantial and should not be underestimated. Deforestation and forest degradation result in a loss of revenue from timber and non-timber forest products, as well as decreased ecosystem services such as carbon sequestration and watershed protection. In 2019, the United Nations Development Programme (UNDP) estimates the annual cost of deforestation and forest degradation to be around USD 2 trillion to USD 4.5 trillion [120]. The loss of these services can lead to an increased risks of climate change that has been linked to numerous health challenges, including increased rates of infectious diseases, malnutrition, and respiratory illness, which in turn exacerbates poverty and inequality.

It is estimated that 31% of modern diseases are a result of deforestation [2]. In addition to the economic and social burden, the loss of forest biomass also has adverse effects on wildlife species, forests are home to a vast array of wildlife species, including many endangered and threatened species. The loss of forest biomass can lead to the fragmentation of ecosystems and the isolation of wildlife populations, leading to species extinction, decreased genetic diversity and reduced resilience to environmental change [13].

To address these issues, various conservation strategies have been implemented, including forest restoration, sustainable forest management, and wildlife protection [8]. However, accurately understanding the effectiveness of these strategies and predicting their impact on forest biomass and wildlife populations requires a comprehensive understanding of forest ecosystem dynamics [10].

Mathematical models, particularly systems of differential equations, have emerged as valuable tools for simulating and predicting the impact of human activities on forest ecosystems. These models offer insights into the intricate interplay between forest biomass and dependent wildlife populations, garnering significant interest among researchers in forest management. By leveraging mathematical models, policymakers and forest managers can make informed decisions to conserve forest ecosystems for future generations. Nevertheless, formulating mathematical models for complex ecological systems is very challenging, primarily due to the non-linear dynamics characterizing ecological interactions [29]. In such systems, multiple variables and

feedback loops can lead to unpredictable and emergent behaviours, making it difficult to develop precise models that fully capture the complexities of ecological processes.

The presence of uncertainty and variability in the data and model parameters further complicates the model development process [35]. Such uncertainties can arise from various sources, including measurement errors, sampling limitations, and incomplete information. Therefore, this study formulates mathematical models to investigate the depletion and conservation of forest biomass and forest-dependent wildlife populations in both crisp and fuzzy environments, effectively capturing the complexities and uncertainties inherent in ecological systems. By incorporating uncertainty and sensitivity analysis, and employing fuzzy logic, the study makes a significant contribution to advancing ecological modelling in dynamic and ever-changing environments. The results and insights gained from this research have the potential to inform policymakers and conservationists, enabling them to make well-informed and effective decisions to protect and preserve forest ecosystems and their dependent wildlife populations sustainably.

## **1.2 Statement of the Problem**

Forests play a crucial role in providing ecological, economic, and social benefits. However, anthropogenic activities such as deforestation, land-use changes, and fragmentation have resulted in the loss of forest biomass and declining populations of forest-dependent wildlife [73]. Significant challenges have emerged, particularly in developing countries, in achieving a balance between forest conservation and exploitation without losing ecological benefits [86]. There is a considerable number of theoretical and experimental studies in the pursuit of understanding the depletion and conservation of forest biomass as well as forest-dependent wildlife population. In the realm of theoretical studies which is the focus of this study, mathematical models have become important tools for analysing the impacts of human population and its associated activities on forest and forest-dependent wildlife population [38, 73]. However, despite the available literature, most of studies have assumed a crisp environment, wherein human-forest interactions are modelled under the assumption of certain parameter values. Nevertheless, in reality, biological parameters are not fixed but rather exhibit variability in response to the surrounding environment. Factors such as soil fertility, presence of pollinators, rainfall patterns, and other variables can significantly influence the growth rate of forests, which are inherently non-uniform. Furthermore, the existing literature lacks comprehensive exploration of the variability of model

parameters, necessitating the development of mathematical models that account for uncertainty and imprecision. Therefore, this study developed and analysed mathematical models in crisp and fuzzy environments to study the impact of anthropogenic activities on forest biomass and the populations of wildlife dependent on forests. Similarly, special attention is given to incorporating technological efforts and economic measures to conserve the resources. Furthermore, an extensive uncertainty and sensitivity analysis is conducted to identify and quantify sources of uncertainty stemming from model parameters.

### **1.3 Rationale of the Study**

The depletion of forests and the decline of forest-dependent wildlife populations are pressing issues that have significant ecological and societal implications. Forests offer important goods and services that enhance people's livelihoods and contribute to a country's economy [79]. Furthermore, these are vital ecosystems that provide a range of ecological services. However, the Global Forest Resources Assessment 2020 report by FAO [27] shows that the world lost 178 million hectares of forest between 1990 and 2020. On average, this represents a loss of about 4.4 million hectares per year over the last three decades. From Fig. 2 it is revealed that there is a continuous increase of human encroachment on forest lands. The persistent rise in this trend highlights the urgent need for sustainable forest management practices and effective conservation strategies to prevent further damage to these valuable ecosystems.

As human populations continue to expand and demand for resources grows, the pressure on forests and other natural habitats is expected to increase. This not only affects the biodiversity of these regions but also has a significant impact on the global climate. The conservation and sustainable management of forests align with several Sustainable Development Goals (SDGs) set by the United Nations. Specifically, SDG 15 (Life on Land) emphasizes the need to protect, restore, and promote sustainable use of terrestrial ecosystems, sustainably manage forests, combat desertification, halt and reverse land degradation, and halt biodiversity loss.

To ensure sustainable forest management, it is essential to study the different components involved in forest dynamics and the relationships between them. Therefore, understanding the impact of human activities on forest biomass and wildlife populations is crucial for developing conservation strategies that promote sustainable forest management, forest restoration, and wildlife conservation, and thus ensure the long-term ecological, social, and economic benefits

of forests. Mathematical models offer a powerful tool for simulating and predicting the impact of human activities on forest ecosystems, providing valuable insights into the complex interactions between forest biomass and dependent wildlife populations, and informing effective conservation strategies.

## **1.4 Research Objective**

### **1.4.1 General objective**

The general objective of this study is to develop and analyse mathematical models in crisp and fuzzy environments to study the impacts of anthropogenic activities on forest biomass and forest-dependent wildlife population with management measures.

### **1.4.2 Specific objectives**

The specific objectives of the study are:

- (i) To formulate and analyse a univariate time series model for forest land loss forecast in Tanzania,
- (ii) To develop and analyse a mathematical model in crisp environment that describe the impacts of anthropogenic activities on forest biomass and forest-dependent wildlife population,
- (iii) To examine how the uncertain parameters represented by fuzzy numbers affect the behaviour of the model that simulates forest biomass and forest-dependent wildlife population depletion due to anthropogenic activities,
- (iv) To evaluate the effects of economic measures and technological efforts on conservation of forest biomass and protection of forest-dependent wildlife population.

## **1.5 Research Questions**

The study was guided by the following research questions:

- (i) How well a univariate time series model effectively forecast the rate and pattern of forest land loss in Tanzania?

- (ii) How can a crisp mathematical model represent the dynamics of impacts of anthropogenic activities on forest biomass and forest dependent wildlife population?
- (iii) How do uncertain parameters represented by fuzzy numbers, influence the predictive accuracy and behaviour of a model simulating the depletion of forest biomass and wildlife populations due to anthropogenic activities?
- (iv) What are the impacts of various economic measures and technological interventions on the conservation of forest biomass and the protection of forest-dependent wildlife populations?

## **1.6 Significance of the Study**

The significance of the study are:

- (i) The findings of this study offer a balanced approach to forest and wildlife management, providing practical insights that can be readily adopted by forest resource users, conservationists, and government authorities. By understanding the intricate relationships between forest biomass, wildlife populations, and human activities, stakeholders can make informed decisions that promote sustainable practices while mitigating environmental degradation.
- (ii) This study contributes directly to the achievement of Goal 15 of the United Nations Sustainable Development Goals (SDGs), which aims to protect, restore, and promote sustainable use of terrestrial ecosystems, sustainably manage forests, combat desertification, halt and reverse land degradation, and halt biodiversity loss. By addressing the complex interactions between human activities and forest ecosystems, the research aligns with global efforts to preserve biodiversity and ensure the long-term health of ecosystems.
- (iii) The findings of this study serve as valuable resources for further research and policy formulation in the field of forest and wildlife management. Researchers can build upon the insights gained to delve deeper into specific aspects of forest ecology, biodiversity conservation, and sustainable resource utilisation. Policymakers can also use these findings to develop evidence-based policies that promote environmental conservation, biodiversity protection, and sustainable development.

## **1.7 Delineation of the Study**

Human activities are directly linked to the growth of the human population, and this study reveals a noticeable increase in population growth, as shown in Fig. 3. Nevertheless, the study does not incorporate population control measures as part of the forest conservation strategies. This omission is due to cultural differences and differing perceptions regarding the concept of depopulation. The concept of depopulation is diverse and complex, shaped by a combination of historical, cultural, religious, and socio-economic factors [119]. As such, any population control or family planning strategies should consider the unique contexts and sensitivities of each community to be effective and respectful of their values and beliefs. To address this challenge, economic measures are proposed to control population pressure stemming from the growth of the human population.

## **CHAPTER TWO**

### **LITERATURE REVIEW**

#### **2.1 Introduction**

Mathematical models have been extensively utilised by researchers in various fields to study and analyse complex systems arising from physical, non-physical and natural sciences [29]. In the context of resources management, mathematical models have gained significant attention from scholars. These models provide a quantitative framework for understanding the dynamics of natural systems, such as forests, wildlife populations, and ecosystems [108]. They allow simulation and prediction of the behaviour of these systems under different scenarios, test hypotheses, and evaluate the impacts of various management measures [73]. On the other hand, the models are often used to optimize decision-making, develop sustainable management strategies, and guide policy formulation [10]. To gain a comprehensive understanding of the existing knowledge and identify potential gaps, it was crucial to review the relevant literature. Hence, this chapter focuses on the analysis and discussion of the related works pertaining to the present study.

#### **2.2 Human-Forest Interactions Dynamics**

The relationship between humans and forests is complex, involving various interactions that can have both positive and negative impacts on forest ecosystems and the communities that depend on them. One key aspect of the dynamics in human-forest interactions is resource extraction. Forests have long been a source of timber, fuelwood, non-timber forest products (NTFPs), and other resources crucial for human livelihoods, economic development, and cultural practices [27]. However, unsustainable resource extraction practices, such as illegal logging, overharvesting, and destructive farming methods can lead to deforestation, habitat destruction, and biodiversity loss. Human activities such as agriculture, infrastructure development, and urbanisation also impact forest ecosystems [13]. Deforestation and forest fragmentation, resulting from agricultural expansion, infrastructure development, and urbanisation, can disrupt forest ecosystems, alter species composition, and degrade ecosystem services.

Another aspect of human-forest interactions is forest governance and policies. Effective governance and policies that promote sustainable forest management, community engagement, and

stakeholder participation can have positive impacts on forest conservation and livelihoods [8]. Conversely, weak governance, inadequate policies, and lack of enforcement can lead to unsustainable resource extraction, land grabbing, and forest degradation [115]. Thus, addressing governance and policy challenges is crucial for achieving sustainable human-forest interactions.

Furthermore, the social and economic dimensions of human-forest interactions cannot be overlooked. Forest-dependent communities often face challenges such as food insecurity, and limited access to education and healthcare [112]. Addressing these socio-economic challenges is essential for achieving sustainable forest management and ensuring that forest-dependent communities can benefit from forest resources in a socially equitable manner. On the other hand, the dynamics of human-forest interactions are shaped by cultural values, traditional knowledge, and social practices associated with forests [103]. Forests have deep cultural and spiritual significance for many indigenous and local communities, who have long-established relationships with forests landscapes through traditional and cultural values [113]. These cultural values and practices can influence human behaviours towards forests, including conservation practices, sustainable resource use, and traditional ecological knowledge that contributes to forest resilience and sustainability.

Various theories have been formulated to gain insights into the complexities of human-environment interactions [17]. Among these theories, Human-Environment Systems Theory (HEST) stands out as a significant framework for understanding the intricacies of the relationship between humans and the environment [25]. In the following section, the applicability and significance of this theory to the current study is elaborated upon.

### **2.3 Human-Environment Systems Theory (HEST)**

Human-Environment Systems Theory (HEST) is a multidisciplinary approach that facilitates us to understand the dynamic of interactions and feedback loops between humans and their environment, including natural ecosystems such as forests [5]. It emphasizes the interconnectedness and interdependencies between human and the ecosystems they inhabit, recognizing that humans and the environment are interrelated and mutually influence each other. This theory acknowledges that human activities, such as logging, agriculture, and infrastructure development can have significant impacts on forest ecosystems. The theory highlights importance of studying human-forest interactions as integrated systems, rather than isolated components [61].

Furthermore, it emphasizes the need to understand the complex feedback loops, synergies, and trade-off between human activities and ecosystem dynamics. This approach recognises that changes in one system can have cascading effects on the other, and that effective management and conservation of the environment require considering the social, cultural, economic, and political dimensions of human-environment interactions.

Human-Environment Systems Theory (HEST) is particularly relevant to this study due to its holistic approach towards understanding the interdependencies between humans and the forest. The theory offers a comprehensive framework that allows for the investigation of the complex relationships, feedback mechanisms, and dynamics that shape human interactions with forest. It emphasises the significance of considering the ecological aspects, social, cultural, and economic dimensions when examining human-forest interactions. Several theoretical and experimental studies including [25], [73] and [28] utilised HEST in the pursuit of understanding the dynamics of human-forest interactions. These studies explore the complex relationships and feedback loops between human societies and forest ecosystems, shedding light on the impacts of human activities on forest resources and biodiversity. In particular, mathematical modelling has emerged as a popular tool in the domain of theoretical studies to analyse and quantify the impact of human population and its corresponding pressure on forest biomass. These studies have contributed to the body of knowledge on the complexities of human-forest interactions, shedding light on the underlying dynamics, drivers, and implications. The analysis and discussion of the related works is presented in the next section.

## **2.4 Mathematical Modelling**

Human-forest interaction is characterised by the interdependence of multiple variables, giving rise to complex systems [29]. Mathematical modelling has become a valuable approach to study and analyse these systems [10]. The model provides theoretical representations of human-forest interactions using mathematical concepts, enabling the exploration of system dynamics, the effects of different components, and predictions about system behaviour [30]. By employing mathematical models, researchers can gain insights into the intricacies of human-forest interactions and better understand the underlying patterns and processes that govern them. These models can also be utilised to quantify the impacts of different management strategies on forest sustainability and conservation outcomes. They offer a means to simulate and predict the

effects of various interventions under different conditions, allowing decision-makers to formulate effective strategies [10]. Furthermore, the models provide a quantitative framework that contributes to evidence-based decision-making and supports informed management and conservation practices for sustainable forest ecosystems [37].

Mathematical models have played a crucial role in investigating the complex dynamics of human-forest interactions. The parallels between these interactions and the dynamics of predator-prey relationships have led researchers to utilise mathematical modelling as a valuable tool to address the challenges arising from human population growth and its impacts on forest resources and wildlife species. For instance, Shukla *et al.* [105] proposed a system of autonomous diffusion equations in their first attempt to model the degradation of forests due to human factors, specifically industrialization. The model consists of two compartments, that is, forest biomass and industrialisation pressure. The findings show that without a forestry regeneration program rapid diffusion cannot prevent extinction when industrialisation pressure is too high. This pioneering work highlights the importance of mathematical models in understanding the complex dynamics of human-forest interactions and the potential consequences of human activities on forest sustainability.

The extension of Shukla *et al.* [105] approach were appreciated in understanding the effects of human population, population pressure and different human activities on forest depletion and conservation. While remaining conventional to Shukla *et al.* [105], Sundar *et al.* [111] and Shukla *et al.* [107] proposed systems of non-linear differential equations to investigate the depletion of forestry resources caused by the combination of population growth and industrialisation. They assumed that industrialisation follows a logistic growth model, and that population growth enhances industrialisation. The analysis revealed that as the depletion rate coefficient of forest biomass increases due to industrialisation, the density of forest biomass decreases. This, in turn, has implications for population growth. The findings from these studies provide valuable insights into the field of forest management, highlighting the implications of population increase and industrialisation on forest ecosystems. However, it is noteworthy to mention that these studies primarily focus on industrialization and its impact on forest resources, while overlooking the ecological aspects related to wildlife species that either partially or entirely rely on forests for their survival. Wildlife species, including their habitat requirements, population dynamics, and ecological interactions within the forest ecosystem, are crucial considerations in

understanding the full extent of human-forest interactions and formulating comprehensive management strategies for sustainable forest conservation. Therefore, the present study integrated the ecological aspects of wildlife species into the mathematical models to provide a more holistic understanding of the complex dynamics of human-forest interactions and inform effective forest resource management policies.

Human population growth and its associated activities such as industrialisation, mining and tourism, have been identified as significant contributors to environmental pollution [50]. Thus, Dubey [22] proposed a novel system of non-linear differential equations to study the effects of industrialisation, population growth, and pollution on forest biomass. The model assumes that the dynamics of forest biomass, industrialisation, and human population follow logistic-type equations, which capture the potential saturation or carrying capacity of these variables. Additionally, the model assumes that the emission of pollutants into the environment is influenced by both industrialisation and population density. This assumption acknowledges that the intensity of industrialisation and population increase may impact the extent of pollutant emissions. The findings suggest that uncontrolled increase of human population, coupled with high emission rates of pollutants into the environment can potentially lead to the depletion of forest biomass. This underscores the critical importance of managing population growth and controlling pollution to ensure the sustainability of forest resources, however, this study did not feature the ecological aspects related to wildlife species that rely on forests for their survival.

Pathak *et al.* [91], [56] and [92] studied the impact of deforestation and forest degradation resulting from human population growth and associated pressure on the survival of wildlife species that depend on forests. These studies employed a logistic growth model to describe the dynamics of forest biomass and human population in their natural habitats. They also considered that wildlife species are entirely dependent on forest resources for their survival. The findings of these studies highlight the alarming deforestation consequences caused by human population growth and associated the pressures, leading to the extinction of forest resources and severe threats to the survival of wildlife species. This underline the urgent need for effective conservation strategies to mitigate the negative impacts of human activities on forest ecosystems and safeguard the biodiversity of wildlife species dependent on forests for their survival. However, these studies did not fully consider the dependence of wildlife species on forest resources. For instance, Pathak *et al.* [92] claims that there exists a solution in which wildlife

species can survive in the absence of forest biomass, which contradicts the assumption that wildlife species solely depend on forest resources for their survival. To address this limitation, the current study incorporates the aspect of forest dependence by defining the growth rate of the wildlife population as a function of forest biomass. This approach allows for a more realistic and comprehensive assessment of the intricate relationship between forest resources, human population, wildlife species, and their survival, shedding light on the complex dynamics that govern human-forest-wildlife interactions. By accounting for the ecological dependencies of wildlife species on forest resources, this study provides a more robust and nuanced understanding of the challenges and implications of human activities on forest ecosystems and wildlife conservation efforts.

Recently, Sinha *et al.* [108] proposed a model to investigate the impacts of industrial activities on forest biomass and forest-dependent wildlife species. The study categorised industries into forest-based and non-forest-based industries, with the objective of identifying mathematical conditions under which non-forest-based industries do not simply replicate the effects of forest-based industries. The findings revealed that pollutants emitted by non-forest-based industries are more detrimental to the environment. Moreover, the study concluded that for the long-term survival of wildlife populations, the expansion of forest biomass should outweigh their utilisation by industries to meet human needs. However, if the continuous depletion of forest biomass by both forest-based and non-forest-based industries in the forest region persists, it could lead to the extinction of forest biomass. This would pose a risk to the stability of forest-dependent wildlife populations, resulting in disruptions to the forest ecosystem, negative impacts on wildlife, and economic repercussions.

The study did not consider the role of human population in the growth of industries, and the potential saturation of wildlife population in the study region. This could be a limitation of the study, as human population dynamics and their interaction with industrial activities can have significant impacts on forest biomass and wildlife populations. Furthermore, understanding the potential saturation of wildlife populations in the study region could provide valuable insights into the long-term sustainability of wildlife populations and the overall ecological balance.

Shukla *et al.* [106], [74] and [76] proposed mathematical models for forest conservation that incorporate technology and economic incentives as conservation strategies. The analysis of these models revealed that forest conservation can be achieved up to a certain threshold level

by implementing technological efforts to accelerate afforestation or by reducing population pressure through economic incentives. However, beyond this threshold, additional efforts to increase forest resources, whether through technological interventions or economic measures, may lead to destabilisation of the system, potentially resulting in unintended consequences. On the other hand, Goshu *et al.* [38] proposed a comprehensive conservation model that incorporates technological efforts, economic incentives, as well as measures to control man-made fires and environmental pollution. The findings of the study highlight that when these conservation strategies are applied, the forest biomass can be maintained at an optimal level, resulting in the conservation of forest density over time. In contrast, if no efforts are made to preserve the forest resources, the resource may become extinct.

It can be noted that the findings emphasise the importance of adopting a balanced integrated approach in forest resource management, considering the ecological, social, and economic dimensions of sustainability. Further they provided valuable insights for policymakers and practitioners in formulating effective strategies for forest conservation and livelihood improvement while maintaining the resilience of forest ecosystems. However, these studies did not investigate the impact of forest depletion and conservation on wildlife species that depend on forest resources for their survival. In contrast, the present study proposes a model that incorporates the depletion and conservation of forest biomass while also considering the role of forest resources in supporting wildlife species. This holistic approach takes into account the interconnectedness of forest resources, wildlife populations, and conservation efforts, providing a more comprehensive understanding of the dynamics between forests and wildlife species.

In addition to dynamical system models, data analysis models are also used to investigate the impact of population and population pressure on biodiversity, a comprehensive review can be found in [28]. For instance, Freitas *et al.* [33] researched the effects of human activities on forest cover dynamics while Lingner *et al.* [60] utilised regression models to study the relationship between anthropogenic disturbance and forest characteristics. Bologna [7] used quantitative analysis to investigate the sustainability of the world human population in light of deforestation. Despite the detailed and nuanced insights that data analysis models can provide, they are limited by the quality and quantity of available data and may not fully capture the complexity of environmental systems [52]. Thus, the present study employed dynamical system models, which have the capability to explore the underlying mechanisms of human-forest-wildlife

species interactions.

## **2.5 Conclusion and Research Agenda**

In the reviewed literature, it is evident that the use of mathematical models, specifically systems of differential equations have gained significant attention from researchers seeking to address challenges in human-forest interactions systems. These models offer a unique opportunity to simulate scenarios, test interventions, and evaluate the effectiveness of conservation efforts, ultimately aiding decision-makers in the development of sustainable management strategies. However, it can be noted that, most of works modelled human-forest interactions in a crisp environment under the assumption that all parameters are constant. But, in reality, biological parameters are not always fixed as they vary with the surrounding environment. For instance, the growth rate of forests can differ based on factors such as soil fertility, presence of pollinators, rainfall, and other variables, which are not uniform throughout. Given this perspective, the model developed in fuzzy environment is more meaningful than the crisp model [20]. In doing so, the parameters can be treated as fuzzy numbers [88], and the formulated model will take into account the variability and uncertainty of the parameters that influence the dynamics of forest-human interactions.

Nevertheless, regular review and improvement of available models is crucial to enhance their usefulness. Since, even the best models have some limitations or imperfections, hence the need for periodic review and improvement to address the emerging issues [66]. This is particularly important in the face of growing population pressures, climate change, and socio-economic factors that can impact forest ecosystems. The model should be able to predict the potential consequences of human activities such as deforestation, land-use changes, and fragmentation on forest ecosystems and their dependent wildlife species. In addition, it should provide insights on appropriate management strategies that can be employed to mitigate the negative impacts of anthropogenic activities on forest ecosystems and their dependent wildlife species.

Therefore, motivated by the existing works, this study developed and analysed models to study the depletion and conservation of forest biomass and forest-dependent wildlife population in both crisp and fuzzy environments to account for uncertainty and variability of the model parameters. Uncertainty and sensitivity analyses were performed to identify the sources of uncertainty and to quantify the impact of parameter variability on the model output.

## CHAPTER THREE

### MATERIALS AND METHODS

#### 3.1 Introduction

This chapter describes the overall materials and methods that were used in model formulation model analysis and parameter estimation. They are presented in the order of the models developed, that is time series model and dynamical system models. The chapter provides comprehensive and lucid understanding of the methodology adopted in this study. This detailed exploration ensures transparency and clarity in research approach, facilitating a deeper comprehension of the results and implications.

#### 3.2 Univariate Time Series Model

The impact of human activities on forests in Tanzania is a matter of great concern. To accurately assess the extent of the damage, the Autoregressive Integrated Moving Average (ARIMA) model for predicting the loss of forest land in Tanzania is formulated and analysed. This predictive model has been formulated by using the Box-Jenkins methodology, which is a widely accepted statistical approach for time series modelling [48]. This methodology has been chosen because it provides a structured and systematic framework for developing accurate time series models.

The ARIMA is the algorithm that analyses and forecast equally spaced stationary univariate time series. The model predicts an observation in a response time series as a linear combination of its own past observations as well as current and past errors of the same series. The time series is stationary if the statistical properties of the series such as mean, variance and autocorrelation structure do not change over time. If the series is not stationary, regular differencing can be applied to make the series stationary. For example, let  $Z_t$  be a non-stationary time series data, then stationary time series data ( $X_t$ ) obtained by differencing time series data ( $Z_t$ ) by order  $d \in \mathbb{Z}$  is expressed mathematically as:

$$X_t = \nabla^d Z_t = (1 - \mathbf{L})^d Z_t, \quad (1)$$

where  $\mathbf{L}$  is a time lag operator (backshift operator).

Different statistical tests such as Dickey-Fuller and Phillips-Perron are used to test the stationarity status. Time series, let say,  $X_1, X_2, \dots, X_{n-1}, X_n$  can be assumed as a particular form of a stochastic process which can be described by  $n$ -dimensional probability distribution  $p(X_1, X_2, \dots, X_{n-1}, X_n)$ . By assuming the joint normality, there are  $n$  observations with  $(n + 1)$  unknown parameters. To infer such a general probability structure from just one realisation of the stochastic process will be impossible. Thus, the condition for stationarity of a time series is used to simplify the probability structure by making it stable with respect to time, as follows:

$$p(X_{t_1}, \dots, X_{t_m}) = p(X_{t_1+k}, \dots, X_{t_m+k}).$$

The univariate time series data for primary forest loss from Tanzania measured in Hectare (ha) were used to illustrate forecast modelling process by using ARIMA model. The data set for the 31 consecutive years (1991-2021) were obtained from Global Watch Forest (`shorturl.at/anzQT`) in which 80% of the observations were used for model training while 20% of the observations were used for model testing. The modelling process follows Box and Jenkins conceptual framework which consists of three steps; model identification, parameter estimation as well as model diagnostic and forecast [96]. These steps offer a straightforward method for expressing time series using mathematical models and doing prediction as well as forecasting.

### 3.2.1 Model identification

The Auto-Regressive Moving Average model of order  $p$  and  $q$  denoted by  $ARMA(p, q)$  is defined mathematically as:

$$X_t = C + \sum_{i=1}^p \varphi_i X_{t-i} + \sum_{i=1}^q \vartheta_i \epsilon_{t-i}, \quad \epsilon_t \sim \mathcal{N}(0, \sigma_{\epsilon_t}), \quad (2)$$

where  $X_t$  and  $X_{t-i}$  represent the values of the time series at time  $t$  and  $t - 1$ , respectively,  $C$  represents constant term of the model,  $\varphi$  represents the Autoregressive (AR) coefficients and  $\vartheta$  represents the Moving Average (MA) coefficients.

In the Box-Jenkins methodology, identifying the suitable model for a given time series is the first step. This involves selecting appropriate values of the model parameters, which include the order of the autoregressive (AR) and moving average (MA) components of the model, denoted by  $p$  and  $q$ , respectively. The goal is to choose the values of  $p$  and  $q$  that capture the underlying dynamic features of the time series, while also ensuring that the model is parsimonious and can be estimated reliably.

When selecting the values of  $p$  and  $q$  the consideration of the stationarity of the time series is of utmost important. A stationary time series is one whose statistical properties, such as the mean and variance, remain constant over time. In contrast, a non-stationary time series may exhibit trends, seasonality, or other patterns that change over time, making it difficult to model accurately. This study used Augmented Dickey-Fuller (ADF) test to test the stationarity of the series due to that the test has a well-established statistical properties and has been extensively studied in the literature [78]. Suppose a stochastic process which is the first order Autoregressive model is given by equation (3):

$$X_t = \varphi X_{t-1} + \epsilon_t, \quad (3)$$

such that  $|\varphi| \leq 1$  and  $\epsilon_t$  is white noise. The unit root is obtained when  $|\varphi| = 1$  and the process becomes non-stationary. Conversely, when  $|\varphi| < 1$  the process becomes stationary, otherwise, the process becomes explosive. The ADF test aimed at checking if the process has a unit root. Differentiating equation (3) gives:

$$\Delta X_t = \varrho X_{t-1} + \epsilon_t, \quad (4)$$

where,

$$\Delta X_t = X_t - X_{t-1} \text{ and } \varrho = \varphi - 1 : \varrho \leq 0.$$

The test for parameter  $\varphi$  transformed to  $\varrho$  test, since  $\varrho$  can not be positive, thus if  $\varrho = 0$ , the process becomes random walk, if not and  $-2 < \varrho < 0$  then the process becomes stationary. Thus, the null hypothesis for the ADF test is the process is a random walk (the data are non-stationary,  $\varrho = 0$ ).

After ensuring the stationarity of the data, the model is selected by using the inferences from Autocorrelation Function (ACF), Partial Autocorrelation Function (PACF) [121], and Akaike Information Criteria (AIC) [93]. The Autocorrelation Function (ACF) is the statistical measure which shows the dependency between time series data. The ACF of time series data  $X_t$  at time lag  $k$  is defined mathematically as:

$$\rho_k = \frac{\gamma_k}{\gamma_0},$$

such that,

$$\gamma_k = \text{Cov}(X_t, X_{t-k}), \quad \gamma_0 = \sqrt{(\text{Var}(X_t)\text{Var}(X_{t-k}))}.$$

The Partial Autocorrelation Function (PACF) is the statistical tool which measure the correlation between two lags, say  $X_t$  and  $X_{t-k}$  when the effects of the other time lags,

$1, 2, \dots, (k-2), (k-1)$  have been removed. The PACF is referred as conditional correlation  $\left( \text{Cov} \left( X_t, X_{t-k} | X_{t+1}, X_{t+2}, \dots, X_{t-(k-1)} \right) \right)$  denoted as  $\rho_{kk}$  and is defined as:

$$\rho_{kk} = \frac{\gamma_k - \gamma_{k-1}^2}{\gamma_{k-1}^2}.$$

The inference from ACF and PACF for Autoregressive model  $\text{AR}(p)$  are as follows: consider the Autoregressive model of order  $p$  as presented in equation (5):

$$X_t = \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \dots + \varphi_{1-p} X_{t-(1-p)} + \varphi_p X_{t-p} + \epsilon_t. \quad (5)$$

For ACF, multiplying equation (5) by  $X_{t-k}$  and taking the expectation gives equation (6).

$$\rho_k = \varphi_1 \rho_{k-1} + \varphi_2 \rho_{k-2} + \varphi_3 \rho_{k-3} + \dots + \varphi_{1-p} \rho_{k-(1-p)} + \varphi_p \rho_{k-p}. \quad (6)$$

Equation (6) can be represented in lag operator as  $\Psi(\mathbf{L})\rho_k = 0$ , where,

$$\Psi(\mathbf{L}) = 1 - \varphi_1 \mathbf{L} - \varphi_2 \mathbf{L}^2 - \varphi_3 \mathbf{L}^3 - \dots - \varphi_p \mathbf{L}^p.$$

For different values of  $k = 1, 2, \dots, p$  gives Yule-walker equation [40],

$$\vec{\rho} = \mathbf{P}\vec{\varphi}.$$

By considering a solution of the form  $\rho_k = \tau^k$ , gives:

$$\tau^p - \varphi_1 \tau^{p-1} - \varphi_2 \tau^{p-2} - \varphi_3 \tau^{p-3} - \dots - \varphi_{p-1} \tau - \varphi_p = 0. \quad (7)$$

It can be observed that the solutions of the equation (7) are the reciprocal roots

$$\varpi_1, \varpi_2, \varpi_3, \dots, \varpi_{p-1}, \varpi_p,$$

of the polynomial  $\Psi(\mathbf{L})$ . Thus, the general solution is given as:

$$\rho_k = \sum_{i=1}^p c_i \varpi_i^k = c_1 (\varpi_1)^k + c_2 (\varpi_2)^k + c_3 (\varpi_3)^k + \dots + c_p (\varpi_p)^k,$$

suggesting that  $\rho_k$  has an exponential behaviour or sinusoidal waves if some of the reciprocal roots are complex numbers. For PACF, utilising Cramer's rule, for different values of  $k = 1, 2, \dots, p$ , gives  $\rho_{kk}$ . Thus, for any value of  $k > p$ , gives  $\rho_{kk} = 0$ . Therefore, PACF of an AR model of order  $p$  cuts off to zero after lag  $k = p$ .

Furthermore, the inference from ACF and PACF for Autoregressive model  $MA(q)$  are as follows: Consider the Moving Average model of order one given as:

$$X_t = \epsilon_t + \vartheta \epsilon_{t-1},$$

such that,  $\vartheta$  is the model parameter and  $\epsilon_t$  the error term which are normally distributed with mean zero and constant standard deviation  $\sigma_{\epsilon_t}$  ( $\epsilon_t \sim \mathcal{N}(0, \sigma_{\epsilon_t}^2)$ ). The corresponding ACF is given as:

$$\rho_k = \begin{cases} \frac{\vartheta}{(1 + \vartheta^2)}, & k = 1 \\ 0, & k > 1. \end{cases}$$

Taking the value of  $\rho_k = 0$  for  $k > 1$ , gives the PACF as follows:

$$\rho_{kk} = \frac{\vartheta^k (1 - \vartheta^2)}{1 - \vartheta^{2(k+1)}}, \quad k \geq 1.$$

In contrast to its ACF, the PACF of an  $MA(1)$  model decays exponentially for  $k > 1$ . Thus the general inference from ACF and PACF of  $MA(q)$  model is that, the ACF cut-off to zero for  $k > q$  and the PACF will have exponential behaviour depending on the roots of the polynomial equation  $\Psi(\mathbf{L})$ . The inference from ACF for  $ARMA(p, q)$  is similar to the pure  $AR(p)$  process while the PACF have the similar behaviour as for  $MA(q)$  model. Thus, Table 1 provides a summary of the general inferences from ACF and PACF for  $AR(p)$ ,  $MA(q)$  and  $ARMA(p, q)$  processes.

**Table 1:** General inferences from theoretical ACFs and PACFs

| Model        | ACF characteristics               | PACF characteristics              |
|--------------|-----------------------------------|-----------------------------------|
| $AR(p)$      | Decays exponentially towards zero | Cut-off to zero for $k > p$       |
| $MA(q)$      | Cut-off to zero for $k > q$       | Decays exponentially towards zero |
| $ARMA(p, q)$ | Decays exponentially towards zero | Decays exponentially towards zero |

### 3.2.2 Parameter estimation

After model identification, the next step is parameter estimation. This step is very crucial because the fitted model is obtained through the estimated parameters. Since there is no closed form estimate can be found analytically, the computer aided methods and *statsmodels* python package were used to facilitate the process. The package use Maximum Likelihood Estimation

(MLE) to estimate the parameters of the model. The general MLE approach given as follows:

If the data set is iid with probability density function (pdf)  $f(y; \Theta)$ , then the joint probability density function of the sample data  $Y = (Y_1, Y_2, Y_3, \dots, Y_K)$  is given as the product of probability density function of each observation:

$$f(y; \Theta) = \prod_{j=1}^K f(y_j; \Theta).$$

The *likelihood function*  $\mathbf{L}$  is given as the joint probability density function of  $y$  given  $\Theta$ :

$$\mathbf{L}(y|\Theta) = \prod_{j=1}^K f(y_j; \Theta). \quad (8)$$

The *log-likelihood* ( $\ell$ ) is given as the log of the equation (8):

$$\ell(\Theta) = \log(\mathbf{L}(y|\Theta)) = \sum_{j=1}^K \log f(y_j; \Theta).$$

Thus, the estimated  $\Theta$  denoted by  $\hat{\Theta}$  which optimizes  $\mathbf{L}(y; \Theta)$  is given as:

$$\hat{\Theta} = \arg \max_{\Theta} \sum_{j=1}^K \log(\mathbf{L}(y|\Theta)).$$

Suppose a random variable  $X_k|X_{k-1}$  contains  $\epsilon_t$  as the only random term at time  $t$  which does not depend on anything since it is a white noise, then the joint probability density function of ARMA process (2) is given as:

$$f(X_k; \Theta) = \left( \prod_{t=p+1}^k f(x_t|X_{t-1}, \Theta) \right) f(X_p|(\varphi, \vartheta), \Theta),$$

such that,  $\Theta = (\varphi, \vartheta, \sigma_{\epsilon_t})$  and  $p$  is the order of AR in ARMA( $p, q$ ) process. The corresponding likelihood function is defined as:

$$\mathbf{L}(X_k, \Theta) = \left( \prod_{t=p+1}^k f(x_t|X_{t-1}, \Theta) \right) f(X_p|\Theta). \quad (9)$$

Following equations (8) and (9), the *log-likelihood* function is give as:

$$\ell((\varphi, \vartheta)|X) = \ln(f(X_p|\Theta)) + \sum_{t=1+p}^k f(x_t|X_{t-1}; \Theta).$$

For, MLE estimate, let  $\theta$  be a set of parameters for ARMA model (2) and  $\sigma_{\epsilon_t}$  be the error standard deviation which is assumed to be know. Further, let  $\hat{x}_t|_{t-1}$  be one step forecast which

is the mean of  $x_t|X_{t-1}$ , hence, following equation (2), the prediction error ( $\hat{r}_t$ ) with variance  $\sigma_{\epsilon_t}$  is given by  $r_t^* = x_t - \hat{x}_t|_{t-1}$ . Given that  $\epsilon_t$  is assumed to be white noise, then, the probability density function of  $X_t$  is given as:

$$f(x_t|x_{t-1}, \boldsymbol{\theta}, \sigma_{\epsilon_t}^2) = \frac{1}{\sigma_{\epsilon_t} \sqrt{2\pi}} \exp \left( -\frac{\left( (x_t - \hat{x}_t|_{t-1}) (\boldsymbol{\theta}) \right)^2}{2\sigma_{\epsilon_t}} \right),$$

whereby, the respective conditional likelihood function is defined as:

$$L(X_k, \boldsymbol{\theta}, \sigma_{\epsilon_t}) = (2\pi\sigma_{\epsilon_t})^{(k-p)/2} \exp \left( -\frac{1}{2\sigma_{\epsilon_t}} \sum_{t=p+1}^k (r^*(\boldsymbol{\theta}))^2 \right). \quad (10)$$

Therefore, the LME estimate  $\hat{\boldsymbol{\theta}}$  is the parameter which optimise equation (10). Note that, the value of  $L(X_k, \boldsymbol{\theta}, \sigma_{\epsilon_t})$  is maximum provided  $\sum_{t=p+1}^k (r^*(\boldsymbol{\theta}))^2$  is minimum. Thus, the set of parameters  $\hat{\boldsymbol{\theta}}$  which maximise equation (10) is given as the minimum of  $SSE(\boldsymbol{\theta})$ :

$$SSE(\boldsymbol{\theta}) = \sum_{t=p+1}^k (r^*(\boldsymbol{\theta}))^2.$$

After obtaining the estimated parameters which are the coefficients of the model the next step is model validation.

### 3.2.3 Model validation

After model selection and parameter estimation, the next step is to perform model diagnosis to check whether the proposed model explains the process. The residual standard error plot, a normal qq, the ACF of the residuals, the probability density function, and a histogram of the residuals were used to analyse the residuals. The plot of standardised residuals are used to see if they are stationary as well as independent and identically distributed with a mean of zero and variance of one. A normal probability plot and the histogram plot test the normality of the residuals. Furthermore, the ACF of the residuals check the independence of residuals.

## 3.3 Dynamical System Models

The dynamical system models are models used to define and predict the interactions between several compartments overtime of a phenomenon that is viewed as a system [30]. Mathematically, dynamical system is defined as:

$$\dot{\mathbf{X}} = \mathcal{G}(\mathbf{X}, \boldsymbol{\Lambda}, t), \quad (11)$$

such that,

$$\mathbf{X}(0) = \mathbf{X}_0, \quad \mathbf{X} \in \mathbb{R}^n \quad \text{and} \quad \mathbf{A} \in \mathbb{R}^m.$$

Here,  $\mathbf{X}$  is the vector of the model compartments while  $\mathbf{A}$  is the set of parameters which define the relationships between the compartments over time  $t$ . The dynamical system models for the depletion and conservation of forest biomass and forest-dependent wildlife population were formulated as follows:

### 3.3.1 Modelling Depletion of Forest Biomass and Wildlife Population in a Crisp Environment

The model is formulated by considering the interrelationships between forest biomass, forest-dependent wildlife population, and human population, with the aim of understanding the impact of human activities on forest ecosystems. Notably, the model is formulated under the assumption of fixed parameters rather than fuzzy variables, ensuring a clear and defined framework for understanding the dynamics at play. Following equation (11), the variables of the model are given by:

$$\mathbf{X} = \{B, W, N, H\},$$

the descriptions are given in Table 2. Furthermore, the set of parameters ( $\mathbf{A}$ ) which are used to describe the relationships between the variables are given in Table 3.

**Table 2:** Model variables and description

| Variable | Description                                                 | Dimension   |
|----------|-------------------------------------------------------------|-------------|
| $B$      | Density of forest biomass at time $t$                       | Biomass     |
| $W$      | Density of forest-dependent wildlife population at time $t$ | Individuals |
| $N$      | Human population in the area of consideration at time $t$   | Individuals |
| $H$      | Human activities at time $t$                                | Area        |

Using the variables in Table 2 and the model parameters in Table 3, a system of ordinary differential equations was formulated by using the following set of assumptions: Forest biomass ( $B$ ) grows logistically due to natural processes, but is depleted by an increase in human population, human activities, and forest-dependent wildlife population. In this context, vegetation such as trees, woodlands, and shrubs have been quantified as the density of forest biomass.

**Table 3:** Model parameter symbol and description

| Parameter        | Description                                                                                         | Units                                                               |
|------------------|-----------------------------------------------------------------------------------------------------|---------------------------------------------------------------------|
| $s$              | Forest biomass intrinsic growth rate                                                                | 1/Year                                                              |
| $L$              | Forest carrying capacity                                                                            | Biomass                                                             |
| $\beta_0$        | The coefficient of forest biomass depletion rate due to wildlife population under natural processes | $\frac{\text{Individuals}^{-1}}{\text{Year}}$                       |
| $\beta_1$        | The coefficient of forest biomass depletion rate due to human population                            | $\frac{\text{Individuals}^{-1}}{\text{Year}}$                       |
| $\beta_2$        | The coefficient of forest biomass depletion rate due to human activities                            | $\frac{\text{Biomass}^{-1} \times \text{Area}^{-1}}{\text{Year}}$   |
| $\nu_1$          | The coefficient of wildlife population depletion rate due to human population                       | $\frac{\text{Individuals}^{-1}}{\text{Year}}$                       |
| $\nu_2$          | The coefficient of wildlife population depletion rate due to human activities                       | $\frac{\text{Area}^{-1}}{\text{Year}}$                              |
| $\theta$         | Human population intrinsic growth rate                                                              | 1/Year                                                              |
| $M$              | Human population carrying capacity                                                                  | Individuals                                                         |
| $\lambda\beta_1$ | The coefficient of human population growth rate due to forest biomass                               | $\frac{\text{Individuals}^{-1}}{\text{Year} \times \text{Biomass}}$ |
| $\sigma$         | The coefficient of human population depletion rate due to wild animals                              | $\frac{\text{Individuals}^{-1}}{\text{Year}}$                       |
| $\gamma$         | The coefficient of human activities growth rate due to human population                             | $\frac{\text{Area} \times \text{Individuals}^{-1}}{\text{Year}}$    |
| $\gamma_1$       | The coefficient of natural depletion rate of human activities                                       | 1/Year                                                              |

The forest-dependent wildlife population ( $W$ ) entirely depends on forest biomass, which grows logistically but is simultaneously depleted by an increase in human population and its corresponding activities as well as a decrease in forest biomass. Moreover, the human population ( $N$ ) grows logistically, and their growth is enhanced by the use of forest resources. As the human population increases, the demand for food, forest products, human settlements, roads, and industries tends to increase. Therefore, it was assumed that the growth of human activities ( $H$ ) is proportional to the size of the human population, and the availability of forest biomass

motivates human activities. All parameters are assumed to be positive.

Human activities are viewed as a composite variable that encapsulates a range of anthropogenic actions, such as agriculture, industrialization, road construction, urbanisation, mining, and other forms of land-use changes. While it is accurate to identify these as actions performed by humans, from an ecological modelling standpoint, it is crucial to represent these activities as a distinct variable ( $H$ ) that influences the dynamics of the ecosystem. The assumption that the growth of human activities is proportional to their density aligns with observations in various ecological systems, where human activities tend to increase in areas of higher human population density. Furthermore, the irreversible destruction of forest due to these activities is a key aspect of our model. The inability of the forest to regenerate post-destruction is a critical consideration when analysing the long-term impact of human activities on the ecosystem.

Taking into account these considerations, a formulated model and its analysis is presented in section 4.2

### **3.3.2 Formulation of fuzzy parameter based model on depletion of forest biomass and forest-dependent wildlife population**

Fuzzy modelling is a mathematical technique that deals with uncertainty and imprecision in data by assigning degrees of membership to different intervals [123]. Input parameters and model outputs are described using fuzzy sets, which are sets of elements with degrees of membership. The approach of using fuzzy parameters in modelling forest depletion is especially useful when considering the impact of human activities on the environment. Human activities such as deforestation, overgrazing, and hunting can have a significant impact on the growth of forest and forest-dependent wildlife population. These activities are often influenced by a range of factors that can vary widely in different contexts, such as economic incentives, cultural practices, and regulatory frameworks. On the other hand, the growth rate of forest biomass can differ based on factors such as soil fertility, presence of pollinators, rainfall, and other variables, which are not uniform throughout. Given this perspective, it is reasonable to take into account a range of values for the parameters instead of a single fixed value. In doing so, the parameters can be treated as fuzzy numbers, and the formulated model will take into account the variability and uncertainty of the parameters that influence the dynamics of forest biomass and wildlife population [68].

The model in fuzzy environment is formulated by assuming that parameters are triangular fuzzy numbers (Table 4) due to their intuitive nature, ease of use, computational simplicity, and ability to facilitate representation [87].

**Table 4:** Description of fuzzy parameters

| Parameter          | Description                                                                                                         |
|--------------------|---------------------------------------------------------------------------------------------------------------------|
| $\tilde{s}$        | Fuzzy intrinsic growth rate coefficients of forest biomass                                                          |
| $\tilde{\beta}_0$  | Fuzzy depletion rate coefficients of forest biomass due to wildlife population under natural processes              |
| $\tilde{\eta}$     | Fuzzy proportionality constants which represent the growth of wildlife population in the presence of forest biomass |
| $\tilde{\beta}_1$  | Fuzzy depletion rate coefficients of forest biomass due to human population                                         |
| $\tilde{\beta}_2$  | Fuzzy depletion rate coefficients of forest biomass due to human activities                                         |
| $\tilde{\nu}_1$    | Fuzzy depletion rate coefficients of wildlife population due to human population                                    |
| $\tilde{\nu}_2$    | Fuzzy depletion rate coefficients of wildlife population due to human activities                                    |
| $\tilde{\theta}$   | Fuzzy intrinsic growth rate coefficients of human population                                                        |
| $\tilde{\lambda}$  | Fuzzy proportionality constants which represent the growth of human population in the presence of forest biomass    |
| $\tilde{\sigma}$   | Fuzzy depletion rate coefficients of human population in the presence of wild animals                               |
| $\tilde{\gamma}$   | Fuzzy growth rate coefficients of human activities due to human population                                          |
| $\tilde{\gamma}_1$ | Fuzzy natural depletion rate coefficients of human activities                                                       |

Furthermore, the following mathematical definitions were used in formulation and analysis of the model as presented in section 4.3.

**Definition 3.1** (Fuzzy set [123, 87])

A fuzzy set  $\tilde{A}$  in a universe of discourse  $X$  is defined as the following set of pairs  $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ . The mapping  $\mu_{\tilde{A}}(x) : X \mapsto [0, 1]$  is the membership function of fuzzy set  $A$ , and  $\mu_{\tilde{A}}(x)$  is the membership value or degree of membership of  $x \in X$  in a fuzzy set  $\tilde{A}$ .

**Definition 3.2** (Triangular Fuzzy Number (TFN) [90])

A Triangular Fuzzy Number  $\tilde{A} = (a_1, a, a_2)$  (Fig. 4a) is fuzzy set of the real number  $\mathbb{R}$  characterized by its continuous membership function  $\mu_{\tilde{A}}(x) : X \mapsto [0, 1]$  as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - a_1}{a - a_1}, & \text{if } a_1 \leq x \leq a, \\ \frac{a_2 - x}{a_2 - a}, & \text{if } a \leq x \leq a_2, \\ 0 & \text{Otherwise.} \end{cases}$$

**Definition 3.3** ( $\alpha$ -cut of a fuzzy number [90])

The  $\alpha$ -cut of a fuzzy number  $\tilde{A}$  in  $X$  (Fig. 4b) is the crisp set:

$$A(\alpha) = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\} \forall \alpha \in (0, 1],$$

for

$$\alpha = 0, A(0) = \sup(\tilde{A}) = \overline{\{x \in X : \mu_{\tilde{A}}(x) > 0\}}.$$

$A(\alpha)$  is a non empty bounded closed interval contained in  $X$  defined as  $A(\alpha) = [A_l(\alpha), A_r(\alpha)]$ , such that  $A_l(\alpha)$  and  $A_r(\alpha)$  are lower and upper bounds of the closed interval, respectively. Thus, from the definition, the  $\alpha$ -cut of a TFN  $\tilde{A}$  is the closed and bounded interval  $[A_l(\alpha), A_r(\alpha)]$ , such that  $A_l(\alpha) = \inf\{x : \mu_{\tilde{A}}(x) \geq \alpha\} = a_1 + \alpha(a - a_1)$ , and  $A_r(\alpha) = \sup\{x : \mu_{\tilde{A}}(x) \geq \alpha\} = a_2 - \alpha(a_2 - a)$ .

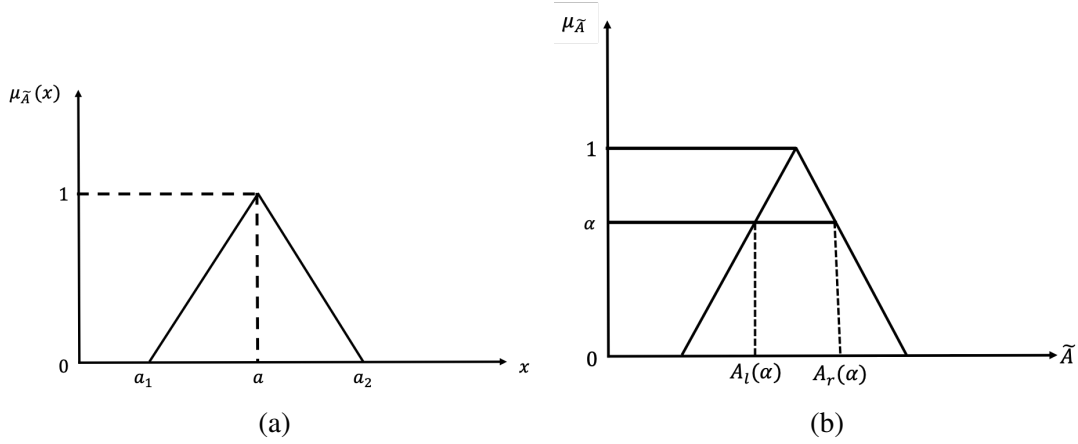


**Figure 4:** (a) Triangular fuzzy number (b) The  $\alpha$ -cut of a Triangular fuzzy number

**Definition 3.4** (Interval Arithmetic [80])

Let  $[P_l, P_r]$  and  $[Q_l, Q_r]$  be two interval numbers, the addition and subtraction of the interval numbers is given as follows:

$$[P_l, P_r] \pm [Q_l, Q_r] = [P_l \pm Q_l, P_r \pm Q_r].$$

**Definition 3.5** (Signed distance of fuzzy numbers [20])

The signed distance  $d$  between two fuzzy numbers is a measure of the dissimilarity or similarity between the numbers. Let  $\tilde{G}$  be a fuzzy number, then the signed distance of  $\tilde{G}$  measured from  $\tilde{0}$  denoted by  $d(\tilde{G}, \tilde{0})$  is given by:

$$d(\tilde{G}, \tilde{0}) = \frac{1}{2} \int_0^1 (\tilde{G}_l(\alpha) + \tilde{G}_r(\alpha)) d\alpha.$$

**Remark 3.1**

Let  $\tilde{P}_i = (a - \Delta_i, a, a + \Delta_{i+1})$  and  $\tilde{Q}_j = (b - \Delta_j, b, b + \Delta_{j+1})$  be TFN, where  $0 < \Delta_i < a, \Delta_{i+1} > 0$  and  $0 < \Delta_j < b, \Delta_{j+1} > 0$ . Then following definition 3.5, the signed distance of  $\tilde{P}_i$  to  $\tilde{0}$  is given by:

$$d(\tilde{P}_i, \tilde{0}) = a + \frac{1}{4}(\Delta_{i+1} - \Delta_i),$$

$\tilde{Q}_j$  to  $\tilde{0}$  is given as:

$$d(\tilde{Q}_j, \tilde{0}) = b + \frac{1}{4}(\Delta_{j+1} - \Delta_j),$$

while the signed distance between  $\tilde{P}_i \otimes \tilde{Q}_j$  and  $\tilde{0}$  is given by:

$$d(\tilde{P} \otimes \tilde{Q}, \tilde{0}) = \frac{ab}{2} + \frac{a}{4}(\Delta_{j+1} - \Delta_j) + \frac{b}{4}(\Delta_{i+1} - \Delta_i) + \frac{1}{4}(\Delta_i \Delta_j + \Delta_{i+1} \Delta_{j+1}).$$

**Definition 3.6** (Hukuhara derivative [94])

Suppose  $f : [v_1, v_2] \rightarrow X$  and  $x_0 \in [v_1, v_2]$ . The function  $f$  is Hukuhara differentiable at the point  $x_0$  if there exists an element  $f'(x_0) \in X$  such that the limits:

$$\lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \text{ and } \lim_{h \rightarrow 0^-} \frac{f(x_0) - f(x_0 - h)}{h} \text{ exist and equal to } f'(x_0).$$

**3.3.3 Formulation of forest biomass and forest-dependent wildlife population conservation model**

The conservation model is formulated by extending the model formulated in subsection 3.3.1 by adding three compartments, that is population pressure  $P$  (individuals/area), technological efforts  $T_e$  (cost) and economic measures  $E_f$  (cost). Similarly, parameters in Table 5 were added to the list of the parameters in Table 3.

Following the addition of compartments and parameters, the conservation model is formulated based on the following set of assumptions: Forest biomass grows logistically but is depleted by wildlife population ( $W$ ) under natural processes, the human population ( $N$ ), and its associated activities ( $H$ ). Technological efforts ( $T_e$ ) aimed at accelerating the rate of reforestation are implemented to maintain an appropriate amount of forest biomass. The pace of implementation is assumed to be proportional to the difference between forest carrying capacity ( $L$ ) and the present forest biomass ( $B$ ).

**Table 5:** Model Parameter and description

| Parameter  | Description                                                                        | Units                                                                            |
|------------|------------------------------------------------------------------------------------|----------------------------------------------------------------------------------|
| $\phi$     | Coefficient of population pressure due to human population                         | $\frac{\text{Area} \times \text{Individuals}^{-2}}{\text{Year}}$                 |
| $\phi_0$   | Coefficient of natural depletion rate of population pressure                       | 1/Year                                                                           |
| $\phi_1$   | Coefficient of population pressure depletion rate as a result of economic measures | $\frac{\text{Cost}^{-1}}{\text{Year}}$                                           |
| $\phi_2$   | Coefficient of human activities growth rate due to population pressure             | $\frac{\text{Individuals/Year}}{\text{Cost}}$                                    |
| $\rho$     | Coefficient of technological efforts implementation rate                           | $\frac{\text{Biomass} \times \text{Year}}{\text{Cost}}$                          |
| $\rho_1$   | Coefficient of technological efforts natural depletion rate                        | 1/Year                                                                           |
| $\rho_2$   | Coefficient of technological depletion rate due to the level of forest biomass     | $\frac{1/\text{Year} \times \text{Cost}}{\text{Individuals} \times \text{Cost}}$ |
| $\omega$   | Coefficient of economic measures implementation rate                               | $\frac{\text{Area} \times \text{Year}}{\text{Cost}}$                             |
| $\omega_1$ | Coefficient of natural depletion rate of economic measures                         | 1/Year                                                                           |

The forest-dependent wildlife population ( $W$ ), which entirely depends on forest biomass, grows logistically but is simultaneously depleted by an increase in human population and its corresponding activities as well as a decrease in forest biomass. Population pressure ( $P$ ) on forest resources was assumed to be driven by horizontal population growth and advancements in resource exploitation knowledge [76]; its growth was thought to be directly proportional to the size of the human population and simultaneously decreases naturally and with the application of economic measures [55]. Human activities ( $H$ ) are assumed to grow proportionally to the size of the human population and population pressure. To reduce the intensity of population pressure on resources, various economic measures ( $E_f$ ) in the form of incentives are instituted [24]. The pace of its implementation is assumed to be proportionate to the intensity of population pressure. Furthermore, all model parameters are assumed to be positive.

The theory of dynamical systems is largely concerned with establishing qualitative predictions about the behaviour of systems that evolve over time when the system's control parameters and the initial conditions change [30]. Most of the models developed in practise cannot be solved using analytic techniques; hence, numerical simulations are critical for gaining a bet-

ter understanding [110]. Therefore, the formulated models are analysed both qualitatively and quantitatively. Since most system equations that result from dynamical systems do not possess outright analytical solutions, the stability theory of differential equations was used to perform qualitative analysis [4]. The analysis is performed to fully comprehend the system's ultimate behaviours by thoroughly investigating the underlying conditions such as the existence of equilibrium points, the system's local and global stability, and system permanence.

### 3.3.4 Equilibrium and stability analysis

Determining the equilibrium points and examining their stabilities is crucial for comprehending the behaviour of a dynamical system. This is because it allows the prediction of the long-term behaviour of the system, designing the controls to stabilise the system around a given equilibrium point, identify the critical points, that is, extinction and persistent as well as validation of the accuracy of the model. Thus, for the purpose of understanding the behaviour of the developed dynamical system models, the equilibrium points and their stabilities were analysed.

The process of obtaining the equilibrium points for a system involved solving the equations by setting all derivative terms equal to zero. This allows for the identification of the values of the variables where the system remains constant over time. Following Morgan [81], the local stability analysis of the boundary equilibria was determined using the sign of the eigenvalues of the corresponding Jacobian matrix, whereas for the interior equilibrium point, the direct the Lyapunov method [56] was used after linearising the system around the point. For the the global stability, both the theory of the Lyapunov function [56] and the Volterra-Lyapunov matrix [16, 66] were used after linearising the system around the interior equilibrium point.

The inferences from the sign of the eigenvalues is given as follows: Given a two dimensional system:

$$\begin{aligned} \dot{y}_1(t) &= \theta_1 y_1(t) + \theta_2 y_2(t), \\ \dot{y}_2(t) &= \theta_3 y_1(t) + \theta_4 y_2(t), \end{aligned}$$

with a characteristic equation  $\mathcal{C}_1 \lambda^2 + \mathcal{C}_2 \lambda + \mathcal{C}_3 = 0$ . If the corresponding roots (called eigenvalues) of the characteristic equation are  $\lambda_1$  and  $\lambda_2$ , then Table 6 provides inferences that allow for the determination of the stability of a system. Thus, conclusions can be made about the stability of the system based on the analysis of its eigenvalues.

**Table 6:** General system stability inferences from eigenvalues

| Eigenvalues                                                  | Type of critical point | Stability analysis    |
|--------------------------------------------------------------|------------------------|-----------------------|
| $\lambda_1 \leq \lambda_2 < 0$                               | Sink                   | Asymptotically stable |
| $\lambda_1 \leq \lambda_2 > 0$                               | Source                 | Unstable              |
| $\lambda_1 < 0 < \lambda_2$                                  | Saddle                 | Unstable              |
| $\lambda_{1,2} = \alpha_0 \pm \beta i$ , with $\alpha_0 < 0$ | Inward spiral          | Asymptotically stable |
| $\lambda_{1,2} = \alpha_0 \pm \beta i$ , with $\alpha_0 > 0$ | Outward spiral         | Unstable              |
| $\lambda_{1,2} = \pm \beta i$ ,                              | Center                 | Stable                |

For the interior equilibrium stability analysis, Theorem 3.1 provides the foundation of the analysis.

**Theorem 3.1 (Lyapunov stability theory [21]).** *Let  $(x_1^*, x_2^*) = (0, 0)$  be the equilibrium point of  $\dot{x} = f(x_1, x_2)$  and  $V(x_1, x_2)$  be a continuously differentiable positive definite function in the neighbourhood of the origin. The following conclusions can be made:*

- *If  $V'(x_1, x_2) \leq 0, \forall x_1, x_2 \in \mathbb{U} - \{0\}$ , then the origin is stable.*
- *If  $V'(x_1, x_2) < 0, \forall x_1, x_2 \in \mathbb{U} - \{0\}$ , then the origin is uniformly asymptotically stable.*
- *If  $V'(x_1, x_2) > 0, \forall x_1, x_2 \in \mathbb{U} - \{0\}$ , then the origin is unstable.*

### 3.3.5 The Volterra-Lyapunov matrix theory for global stability analysis

The Lyapunov direct method is commonly used to establish the conditions for the global stability behaviour of the interior equilibrium point due to its simplicity and require little theoretical knowledge to implement [59]. However, because there is no systematic way to construct Lyapunov functions (especially determining the appropriate coefficients of the function), its effectiveness is mostly dependent on trial and error as well as specific problems [16]. Thus, the Volterra-Lyapunov stable matrices approach was applied to establish the conditions for interior equilibrium point to be globally stable. The Approach incorporates the Volterra matrix theory into Lyapunov functions, which avoids the necessity to determine the coefficients under specific conditions [100]. Thus, following Chien [16] and Mayengo [66], the following definitions, notations, and lemmas were used to facilitate the analysis.

**Notation 3.1**

If matrix  $\mathcal{B}$  has the property of symmetry and is a positive (negative) definite, then the matrix is written as  $\mathcal{B} > 0 (< 0)$ . Thus, the following fundamental results of matrix stability are established by Lyapunov.

**Lemma 3.1.** *Let  $\mathcal{B}$  be  $n \times n$  real matrix. Then all eigenvalues of  $\mathcal{B}$  have negative (positive) real parts if and only if there exists a matrix  $\mathcal{K} > 0$  such that  $\mathcal{K}\mathcal{B} + \mathcal{B}^T\mathcal{K}^T < 0 (> 0)$ .*

**Definition 3.7**

A nonsingular matrix  $\mathcal{B}$  is Volterra-Lyapunov stable if there exists a positive diagonal  $n \times n$  matrix  $\mathcal{C}$  such that  $\mathcal{C}\mathcal{B} + \mathcal{B}^T\mathcal{C}^T < 0$ .

Lemma 3.2 states the conditions for all  $2 \times 2$  Volterra-Lyapunov stable matrices.

**Lemma 3.2** (Rinaldi [100]). *Let  $D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$  be a  $2 \times 2$  matrix. Then matrix  $D$  is Volterra-Lyapunov stable if and only if  $d_{11} < 0$ ,  $d_{22} < 0$  and  $\det(D) = d_{11}d_{22} - d_{12}d_{21} > 0$ .*

The following definition is required for the characterization of Volterra-Lyapunov stable matrices of higher dimensions.

**Definition 3.8**

A nonsingular  $n \times n$  matrix  $\mathcal{B}$  is diagonally stable (or positive stable) if there exists a positive diagonal  $n \times n$  matrix  $\mathcal{C}$  such that  $\mathcal{C}\mathcal{B} + \mathcal{B}^T\mathcal{C}^T > 0$ .

Therefore, from definitions (3.7) and (3.8), a matrix  $\mathcal{B}$  is Volterra-Lyapunov stable if and only if its negative matrix,  $-\mathcal{B}$  is diagonally stable.

**Notation 3.2**

For any  $n \times n$  matrix  $\mathcal{B}$ , let  $\check{\mathcal{B}}$  denote the  $(n-1) \times (n-1)$  matrix obtained from  $\mathcal{B}$  by deleting its last row and column.

**Lemma 3.3** (Redheffer [97, 98]). *Let  $D = [d_{ij}]$  be a nonsingular matrix  $n \times n$  matrix ( $n \geq 2$ ) and  $\mathcal{C} = \text{diag}\{c_1, \dots, c_m\}$  be a positive diagonal  $n \times n$ . Let  $\mathbf{H} = D^{-1}$ . Then if  $d_{nn} > 0$ ,  $\check{\mathcal{C}}\check{\mathbf{H}} + (\check{\mathcal{C}}\check{\mathbf{H}})^T > 0$  and  $\check{\mathcal{C}}\check{D} + (\check{\mathcal{C}}\check{D})^T > 0$ , it is possible to choose  $c_m > 0$  such that  $\mathcal{C}D + D^T\mathcal{C}^T > 0$ .*

All steps used to prove the Volterra-Lyapunov stability of matrix  $A_{4 \times 4}$  are summarized in Fig. 5.

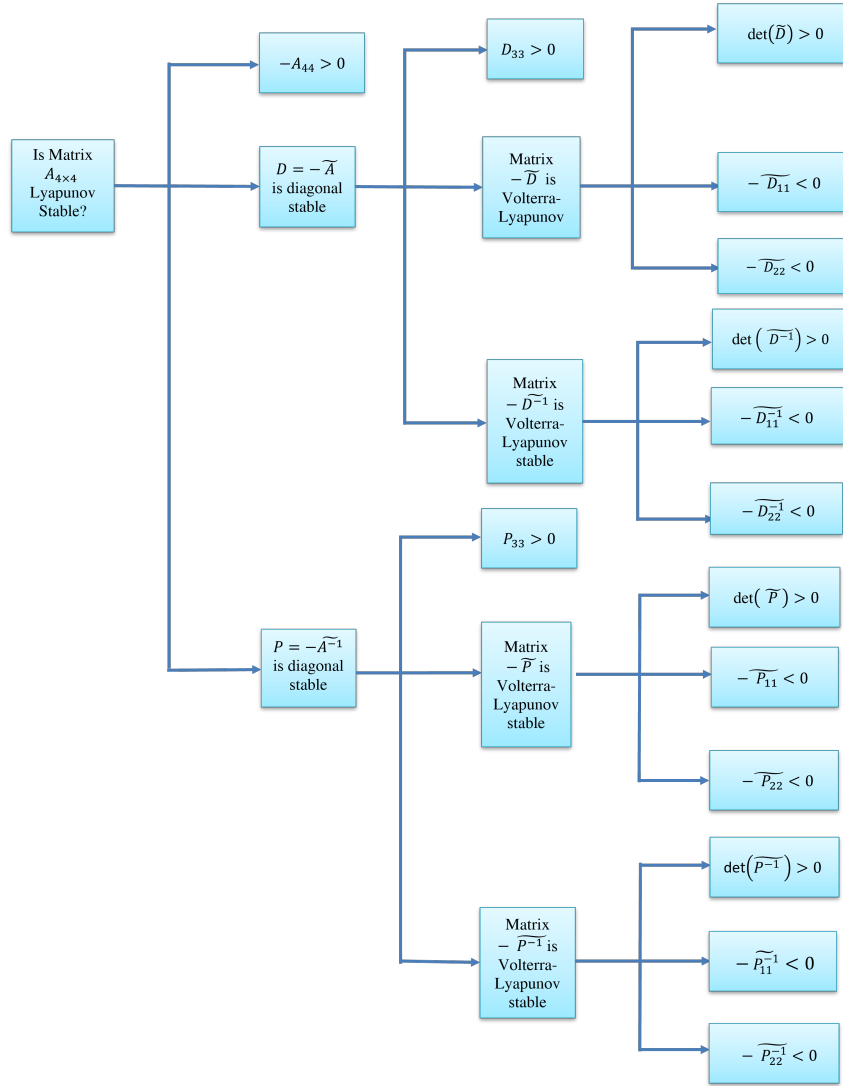


**Figure 5:** Algorithm for proving Volterra–Lyapunov stability of  $4 \times 4$  matrix  $A$

### 3.3.6 System permanence

Ecologically, system permanence means that all species will be extant in the future and that none will fall extinct if they are existing initially [32]. Therefore, it was imperative to determine the conditions for the system permanence. Mathematically, permanence can be analysed using dynamical systems theory, which involves studying the behaviour of differential equations, following Mukherjee [82], let the system of  $n$  species interactions be defined by the differential equation (12):

$$\dot{y}_i = g_i(y_1, y_2, y_3, \dots, y_n), \quad \text{for } i = 1, 2, 3, \dots, n \quad (12)$$

on  $\mathbb{R}_+^n$ , such that  $\mathbb{R}_+^n = \{y \in \mathbb{R}^n | y_i > 0, 1 \leq i \leq n\}$  and  $y_i$  denote the density of the  $i^{th}$  species. The system (12) is said to be permanent (uniformly persistent) if there exist constants  $\delta_1$  and  $\delta_2$  such that for all solutions  $y_i(t)$  with initial conditions  $y_i(0) > 0$  satisfy  $\delta_1 \leq y_i(t) \leq \delta_2$  for all  $t \geq t_0$ . For this case,  $t_0$  may depend on initial condition  $y_i(0)$ , however,  $\delta_1$  and  $\delta_2$  do not. More precisely, the following definition follows:

**Definition 3.9**

The system (12) is permanent if there exist a compact set in  $\mathbb{R}_+^n$  that contains  $\omega$ -limit points of all orbits initiating in  $\mathbb{R}_+^n$ , that is, if there exists a constant  $\delta$  such that  $\liminf_{t \rightarrow \infty} y_i(t) \geq \delta$ , where  $y_i(t) > 0 \forall i$ .

**Definition 3.10**

A solution  $y(t) = \{y_1(t), y_2(t), \dots, y_n(t)\}$  of equation (12) is weakly persistent provided equation (13) holds.

$$\limsup_{t \rightarrow \infty} y_i(t) > 0 \quad \forall i. \quad (13)$$

**Definition 3.11**

The system (12) which describes the evolution of  $y(t)$  is dissipative if the trajectories are asymptotically uniformly bounded in positive  $t$ , that is, if there exists a constant  $\xi$  such that

$$\limsup_{t \rightarrow \infty} \|y(t)\| \leq \xi.$$

### 3.4 Parameter Estimation

After understanding the asymptotic behaviour and long-term qualitative results of the model system, the numerical simulation was performed to obtain the numerical solutions of the system and predict the variations of the dynamics over time for different parameter values. Since this process depends on the numerical values of parameters, Least squares and Markov Chain Monte Carlo (MCMC) were used to estimate the numerical values of the model parameters through coding in MATLAB R2022a software.

#### 3.4.1 Least Squares

Least squares is a popular method used in parameter estimation, which involves finding the best-fit parameters for a model to a given set of data. The goal of this method is to minimize the sum of the squares of the differences between the predicted values of the model and the observed

data. By minimizing this error, the estimated parameters will better represent the true parameters of the system. It worth noting that the formulated models are time dependent differential equations with the form of equation (11). The set of the parameters  $\mathbf{A}$  were estimated after simulating the model by using literature and assumed values, thereafter independent and identically distributed (iid) Gaussian noise with variance 0.02 were added to corrupt the simulated data. The corrupted data were assumed to be true observations.

Suppose  $y_i$  is the observed data and  $f(\mathbf{x}_i, \mathbf{A}, t)$  represents the model prediction for input  $\mathbf{x}_i$ ,  $\mathbf{A}$  and  $t$ , then the residuals denoted by  $\mathbf{r}_i$  is defined as follows:

$$\mathbf{r}_i = y_i - f(\mathbf{x}_i, \mathbf{A}, t).$$

The Sum of Squared Residuals ( $SSR$ ) is given by:

$$SSR = \sum_{i=1}^n (\mathbf{r}_i)^2 = \sum_{i=1}^n (y_i - f(\mathbf{x}_i, \mathbf{A}, t))^2,$$

$n$  is the number of the data points. The sum of the squares of the residuals is minimized by adjusting the parameter values until the best fit is achieved, that is

$$\hat{\mathbf{A}} = \arg \min_{\mathbf{A}} \sum_{i=1}^n (y_i - f(\mathbf{x}_i, \mathbf{A}, t))^2.$$

This implies, the best fitting set of parameters  $\hat{\mathbf{A}}$  are obtained after solving the partial derivative of  $SSR$  with respect to  $\mathbf{A}$  equal to zero.

$$\frac{\partial SSR}{\partial \mathbf{A}} = \frac{\partial \left( \sum_{i=1}^n (y_i - f(\mathbf{x}_i, \mathbf{A}, t))^2 \right)}{\partial \mathbf{A}} = 0.$$

Due to the non-linearity behaviour of the formulated models and number of the parameters to be estimated it is difficult to find the exact solutions. Thus, the built in MATLAB R2022a function `fminsearch` which uses Nelder-Mead simplex algorithm to obtain the local minimizers of the sum of the squares of residuals was used (Algorithm 1) [54].

### 3.4.2 Markov Chain Monte Carlo Method

The Markov Chain Monte Carlo (MCMC) method is a Bayesian technique that is used to estimate parameter values through generating samples from the posterior distribution of the parameters given the observed data [67]. Given equation (11), the posterior distribution of parameters  $\mathbf{A}$  is given by:

$$p(\mathbf{A} | y_1, y_2, y_3, \dots, y_n).$$

---

**Algorithm 1** (fminsearch)

---

**Require:** Function handle  $fun$ , initial guess  $x_0$ , and termination tolerance  $\epsilon$

**Ensure:** Optimal solution  $x^*$  and objective function value  $f(x^*)$

```
1:  $x \leftarrow x_0$                                 ▷ Set the initial guess for the optimal solution
2:  $f_x \leftarrow fun(x)$                           ▷ Evaluate the objective function at the initial guess
3:  $i \leftarrow 0$                                 ▷ Set the iteration counter to zero
4: while  $\|\nabla f_x\| > \epsilon$  and  $i < i_{\max}$  do    ▷ Repeat until convergence or maximum iterations reached
5:    $s_d \leftarrow -\nabla f_x / \|\nabla f_x\|$           ▷ Calculate the search direction
6:    $x_{\text{new}} \leftarrow x + \alpha_0 s_d$           ▷ Update the solution with a step in the search direction
7:    $f_{\text{new}} \leftarrow fun(x_{\text{new}})$           ▷ Evaluate the objective function at the new solution
8:   if  $f_{\text{new}} < f_x$  then                      ▷ If the new solution improves the objective function, accept it
9:      $x \leftarrow x_{\text{new}}$                         ▷ Update the optimal solution
10:     $f_x \leftarrow f_{\text{new}}$                       ▷ Update the objective function value
11:   else                                          ▷ If the new solution doesn't improve the objective function, reduce the step size
12:      $\alpha_0 \leftarrow \alpha_0 / 2$               ▷ Reduce the step size
13:   end if
14:    $i \leftarrow i + 1$                             ▷ Update the iteration counter
15: end while
16:  $x^* \leftarrow x$                                 ▷ Set the optimal solution to the final value of  $x$ 
17:  $f(x^*) \leftarrow f_x$                           ▷ Set the objective function value to the final value of  $f_x$ 
```

---

Due to complexity and non-linearity of the model, the posterior distribution is often analytically intractable, so the MCMC method uses a Markov chain to simulate the posterior distribution. The chain is designed to converge to the posterior distribution, and the samples drawn from the chain can be used to estimate the parameters of the model. The approach involves generating a sequence of samples from a proposal distribution denoted by  $q(\Lambda^* | \Lambda^{(i)})$  that forms a Markov chain and may depend on the current value.

An accept/reject rule is then applied to correct the proposal mechanism and simulate from the target distribution of interest, which is an invariant distribution that the Markov chain is designed to converge to. The resulting sequence of samples can be used to approximate the distribution or integrals of interest. This approach is particularly useful when direct sampling from the target distribution is difficult, and the Markov chain provides an efficient way to approximate it. Suppose  $\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}, \Lambda^{(4)}, \dots, \Lambda^{(n)}$ , are the MCMC samples, the the approximated

expectation of function  $g(\mathbf{A})$  is given by:

$$\mathbf{E} \left[ (g(\mathbf{A})) \mid y_1, y_2, y_3, \dots, y_n \right] \approx \frac{1}{n} \sum_{i=1}^n g \left( \mathbf{A}^{(i)} \right).$$

There are several popular MCMC algorithms available, such as Gibbs sampling, Hamiltonian Monte Carlo (HMC), Metropolis-Hastings algorithm, No-U-Turn Sampler (NUTS), Differential Evolution Markov Chain Monte Carlo (DE-MCMC), Slice sampling, and Sequential Monte Carlo (SMC) [67]. In this study, the Adaptive Metropolis-Hastings algorithm was used due to its ability to automatically tune the proposal distribution, which overcomes the need for tedious trial and error in finding a suitable proposal distribution [67, 117]. This feature is especially useful for complex models where finding a suitable proposal distribution is a challenging and time-consuming task.

The proposal distribution in MCMC often uses a Gaussian distribution, and a well-tuned covariance matrix is crucial for a successful sampling process. To achieve this, the adaptive Metropolis algorithm updates the covariance matrix on-the-fly using the empirical distribution of generated MCMC samples [42]. The covariance matrix in this algorithm is computed using a formula which takes into account the covariance of the generated MCMC samples up to a certain iteration. Specifically, the formula used is given by Haario [42] as follows: first, the covariance of the available generated MCMC samples up to the current iteration is computed, denoted as  $\Sigma_i$ . Then, a small positive value  $e$  is added to a scaled identity matrix  $\mathbf{I}_m$ , where  $m$  is the number of parameters to be estimated. This is done to ensure that the resulting matrix is not singular. Mathematically the formula for computing the covariance matrix is given by:

$$\Sigma_i = \text{cov} \left( \mathbf{A}^{(0)}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \mathbf{A}^{(4)}, \dots, \mathbf{A}^{(i)} \right) + e\mathbf{I}_m. \quad (14)$$

Algorithm 2 was used to implement the Adaptive Metropolis-Hastings in MATLAB R2022a software for estimating the parameter values.

### 3.5 Uncertainty and Sensitivity Analysis of the Input Model Parameters

Mathematical models formulated to study human-forest interaction are theoretical representations of physical systems that involve uncertain input parameters and initial conditions [63]. Uncertainties may arise from various sources such as measurement errors, model simplifications, and incomplete knowledge of the system.

---

**Algorithm 2** (Adaptive Metropolis-Hastings)

---

1: Generate the initial point  $\mathbf{A}^{(0)}$  from the probability distribution  $p_0(\mathbf{A})$ . Set the non-adaptive period  $n_0$  and select the initial covariance matrix  $\Sigma_0$ .

2: **for**  $i = 1, 2, \dots$ , **do**

3:     Sample a candidate point  $\mathbf{A}^*$  from the proposal distribution  $q(\mathbf{A}^*|\mathbf{A}^{(i-1)})$ .

4:     Compute the probability of accepting a proposed sample ( $\beta$ )

$$\beta(\mathbf{A}^{(i-1)}, \mathbf{A}^*) = \min \left\{ 1, \frac{p(\mathbf{A}^*|y_1, y_2, \dots, y_n) q(\mathbf{A}^{(i-1)}|\mathbf{A}^*)}{p(\mathbf{A}^{(i-1)}|y_1, y_2, \dots, y_n) q(\mathbf{A}^*|\mathbf{A}^{(i-1)})} \right\}.$$

5:     Establish acceptance/rejection rule: set

$$\mathbf{A}^{(i)} = \begin{cases} \mathbf{A}^* & \text{if } u < \beta(\mathbf{A}^*|\mathbf{A}^{(i-1)}), \quad \text{where } u \sim \mathbb{U}(0, 1), \\ \mathbf{A}^{(i-1)} & \text{otherwise.} \end{cases}$$

6:     Update the covariance matrix after every  $n_0^{th}$  iterations ( $i \geq n_0$ ) by using equation (14).

7:      $i = i + 1$ .

8: **end for**

---

Therefore, uncertainty and sensitivity analysis were carried out to quantify the effects of uncertainty and variability of the input parameters on the model outputs. The analysis involves the characterisation of the sources of uncertainty in a model and propagating that uncertainty through the model and identification of input parameters that have significant impact on the model outputs [6]. In this study, a combination of two statistical techniques, Latin Hypercube Sampling (LHS) and Partial Rank Correlation Coefficients (PRCC), were used to investigate the behaviour of the model system (73) and account for the uncertainty in the input parameters.

### 3.5.1 Latin Hypercube Sampling and Partial Rank Correlation Coefficients

Latin Hypercube Sampling (LHS) and Partial Rank Correlation Coefficients (PRCC) are both uncertainty and sensitivity analysis methods that can be used together to gain insights into the behaviour of a model and identify important input parameters [63]. The LHS is a sampling technique that generates a set of input parameter values that are representative of the distribution of each parameter. This allows for a systematic exploration of the parameter space and reduces the likelihood of missing important regions. The method involves dividing each input parameter into equally likely intervals and then randomly selecting one value from each interval to form a

sample [46]. The reason for selecting the LHS sampling technique over other techniques is that it provides an impartial estimation of the average model output and requires a lower number of samples to achieve a comparable level of accuracy [6, 99].

The PRCC is the sensitivity analysis technique used to quantify the degree of correlation between model input parameters and output responses. It is a dimensionless quantity that ranges from -1 to 1 and measures the linear relationship between two variables while controlling for the effects of other variables. A positive value indicates a positive correlation, while a negative value indicates a negative correlation. The PRCC was selected with the assumption that the model output and input parameters exhibit non-linear and monotonic relationships [99].

Consider equation (11), let the expected model output be  $y = f(\mathbf{X}, \mathbf{A})$ , and suppose the parameter space is given by  $\mathbf{A} \in \mathbb{R}^2$ , that is,  $\mathbf{A} = (\vartheta, \theta)$ , then Uncertainty and Sensitivity analysis of the input model parameters by using LHS/PRCC approach follows the following steps:

- (i) Define the input parameters of the model and their respective ranges.
- (ii) Use LHS to generate a sample set of input parameters. The range of each parameter is divided into equal intervals, and one sample is randomly drawn from each interval, ensuring that the samples are representative of the input parameter space. Suppose the input parameters  $\theta$  and  $\vartheta$  are uniformly and normally distributed, respectively, as shown in Fig. 6 that is

$$\theta \sim Unif(\theta_{\min}, \theta_{\max}), \quad \vartheta \sim \mathcal{N}(\mu_{\vartheta}, \sigma_{\vartheta}^2),$$

then, Fig. 6 shows how  $n$  samples can be drawn from the distribution by using LHS.

- (iii) Form the LHS matrix of the sampled parameters, let say matrix  $\mathbf{X}$ , further, run the model for each set of input parameters in the sample set, and record the model output. Let the model output be matrix  $\mathbf{Y}$ :

$$\mathbf{X} = \begin{pmatrix} \theta_1 & \vartheta_1 \\ \theta_2 & \vartheta_2 \\ \theta_3 & \vartheta_3 \\ \vdots & \vdots \\ \theta_n & \vartheta_n \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} f(\theta_1, \vartheta_1) \\ f(\theta_2, \vartheta_2) \\ f(\theta_3, \vartheta_3) \\ \vdots \\ f(\theta_n, \vartheta_n) \end{pmatrix}.$$

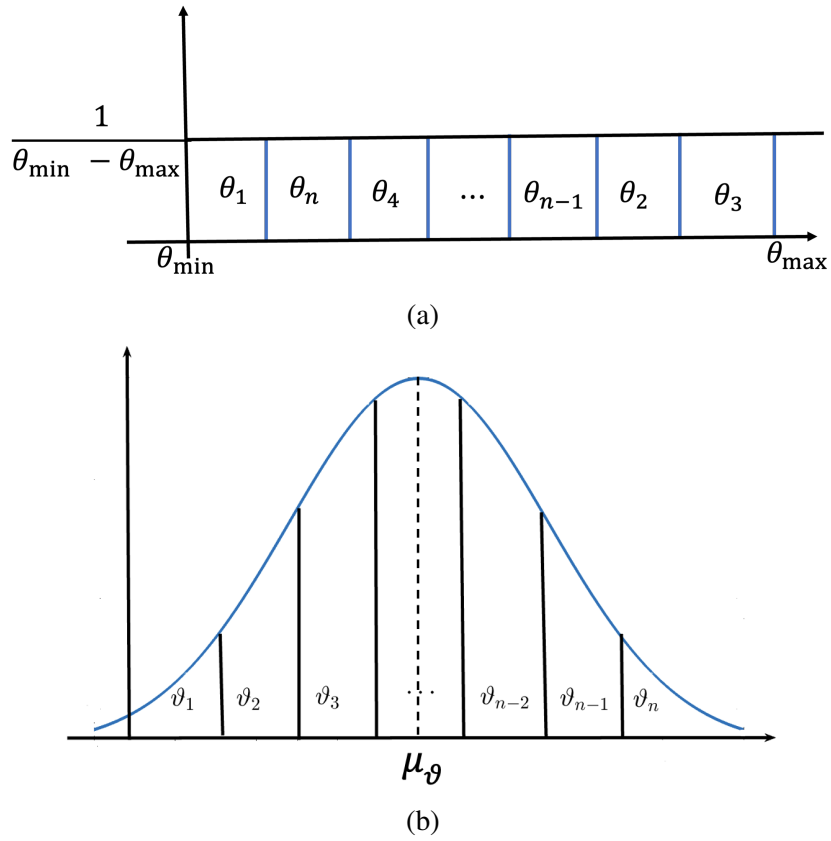


**Figure 6:** Probability distributions for input parameters  $\theta$  and  $\vartheta$

- (iv) Calculate the correlation coefficient (PRCC) between each input parameter and the model output. This is done by ranking the samples based on the input parameter of interest, and then calculating the correlation between the ranked samples and the model output as follows:

$$r_{x_j, y} = \frac{\text{Cov}(y, x_j)}{\sqrt{\text{Var}(y) \text{Var}(x_j)}}.$$

- (v) Analyse the results to identify the most influential input parameters and their impact on the model output. This can be done by plotting the PRCC values or conducting a variance-based sensitivity analysis. In this study, the plots of the PRCC values approach is adopted [1].

### 3.6 Numerical Simulation

In many cases, analytical solutions to mathematical models are not available or are too difficult to derive [58]. In these cases, numerical methods are used to approximate the solutions. In this study, the built-in MATLAB2022a `ode45` function which is a powerful numerical method for

solving ordinary differential equations was used to approximate the solutions of the formulated models. The algorithm is a popular numerical method that uses a variable step size and variable order Runge-Kutta method to approximate the solution of the ODE system [3]. It starts with an initial step size  $h_0$  and iteratively computes the solution at subsequent time steps. At each step, the algorithm estimates the error using a fourth-order and fifth-order solution and adjusts the step size for the next iteration to achieve a specified tolerance. The use of a variable step size allows for efficient and accurate computation of the solution while minimizing the number of function evaluations.

Given an initial value problem of the form:  $y'(t) = f(t, y(t))$ ,  $y(t_0) = y_0$ , where  $t_0$  is the initial time and  $y_0$  is the initial value of  $y(t)$ , the `ode45` function computes an approximation to the solution  $y(t)$  over a specified time interval  $[t_0, t_{\text{final}}]$ , where  $t_{\text{final}}$  is the final time. The basic steps of the `ode45` are described in Algorithm 3.

---

**Algorithm 3** (`ode45`)

---

**Require:** ODE system  $y' = f(t, y)$ , initial condition  $y_0$ , final time  $t_{\text{final}}$ , and maximum step size  $h_{\text{max}}$

**Ensure:** Approximate solution  $y(t)$  at time  $t_{\text{final}}$

```

1: Set  $t \leftarrow 0$  and  $y \leftarrow y_0$                                 ▷ Set initial time and initial condition
2: while  $t < t_{\text{final}}$  do                                          ▷ Loop until final time reached
3:    $h \leftarrow \min(h_{\text{max}}, t_{\text{final}} - t)$                     ▷ Set step size based on maximum step size and remaining time
4:    $k_1 \leftarrow f(t, y)$ 
5:    $k_2 \leftarrow f(t + \frac{1}{2}h, y + \frac{1}{2}hk_1)$ 
6:    $k_3 \leftarrow f(t + \frac{1}{2}h, y + \frac{1}{2}hk_2)$ 
7:    $k_4 \leftarrow f(t + h, y + hk_3)$ 
8:    $y \leftarrow y + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$               ▷ Update solution using 4th order Runge-Kutta method
9:    $t \leftarrow t + h$                                             ▷ Update time
10: end while
11: return  $y$                                                     ▷ Return approximate solution at final time

```

---

Based on methods and materials presented in this chapter, the results and discussion are presented in the next chapter.

## CHAPTER FOUR

### RESULTS AND DISCUSSION

#### 4.1 ARIMA model for Forest Loss Forecasting in Tanzania

The forecasting of forest loss in Tanzania has been made possible with the use of the ARIMA model. To ensure accuracy in the modelling process, the conceptual framework of Box and Jenkins methodology has been followed (Section 3.2).

##### 4.1.1 Data exploration

The plot of the time series data, illustrated in Fig. 7a, provides valuable insights into the historical forest loss pattern. Upon close examination of the figure, a prominent trend emerges, indicating a consistent and notable increase in forest loss over the years. This observable upward trajectory raises important concerns about the long-term ecological health of the forested areas. The identified trend is further emphasized through the moving average plot presented in Fig. 7b. Utilising a smoothing window with a size of 8, this plot effectively filters out short-term fluctuations and highlights the underlying trend with enhanced clarity. The application of the moving average technique not only aids in revealing the trend's direction but also underscores the persistence of the observed forest loss increase.

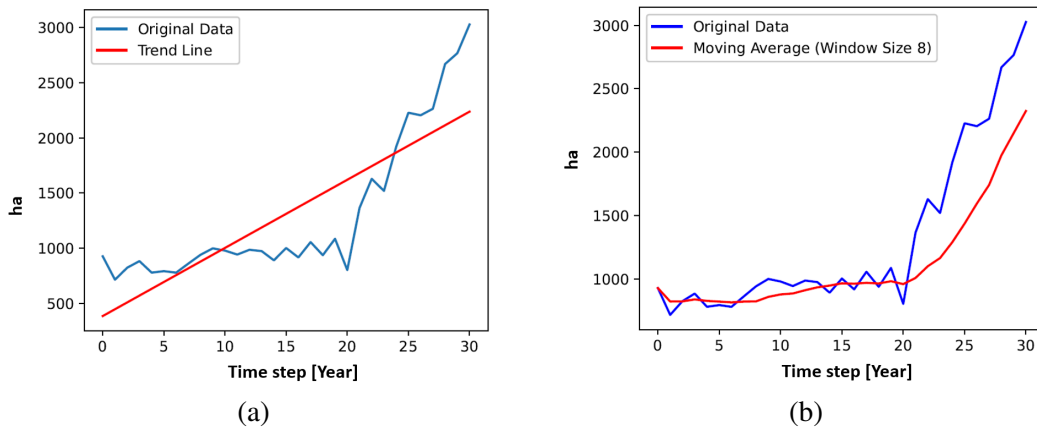


**Figure 7:** (a) Plot showing the forest loss in hectares (ha) from 1991 to 2021 with trend line.  
(b) Moving average plot of the forest loss data over the same interval

This persistent upward trend is a matter of significant environmental significance. The continuous degradation of forested regions, as evidenced by the time series and moving average plots,

has far-reaching implications. It portends potential ecological imbalances, wherein the delicate interactions between flora and fauna could be disrupted. Additionally, the escalating forest loss raises alarms about the potential loss of biodiversity, as numerous plant and animal species rely on these habitats for survival.

To identify temporal dependencies and autocorrelation patterns in the univariate time series data, a lag scatter plot is presented in Fig. 8. This plot displays the relationships between the current value and its lagged values at lags 1, 2, and 3. It is evident from the plot that the current value exhibits a strong correlation with its immediate past values, indicating a notable influence of previous data points on the current value. Such autocorrelation patterns suggest that the time series data is not entirely random, and past observations play a significant role in shaping its behaviour over time.

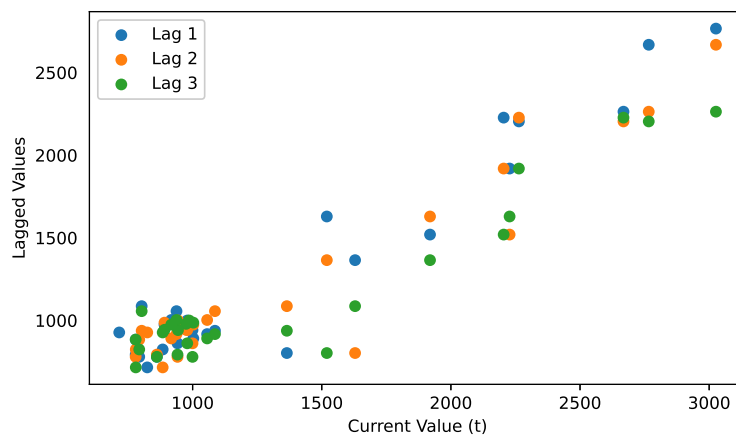


**Figure 8:** Lag scatter plot showing strong correlation among the lags

The Box and Jenkins methodology is effective when dealing with a stationary dataset. To assess stationarity, a stationary test was conducted using the Augmented Dickey Fuller (ADF) test. The null hypothesis for the test is that there exists a unit root (non-stationarity) in the series. Using the built-in Python function `adfuller`, the obtained ADF statistic is 1.243806 with a p-value greater than 0.05. Therefore, based on the p-value, the original series is not stationary ( $d \neq 0$ ).

To address non-stationarity, the value of  $d$  was determined by observing the nature of the Auto-correlation Function (ACF) plots of the differenced series. From Fig. 9, it can be observed that the time series reaches stationarity after order two of differencing. As a result, the value of  $d$  was set to 2 to achieve a stationary dataset for further analysis and modelling.

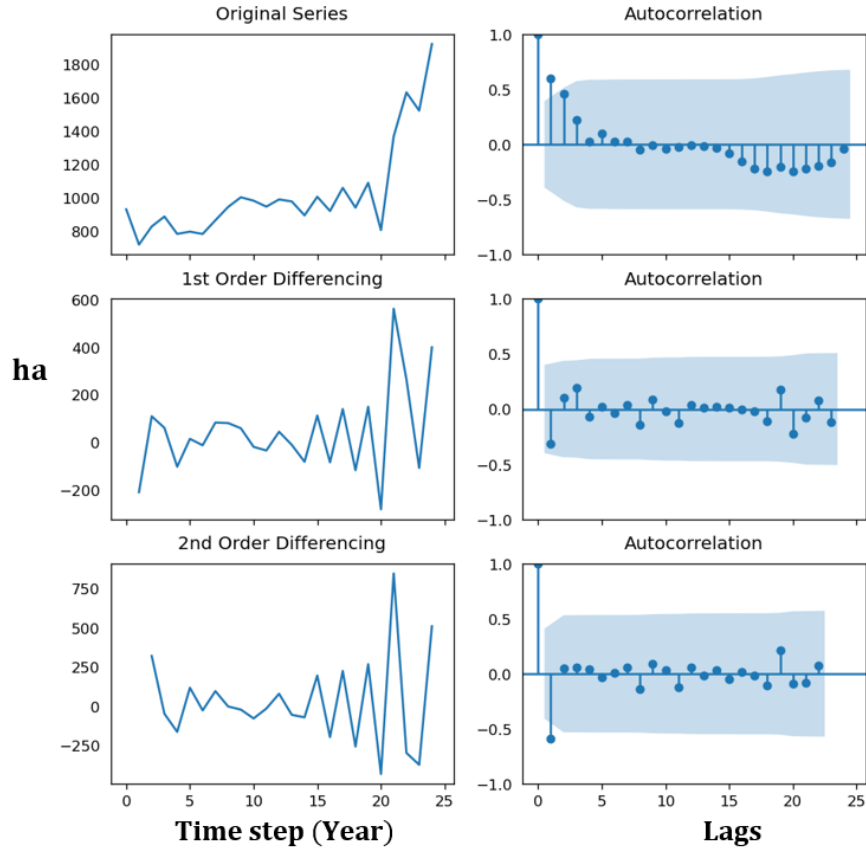


**Figure 9:** The order of differencing with respective ACF plots

#### 4.1.2 Model identification

The ARIMA model is usually expressed as  $ARIMA(p, d, q)$ , where  $p$ ,  $d$ , and  $q$  represent the order of the autoregressive (AR) component, the degree of differencing required to make the series stationary, and the order of the moving average (MA) component, respectively.

The parameter  $p$  in the ARIMA model represents the number of lags that cross the significance limit in the Partial Autocorrelation Function (PACF) plot. In Fig. 10, it can be observed that the PACF plot of the second-order differencing exhibits two lags that significantly exceed the significance line. As a result, to prevent over-fitting, the value of  $p$  is arbitrarily set to one ( $p = 1$ ). Similarly, the parameter  $q$  in the is determined from the Autocorrelation Function (ACF) plot of the second differencing series. Upon analysing the ACF plot, it is noticed that only one lag is above the significance limit. Consequently, the value of  $q$  is identified as zero ( $q = 0$ ). Thus, using the ACF and PACF plots of the second-order differencing, an  $ARIMA(1,2,0)$  model is established.

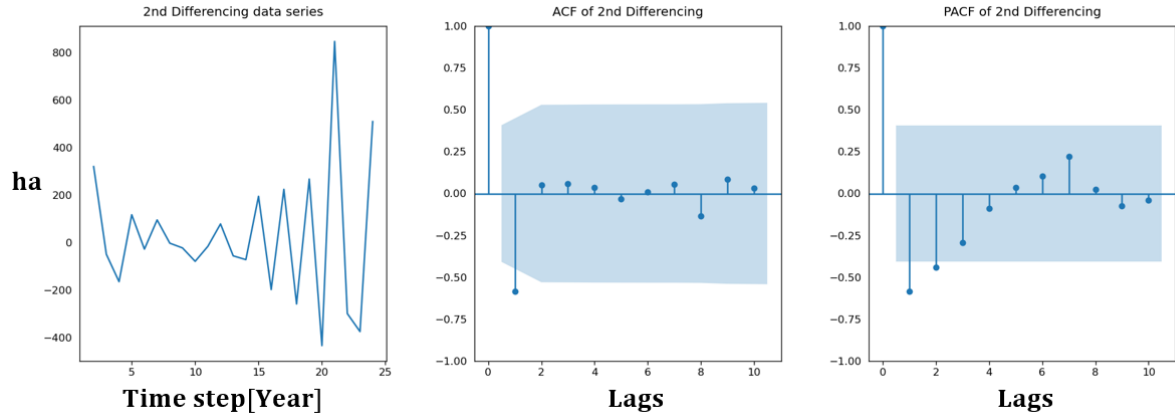


**Figure 10:** Second order differencing with respective ACF and PACF plots

The stepwise search approach was used to select the best ARIMA model. The `auto.arima` Python function was employed to determine the optimal values of  $p$ ,  $d$ , and  $q$ , as well as to estimate the model parameters. The initial values of  $p$  and  $q$  were set to one each, and the Augmented Dickey-Fuller (ADF) test was used to test the stationarity of the series. The results of the model selection process are presented in Table 7.

**Table 7:** Summary of `auto.arima` stepwise search to minimize AIC

| Model                  | AIC value | Model                  | AIC value      |
|------------------------|-----------|------------------------|----------------|
| ARIMA(1,2,1) intercept | inf       | ARIMA(2,2,2)           | inf            |
| ARIMA(0,2,0) intercept | 328.628,  | ARIMA(2,2,0)           | 310.352        |
| ARIMA(1,2,0) intercept | 318.231   | ARIMA(0,2,2)           | inf            |
| ARIMA(0,2,0)           | 326.832   | ARIMA(1,2,2)           | inf            |
| ARIMA(2,2,0) intercept | 311.992   | ARIMA(1,2,0)           | 316.431        |
| ARIMA(3,2,0) intercept | 311.295   | ARIMA(0,2,1)           | 311.321        |
| ARIMA(3,2,1) intercept | 311.808   | ARIMA(4,2,0) intercept | 312.818        |
| ARIMA(2,2,1) intercept | 310.585   | <b>ARIMA(1,2,1)</b>    | <b>309.336</b> |
| ARIMA(3,2,2) intercept | inf       | ARIMA(2,2,1)           | 310.063        |

Based on the AIC, the ARIMA(1,2,1) model was observed to have the lowest AIC value, indicating that it is the best model according to the `auto.arima` function.

### 4.1.3 Parameter estimation for ARIMA(1,2,0) and ARIMA(1,2,1) models

The parameters of the model were estimated using the `statsmodels` Python package [102]. The summary of the model output is presented in Table 8. From the table, it can be observed that the estimated coefficient of the autoregressive term AR(1) is  $-0.6929$ , and its p-value is less than 0.05. The significance of the p-value indicates that the AR(1) term is statistically significant in the model. This means that the previous observation at lag one is relevant and has a significant impact on the current observation. The negative coefficient indicates that there is a negative relationship between the current observation and the observation at lag one. The value of AR(1) close to  $-1$  suggests a strong negative correlation between the current and previous observations, indicating that the series tends to revert to its mean over time.

**Table 8:** ARIMA(1,2,0) model results

| <b>Dep. Variable:</b> | Forest Loss in Ha |         |        |       | <b>No. Observations:</b> | 25       |
|-----------------------|-------------------|---------|--------|-------|--------------------------|----------|
| <b>Model:</b>         | ARIMA(1, 2, 0)    |         |        |       | <b>Log Likelihood</b>    | -156.215 |
|                       |                   |         |        |       | <b>AIC</b>               | 316.431  |
|                       |                   |         |        |       | <b>BIC</b>               | 318.702  |
|                       | coef              | std err | z      | p>  z | [0.025                   | 0.975]   |
| <b>ar.L1</b>          | -0.6921           | 0.113   | -6.123 | 0.000 | -0.914                   | -0.471   |
| <b>sigma2</b>         | 4.569e+04         | 1e+04   | 4.569  | 0.000 | 2.61e+04                 | 6.53e+04 |

The residual plots of the ARIMA(1, 2, 0) model, as shown in Fig. 11, reveal that the residual errors are random and do not exhibit any discernible patterns. The probability density function of the residuals appears to be normally distributed around mean zero, and the variance is approximately uniform. The normality of the residuals suggests that the model adequately captures the underlying patterns in the data, and any remaining deviations are due to random variation rather than systematic errors. Therefore, the characteristics of the residuals confirms the appropriateness of the ARIMA(1, 2, 0) model for predicting the time series data.

The summary of the ARIMA(1,2,1) model is presented in Table 9. The coefficients of AR(1) and MA(1) were found to be highly significant ( $p < 0.05$ ). The residual plot of the ARIMA(1,2,1) model is shown in Fig. 12. From this figure, it can be observed that the residuals are normally distributed around mean zero with uniform variance, suggesting that the predicted values from the ARIMA(1,2,1) model are reliable.

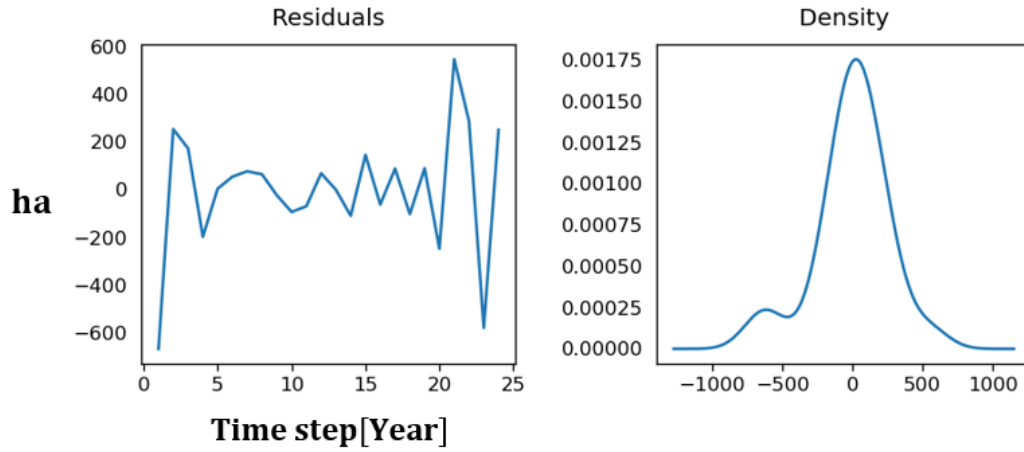


**Figure 11:** Residuals plots for ARIMA(1,2,0) with respectively probability distribution with standardised scale which shows the residual are normally distributed with mean 0

**Table 9:** ARIMA(1,2,1) model results

| <b>Dep. Variable:</b> | Forest Loss in Ha |          |        | <b>No. Observations:</b> | 25       |          |
|-----------------------|-------------------|----------|--------|--------------------------|----------|----------|
| <b>Model:</b>         | ARIMA(1, 2, 1)    |          |        | <b>Log Likelihood</b>    | -151.668 |          |
|                       |                   |          |        | <b>AIC</b>               | 309.336  |          |
|                       |                   |          |        | <b>BIC</b>               | 312.743  |          |
|                       | coef              | std err  | z      | p >  z                   | [0.025   | 0.975]   |
| <b>ar.L1</b>          | -0.4701           | 0.144    | -3.271 | 0.001                    | -0.752   | -0.188   |
| <b>ma.L1</b>          | -0.7619           | 0.207    | -3.689 | 0.000                    | -1.167   | -0.357   |
| <b>sigma2</b>         | 2.977e+04         | 6721.577 | 4.428  | 0.000                    | 1.66e+04 | 4.29e+04 |

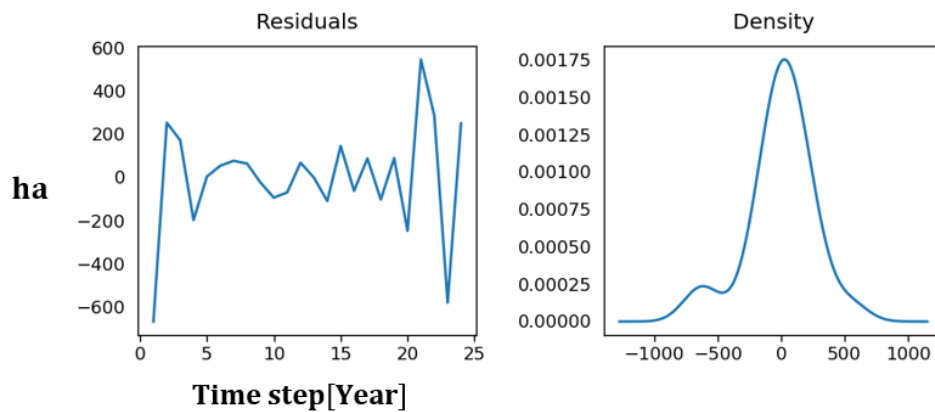


**Figure 12:** Residuals plots for ARIMA(1,2,1) with respectively probability distribution with standardised scale which shows the residual are normally distributed with mean 0

The two models, that is, (ARIMA(1,2,0) and ARIMA(1,2,1)) were validated using test data series, as shown in Fig. 13. Model performance metrics such as Mean Absolute Error (MAE), Mean Absolute Percentage Error (MAPE), and Root Mean Squared Error (RMSE) were used to measure the performance of the two models. The results are presented in Table 10. For ARIMA(1,2,0), MAPE of around 3.8% implies that the model is approximately 96.2% accurate in predicting the next six observations. In comparison, ARIMA(1,2,1) has an accuracy of around 90.6% (MAPE = 9.04%). Thus, ARIMA(1,2,0) proves to be a more reliable model for forest loss forecast, as it has lower error metric values for MAE, MAPE, and RMSE.

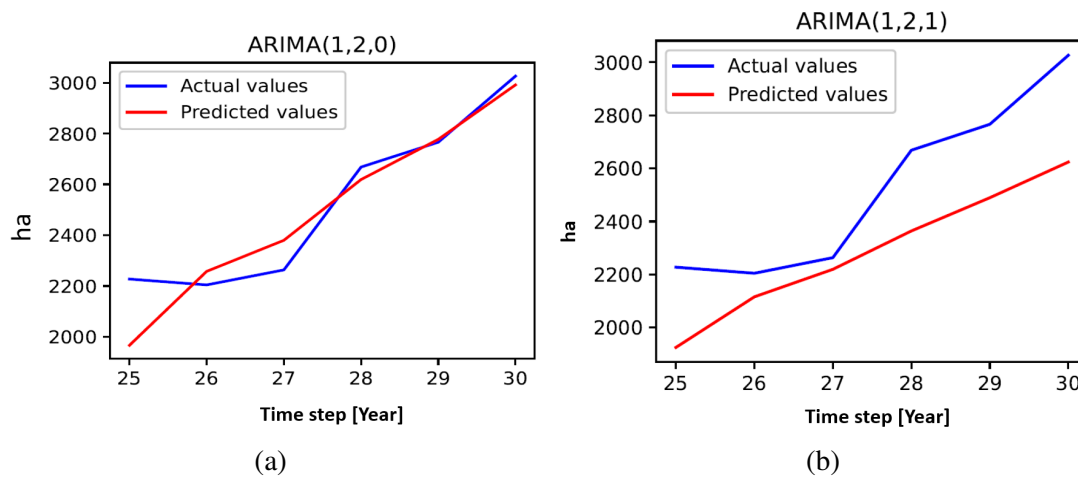


**Figure 13:** Forest land loss prediction: Comparison of ARIMA(1,2,0) and ARIMA(1,2,1)

**Table 10:** Model performance measures

|              | MAE      | MAPE    | RMSE     |
|--------------|----------|---------|----------|
| ARIMA(1,2,0) | 87.6348  | 0.0377  | 121.2642 |
| ARIMA(1,2,1) | 236.5554 | 0.09047 | 268.5683 |

#### 4.1.4 Model diagnostic and forecast

The residuals plots were used to perform a diagnostic on the ARIMA(1,2,0) model, the results are shown in Fig. 14. Starting from the top left corner, the plot of the standardized residuals shows that the residual errors fluctuate around a mean of zero and have a uniform variance. Moving to the top right, the density plot of the residuals suggests that they are normally distributed around mean zero. The bottom left displays the qq-plot, revealing that nearly every dot falls within the line, indicating that the distribution of the residuals is not skewed. Lastly, the

ACF plot shows that the residual errors are not correlated. Absence of autocorrelation indicates that the residual errors do not exhibit any patterns that were not captured by the model. The lack of autocorrelation in the residuals validates the appropriateness of the ARIMA(1,2,0) model in capturing the underlying patterns in the data without any additional predictors.

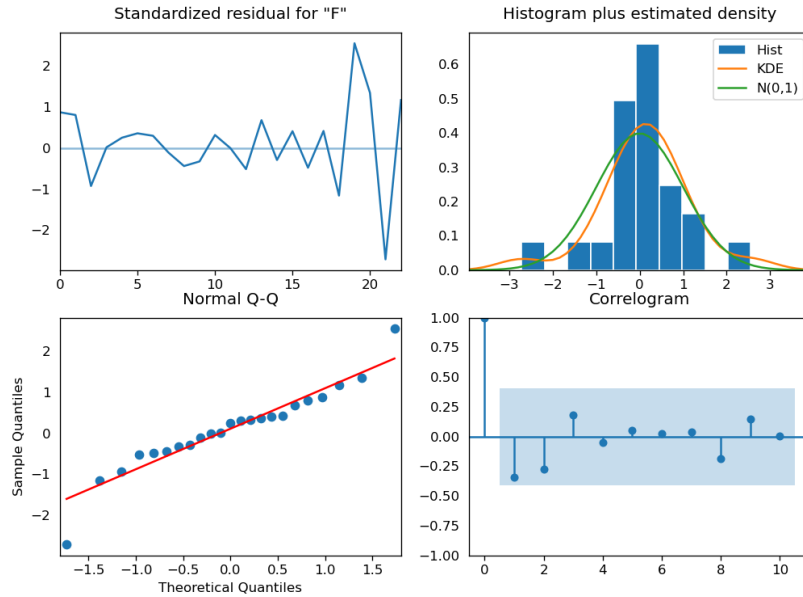


**Figure 14:** Residuals plots, from top left corner is standardise residuals, histogram and estimated density, qq-plot and autocorrelation

The forecast of forest loss using the ARIMA(1,2,0) model is depicted in Fig. 15. The model demonstrates strong predictive capabilities, as evidenced by the close match between the predicted values and actual values, shown in Fig. 15a. This accuracy attests to the effectiveness of the ARIMA approach in forecasting time series data.

#### Remark 4.2

The model's forecast reveals a disconcerting trend, predicting a notable increase in forest loss over the course of the next ten years, as visually depicted in Fig. 15b. This worrisome trajectory raises significant concerns regarding the preservation and sustainable management of our forests. Forests are crucial components of Sustainable Development Goal 15 (SDG 15), which aims to safeguard, restore, and promote the sustainable use of terrestrial ecosystems. The escalated loss of forests outlined by the model's forecast poses a considerable challenge to achieving SDG 15, hampering progress toward a sustainable and balanced ecosystem.

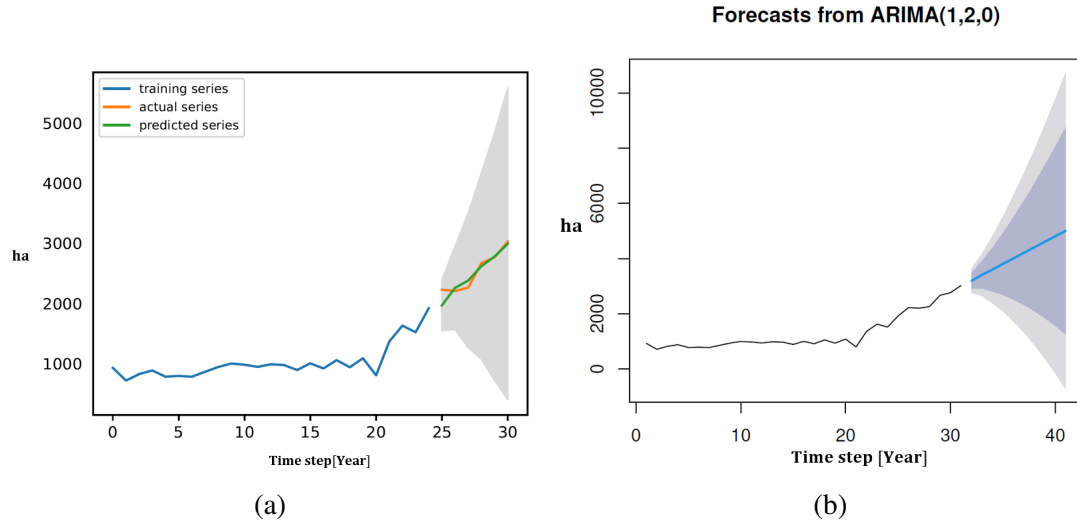


**Figure 15:** Time series forecast plot using ARIMA(1,2,0), the shaded regions signify the confidence interval

This highlights the urgent need for implementing effective conservation strategies to preserve the forests and protect their vital ecological, social, and economic functions. To achieve this, it is crucial to gain a comprehensive understanding of the dynamics of human-forest interactions. Such an understanding is essential for developing conservation strategies that can balance the competing demands on forest resources and promote sustainable land-use practices.

The following sections will delve deeper into the dynamics of human-forest interactions and explore some of the key drivers of deforestation. The challenges and opportunities for promoting sustainable forest management will be explored, aiming to find solutions that safeguard the environment while meeting the needs of human populations.

## 4.2 Mathematical Modelling of Forest Biomass Depletion and its impact on Wildlife Dependent on Forest Ecosystems

Based on the assumptions and descriptions stated in chapter three, the model flow diagram in crisp environment that describes the interplay between the variables (Table 2) is shown in Fig. 16.

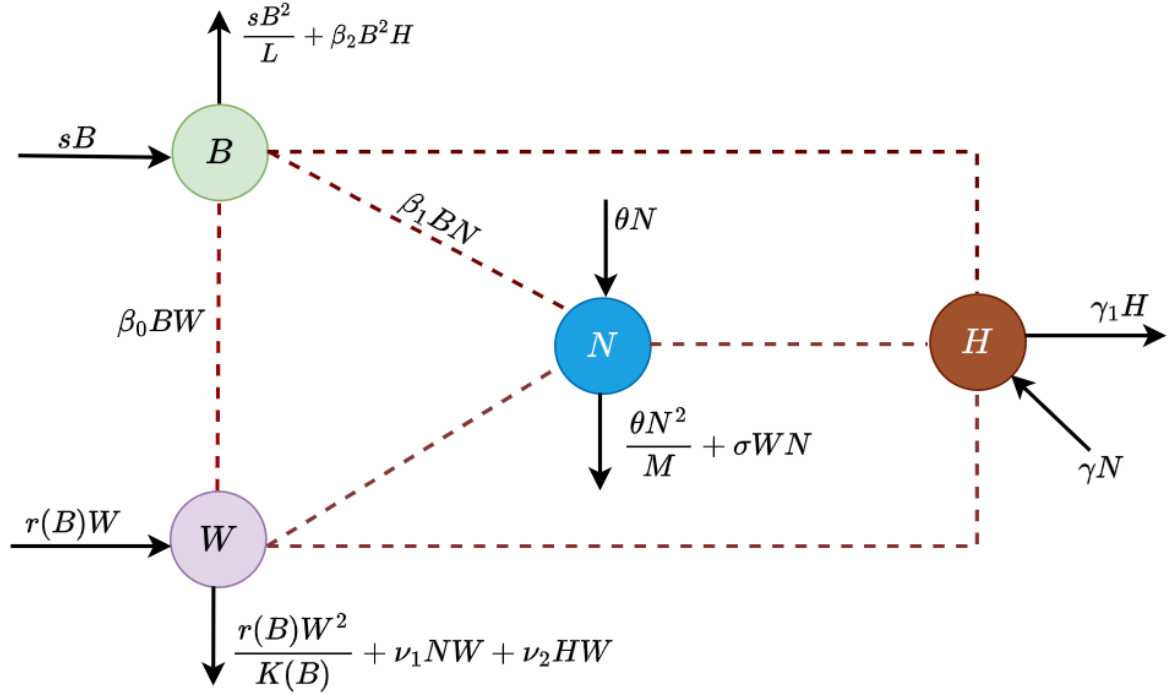


**Figure 16:** Schematic diagram for model system (15)

Consequently, the system of equations derived from the model flow diagram is given as:

$$\begin{cases} \frac{dB}{dt} = sB \left(1 - \frac{B}{L}\right) - \beta_0 BW - \beta_1 BN - \beta_2 B^2 H, & (15a) \\ \frac{dW}{dt} = r(B)W \left(1 - \frac{W}{K(B)}\right) - \nu_1 WN - \nu_2 WH, & (15b) \\ \frac{dN}{dt} = \theta N \left(1 - \frac{N}{M}\right) + \lambda \beta_1 BN - \sigma NW, & (15c) \\ \frac{dH}{dt} = \gamma N - \gamma_1 H & (15d) \end{cases}$$

such that,

$$B(0) \geq 0, W(0) \geq 0, N(0) \geq 0, H(0) \geq 0, 0 \leq \lambda \leq 1.$$

The first equation in the proposed model captures the logistic growth of forest biomass, characterised by an intrinsic growth rate,  $s$  and a carrying capacity,  $L$ . The second term in the equation represents depletion rate of forest biomass due to wildlife population under natural processes, this is modelled through a bilinear interaction ( $\beta_0 BW$ ) [92]. The exploitation of forest resources by human has a direct impact on the growth rate of forest biomass. This phenomenon is quantified by the inclusion of the third term,  $\beta_1 BN$ , in the equation [56]. This exploitation includes activities such as tree cutting for wood and medicinal purposes, as well as grazing.

Deforestation for agricultural, residential, and industrial purposes negatively impact the carrying capacity of forestry resources. This reduction in forest land is explicitly incorporated in the model through the fourth term,  $\beta_2 B^2 H$  [56], which accounts for the loss of forest land that cannot be regenerated. The term  $\beta_2 B^2 H$  suggests that the impact of human activities on the forest biomass is non-linear. The quadratic term represents the vulnerability of the forest ecosystem to human activities. At lower levels of human activities, the ecosystem might be relatively resilient and capable of maintaining forest biomass. However, as human activities increase, the resilience of the ecosystem decreases, potentially leading to a more pronounced decline in forest biomass [56].

Since the growth of forest biomass is governed by logistic equation, that is,

$$\frac{dB}{dt} = sB \left( 1 - \frac{B}{L} \right),$$

by incorporating the term  $\beta_2 B^2 H$  gives

$$\frac{dB}{dt} = sB \left( 1 - \frac{B}{L_\kappa} \right),$$

where  $L_\kappa$  is the modified carrying capacity of forestry biomass and is given by:

$$\frac{1}{L_\kappa} = \frac{1}{L} + \frac{\beta_2 H}{s}.$$

From this relation it is evident that  $L_\kappa$  decreases as human activities ( $H$ ) increases. Thus, the term  $\beta_2 B^2 H$  affect only the carrying capacity of forestry biomass without affecting its intrinsic growth rate.

The second equation of the model describes the dynamics of the forest-dependent wildlife population. It follows a logistic growth pattern, with the growth rate,  $r(B)$  and carrying capacity,  $K(B)$  being dependent on the available forest biomass, reflecting the species' reliance on the forest ecosystem. The direct depletion of wildlife due to activities like poaching or smuggling is represented by the term  $(\nu_1 W N)$  [56]. Additionally, the indirect depletion caused by human activities such as habitat destruction or fragmentation is captured by the term  $(\nu_2 W H)$ .

The third equation of the model describes the dynamics of the human population. It assumes that the population follows logistic growth with an intrinsic growth rate,  $\theta$  and a carrying capacity,  $M$ . The term  $\lambda \beta_1 B N$  represents the additional growth of the human population resulting from their utilisation of forest resources [38]. Human population is impacted by interactions

with wildlife population. These interactions may lead to resources competition and human safety concerns as human populations expand and encroach wildlife habitats [70]. For instance, in Tanzania, statistics show that from 2012 to 2019 the conflicts between human and wildlife claimed 1069 human lives and destroyed 4 1404 hectares of crops [65]. This interaction is explicitly accounted for by the term  $\sigma NW$ . The dynamics of human activities within the forest ecosystem are captured in equations four, with human activities assumed to increase proportionally with the density of the human population ( $\gamma N$ ) [92], and naturally depletes with a depletion rate of  $\gamma_1$ .

In addition to the system of equations (15), the functions  $r(B)$  and  $K(B)$  in equation (15b) are given as follows:

(i)

$$r(B) = \eta\beta_0 B, \quad (16)$$

such that,  $r(B) > 0$  for  $B > 0$  and  $0 \leq \eta \leq 1$ .

(ii)

$$K(B) = K_0 + K_1 B, \quad (17)$$

where  $K_1 > 0$ ,  $K(0) = K_0 > 0$  and  $K(B) > 0$ ,  $K'(B) > 0$  for  $B > 0$ . The constant  $K_0$  represents the baseline carrying capacity, or the minimum level of carrying capacity that the wildlife population can sustain in the absence of additional forest habitat while  $K_1$  represents the rate at which the carrying capacity of the wildlife population increases with an increase in available forest habitat.

#### 4.2.1 Basic properties of the model

The model was checked to see if it meet the basic properties, such as well-posedness and boundedness of the solution, that allow it to be used in the modelling process. The well-posedness of the model system means that non-negative initial conditions lead to non-negative solutions. This has been verified by showing that equation (18) is positively invariant with respect to model system (15).

$$\mathbb{R}_+^4 = \{(B, W, N, H) \mid B \geq 0, W \geq 0, N \geq 0, H \geq 0\}. \quad (18)$$

Following Li [58], the positive invariance of  $\mathbb{R}_+^4$  were verified through examining the direction of the vector field of the system (15) on each coordinate. It can be noted that the vector fields

are either tangent to the coordinate plane or pointing to the interior of  $\mathbb{R}_+^4$  suggesting that no solutions can escape the interior of  $\mathbb{R}_+^4$  through the coordinate planes.

For example, on  $B - W$  plane,  $N = H = 0$  and

$$\left. \frac{dN}{dt} \right|_{N=H=0} = \left. \frac{dH}{dt} \right|_{N=H=0} = 0.$$

This shows that the vector fields on the  $B - W$  plane are tangents to  $B - W$  plane. The tangency implies that  $B - W$  plane is invariant, thus, solutions starting on  $B - W$  plane remain on the same plane as shown in Fig. 17a. Furthermore, the invariances in  $N - H$ ,  $B - N$  and  $B - H$  are shown in Fig. 17b-17d, respectively. This completes the proof that all solutions starting in  $\mathbb{R}_+^4$  remain in  $\mathbb{R}_+^4$  for  $t > 0$ .

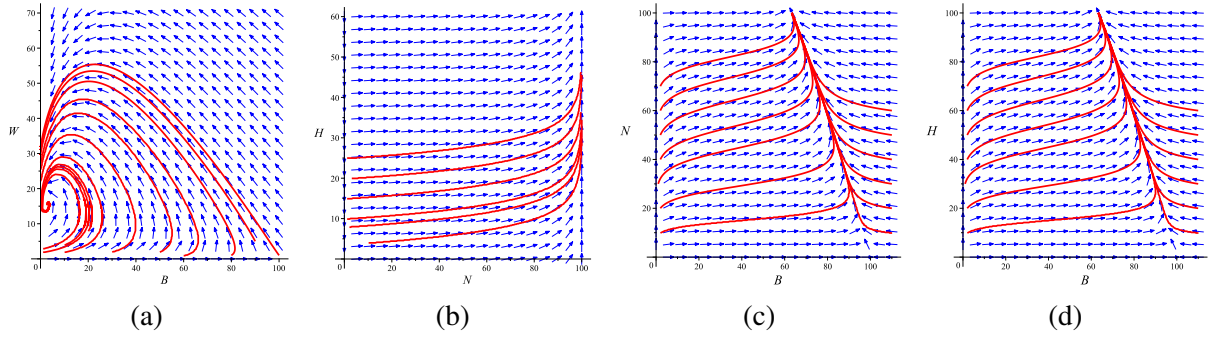


**Figure 17:** Invariance in (a)  $B - W$ , (b)  $N - H$ , (c)  $B - N$  and (d)  $B - H$  planes

According to Freedman [30], the boundedness of the solution confirms the validity of the model system. Following Chen [15] and [14], the region of attraction for the system (15) stated in Lemma 4.4 describes the boundedness of the solution. This is a set of values that indicate how far away from the equilibrium point a trajectory can start and still converge to it.

**Lemma 4.4.** *The set*

$$\Omega = \{(B, W, N, H) \in \mathbb{R}_+^4 : 0 \leq B \leq L, 0 \leq W \leq K(L), 0 \leq N \leq N_m, 0 \leq H \leq H_m\}$$

*is region of attraction for model system (15) and attract all solutions initiating in the interior of positive orthant, where,*

$$N_m = \frac{M}{\theta} (\theta + \lambda\beta_1 L), \text{ and } H_m = \frac{\gamma N_m}{\gamma_1}.$$

*Proof.* On utilising the comparison theorem [44] and the approach in [31], equation (15a) is reduced to

$$\frac{dB}{dt} \leq sB \left(1 - \frac{B}{L}\right).$$

When  $B \rightarrow L$ , the expression in the right hand side of the equation approaches to zero suggesting that

$$\frac{dB}{dt} \leq 0 \text{ for } B \geq L \implies \limsup_{t \rightarrow \infty} B \leq L.$$

Hence,  $0 \leq B \leq L$ .

Similarly, from equations (15b), (15c) and (15d), respectively, applying the theory of differential inequality [77], gives:

$$\begin{aligned} \frac{dW}{dt} &\leq r(L)W \left(1 - \frac{W}{K(L)}\right), \\ \implies \limsup_{t \rightarrow \infty} W &\leq K(L). \end{aligned}$$

$$\begin{aligned} \frac{dN}{dt} &\leq \theta N \left(1 - \frac{N}{M}\right) + \lambda\beta_1 L N, \\ \implies \limsup_{t \rightarrow \infty} N &\leq \frac{M}{\theta} (\theta + \lambda\beta_1 L) = N_m \quad (\text{say}). \\ \frac{dH}{dt} &\leq \gamma N_m - \gamma_1 H, \\ \implies \limsup_{t \rightarrow \infty} H &\leq \frac{\gamma N_m}{\gamma_1} = H_m \quad (\text{say}). \end{aligned}$$

This completes the proof for the boundedness of the system and thus, the system (15) is dissipative.  $\square$

#### 4.2.2 System permanence

Utilising definition 3.9, the conditions for the system permanence are stated in Theorem 4.2.

**Theorem 4.2.** *Assume that Lemma 4.4 exists, then the system (15) is uniformly persistent provided conditions in equation (19) hold:*

$$\left\{ \begin{array}{l} (a) \ s > \beta_0 K(L) + \beta_1 N_m. \\ (b) \ r(B_l) > \nu_1 N_m + \nu_2 H_m. \\ (c) \ \theta > \sigma W_l. \end{array} \right. \quad (19)$$

*Proof.* Following Mukherjee [82], the system is said to be uniformly persistent if there are positive constants  $\Omega_1$  and  $\Omega_2$  such that each positive solutions  $(B, W, N, H)$  of the system with positive initial condition satisfies the following conditions:

$$\Omega_1 \leq \liminf_{t \rightarrow \infty} \mathcal{X}(t) \leq \limsup_{t \rightarrow \infty} \mathcal{X}(t) \leq \Omega_2,$$

where  $\mathcal{X}(t) = (B, W, N, H)$ .

Utilising Lemma (4.4), then  $\Omega_2$  is defined as,  $\Omega_2 = \max \{L, K(L), N_m, H_m\}$ , it follows that  $\limsup_{t \rightarrow \infty} \mathcal{X}(t) \leq \Omega_2$ . On utilising the approach of Chen [15], this is also true for any sufficiently small  $\epsilon_i > 0$ , there exists a  $T_i > 0$ , such that:

$$B < L + \epsilon_1 \forall t \geq T_1, \quad W < K(L) + \epsilon_2 \forall t \geq T_2, \quad N < N_m + \epsilon_3 \forall t \geq T_3, \quad \text{and } H < H_m + \epsilon_4 \forall t \geq T_4.$$

Now taking  $\epsilon = \max \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$  and  $T = \max \{T_1, T_2, T_3, T_4\}$ , it can be concluded that, for every  $\epsilon > 0$ , there exists  $T > 0$  such that for all  $t \geq T$ , the following inequalities holds:

$$B < L + \epsilon, \quad W < K(L) + \epsilon, \quad N < N_m + \epsilon, \quad H < H_m + \epsilon.$$

Hence, the first equation of the model system (15), for all  $t \geq T$  can be written as:

$$\begin{aligned} \frac{dB}{dt} &\geq sB \left(1 - \frac{B}{L}\right) - \beta_0(BK(L) + \epsilon) - \beta_1B(N_m + \epsilon) - \beta_2B^2(H_m + \epsilon) \\ &= (s - \beta_0(K(L) + \epsilon) - \beta_1(N_m + \epsilon)) B - \left(\frac{s}{L} + \beta_2(H_m + \epsilon)\right) B^2. \end{aligned}$$

It follows that

$$\liminf_{t \rightarrow \infty} B(t) \geq \frac{L(s - \beta_0(K(L) + \epsilon) - \beta_1(N_m + \epsilon))}{s + L\beta_2(H_m + \epsilon)}$$

which is true for every  $\epsilon > 0$ , hence, it follows

$$\liminf_{t \rightarrow \infty} B(t) \geq \frac{L(s - \beta_0K(L) - \beta_1N_m)}{s + L\beta_2H_m} = B_l \quad (\text{say}),$$

$B_l$  is positive provided inequality (19(a)) holds.

From the second equation of the system (15), gives:

$$\begin{aligned} \frac{dW}{dt} &\geq r(B_l)W \left(1 - \frac{W}{K(B_l)}\right) - \nu_1W(N_m + \epsilon) - \nu_2W(H_m + \epsilon), \\ &= \left(r(B_l) - (\nu_1(N_m + \epsilon) + \nu_2(H_m + \epsilon))\right) W - \frac{r(B_l)}{K(B_l)}. \end{aligned}$$

It follows:

$$\liminf_{t \rightarrow \infty} W(t) \geq \frac{K(B_l)(r(B_l) - \nu_1(N_m + \epsilon) - \nu_2(H_m + \epsilon))}{r(B_l)},$$

which is true for every  $\epsilon > 0$ , therefore,

$$\liminf_{t \rightarrow \infty} W(t) \geq \frac{K(B_l)(r(B_l) - \nu_1 N_m - \nu_2 H_m)}{r(B_l)} = W_l \quad (\text{say}),$$

$W_l$  is positive whenever inequality (19(b)) holds.

Similarly, the third and fourth equations of the system (15) yield the inequalities (20) and (21), respectively.

$$\begin{aligned} \frac{dN}{dt} &\geq \theta N \left(1 - \frac{N}{M}\right) - \sigma N W_l, \\ &= (\theta - \sigma W_l) N - \frac{\theta N^2}{M}. \end{aligned}$$

Hence, it follows:

$$\liminf_{t \rightarrow \infty} N(t) \geq \frac{M(\theta - \sigma W_l)}{\theta} = N_l \quad (\text{say}), \quad (20)$$

$N_l$  is positive whenever inequality (19(c)) holds.

$$\frac{dH}{dt} \geq \gamma N_l - \gamma_1 H.$$

It follows:

$$\liminf_{t \rightarrow \infty} H(t) \geq \frac{\gamma N_l}{\gamma_1} = H_l \quad (\text{say}), \quad (21)$$

Defining  $\Omega_1 = \min \{B_l, W_l, N_l, H_l\}$ , Theorem 4.2 follows.  $\square$

### Remark 4.3

Following Theorem 4.2, the system may not remain uniformly persistent under the following circumstances:

- (i) If the intrinsic growth rate of forest biomass ( $s$ ) is not particularly large, that is, if forest biomass growth rate is smaller than the utilisation of forest biomass by wildlife and human.
- (ii) If the intrinsic growth rate of forest-dependent wildlife population due to forest biomass ( $r(B)$ ) is insufficient, that is, if the growth rate of forest-dependent wildlife population through forest biomass is small when compared to its depletion rate.

- (iii) If the intrinsic growth rate of human population ( $\theta$ ) is not large enough, that is, the growth rate of human population is small compared to its depletion rate due to forest-dependent wildlife population.

It is noteworthy that the mathematical model under consideration possesses desirable properties, being well-posed, uniformly persistent, and exhibiting bounded solutions. To delve deeper into the underlying dynamics, the subsequent subsections employ the stability theory of differential equations to investigate the existence and stability of equilibrium points. This analytical approach aims to provide a comprehensive understanding of the system's behaviour.

### 4.2.3 Equilibrium analysis

Based on the assumption that wildlife population entirely depends on forest biomass, the possibility of equilibrium point when  $B = 0$  and  $W \neq 0$  was eliminated. Therefore, it can be observed that the system (15) has six equilibrium points, that is five boundary equilibria and one interior equilibrium point, namely  $E_0, E_1, E_2, E_3, E_4$ , and  $E^*$ :

$$\begin{aligned} E_0(0, 0, 0, 0), & \quad E_1(L, 0, 0, 0), & \quad E_2(B_2, W_2, 0, 0), \\ E_3\left(0, 0, M, \frac{M\gamma}{\gamma_1}\right), & \quad E_4(B_4, 0, N_4, H_4), & \quad E^*(B^*, W^*, N^*, H^*). \end{aligned}$$

The existence of  $E_0, E_1$ , and  $E_3$  is obvious and exist without any conditions. The existence of the other equilibria are shown as follows:

Existence of  $E_2(B_2, W_2, 0, 0)$ : It is worth noting that the values of  $B_2$  and  $W_2$  are positive solution of the algebraic equations (22) and (23):

$$s \left(1 - \frac{B_2}{L}\right) - \beta_0 W_2 = 0, \quad (22)$$

$$r(B_2) \left(1 - \frac{W_2}{K(B_2)}\right) = 0. \quad (23)$$

Taking equation (23), after simple algebraic manipulation gives equation (24),

$$W_2 = K(B_2). \quad (24)$$

Upon substitution of equation (24) into equation (22), gives equation (25),

$$0 = s \left(1 - \frac{B_2}{L}\right) - \beta_0 K(B_2) = F(B_2) \quad (\text{say}). \quad (25)$$

From equation (25) the following inferences can be made:

(i) When  $B_2 = 0$  gives  $F(0)$ ,

$$F(0) = s - \beta_0 K_0,$$

$F(0)$  is positive provided condition (26) holds.

$$s > \beta_0 K_0. \quad (26)$$

(ii) When  $B_2 = L$  gives  $F(L)$ ,

$$F(L) = -\beta_0 K(L).$$

Following equation (17),  $K(L) > 0$ , then  $F(L)$  is negative. These inferences (i-ii) suggesting that there exists a positive root  $B_2$  of equation (25) in the interval  $(0, L)$  provided condition (26) hold.

(iii) For uniqueness of  $B_2$ ,  $F'(B_2)$  must be negative, that is,

$$F'(B_2) = -\frac{s}{L} - \beta_0 K'(B_2).$$

From equation (17),  $K'(B_2) > 0$ , hence,  $F'(B_2)$  is negative. This shows that there exists a unique positive root  $B_2$  of equation (25) in the interval  $(0, L)$  provided condition (26) hold.

Once the value of  $B_2$  is known, the value of  $W_2$  can be evaluated from equation (24). This ends the proof of the existence of equilibrium point  $E_2(B_2, W_2, 0, 0)$ . The existence of  $E_2$  gives the conclusion that in absence of human population and its associated activities, forest biomass and wildlife population coexist if the intrinsic growth rate of forest biomass is greater than its depletion rate due to wildlife population under natural processes.

Existence of  $E_4(B_4, 0, N_4, H_4)$ : At this point,  $B_4$ ,  $N_4$  and  $H_4$  are positive solution of the algebraic equations (27)-(29),

$$s \left( 1 - \frac{B_4}{L} \right) - \beta_1 N_4 - \beta_2 B_4 H_4 = 0, \quad (27)$$

$$\theta \left( 1 - \frac{N_4}{M} \right) + \lambda \beta_1 B_4 = 0, \quad (28)$$

$$\gamma N_4 - \gamma_1 H_4 = 0. \quad (29)$$

Taking equation (28), after some algebraic manipulation gives equation (30),

$$N_4 = \frac{M}{\theta} (\theta + \lambda \beta_1 B_4) = g_1(B_4) \quad (\text{say}). \quad (30)$$

Further, taking equations (29) and (30) gives equation (31),

$$H_4 = \frac{\gamma g_1(B_4)}{\gamma_1} = g_2(B_4) \quad (\text{say}). \quad (31)$$

By substituting  $g_1(B_4)$  and  $g_2(B_4)$  into equation (27) gives isocline (32),

$$s \left( 1 - \frac{B_4}{L} \right) - \beta_1 g_1(B_4) - \beta_2 B_4 g_2(B_4) = 0 = G(B_4) \quad (\text{say}). \quad (32)$$

Thus, the following facts can be derived from the isocline (32):

(i) When  $B_4 = 0$  gives,

$$G(0) = s - \beta_1 M,$$

$G(0)$  is positive whenever inequality (33) holds.

$$s > \beta_1 M. \quad (33)$$

(ii) Further, when  $B_4 = L$  gives,

$$G(L) = -\beta_1 g_1(L) - \beta_2 L g_2(L) < 0,$$

Based on inferences (i-ii), there exists a positive root  $B_4$  of equation (32) in  $(0, L)$  whenever inequality (33) holds.

(iii) Further, it is important to note that for uniqueness of  $B_4$ ,  $G'(B_4)$  must be negative, that is

$$G'(B_4) = -\frac{s}{L} - \beta_1 g'_1(B_4) - \beta_2 (B_4 g'_2(B_4) + g_2(B_4)) < 0. \quad (34)$$

This concludes the proof that there exists a unique positive root  $B_4$  of equation (32) in  $(0, L)$  provided inequality (33) is satisfied.

By determining the value of  $B_4$ , it is possible to find the values of  $N_4$  and  $H_4$  using equations (30) and (31), respectively. Hence, this proves the existence of boundary equilibrium point  $E_4(B_4, 0, N_4, H_4)$ . The existence of the equilibrium point  $E_4$  implies that, in the absence of forest-dependent wildlife populations, forest biomass, the human population, and their associated activities can coexist as long as the rate of forest growth outpaces the rate of forest depletion caused by the human population.

Existence of  $E^*(B^*, W^*, N^*, H^*)$ : At this point,  $B^*, W^*, N^*$  and  $H^*$  are positive solution of the algebraic equations (35)-(38).

$$s \left( 1 - \frac{B^*}{L} \right) - \beta_0 W^* - \beta_1 N^* - \beta_2 B^* H^* = 0, \quad (35)$$

$$r(B^*) \left( 1 - \frac{W^*}{K(B^*)} \right) - \nu_1 N^* - \nu_2 H^* = 0, \quad (36)$$

$$\theta \left( 1 - \frac{N^*}{M} \right) + \lambda \beta_1 B^* - \sigma W^* = 0, \quad (37)$$

$$\gamma N^* - \gamma_1 H^* = 0. \quad (38)$$

Taking equations (37) and (38), after some algebraic manipulations, give equations (39) and (41).

$$N^* = \frac{M}{\theta} (\theta + \lambda \beta_1 B^* - \sigma W^*) = h_1(B^*, W^*) \quad (\text{say}), \quad (39)$$

$h_1(B^*, W^*)$  is positive provided inequality (40) holds.

$$\theta + \lambda \beta_1 B^* > \sigma W^*. \quad (40)$$

$$H^* = \frac{\gamma h_1(B^*, W^*)}{\gamma_1} = h_2(B^*, W^*) \quad (\text{say}). \quad (41)$$

Upon substitution of equations (39) and (41) into equations (35) and (36), the isoclines (42) and (43), respectively, were obtained.

$$s \left( 1 - \frac{B^*}{L} \right) - \beta_0 W^* - \beta_1 h_1(B^*, W^*) - \beta_2 B^* h_2(B^*, W^*) = 0 = H_1(B^*, W^*) \quad (\text{say}), \quad (42)$$

$$r(B^*) \left( 1 - \frac{W^*}{K(B^*)} \right) - \nu_1 h_1(B^*, W^*) - \nu_2 h_2(B^*, W) = 0 = H_2(B^*, W^*) \quad (\text{say}). \quad (43)$$

The following inferences were derived from isocline (42):

(a) When  $W^* = 0$ ,  $B^*$  is given by equation (44),

$$H_1(B^*, 0) = s \left( 1 - \frac{B^*}{L} \right) - \beta_1 h_1(B^*) - \beta_2 B^* h_2(B^*) = R_1(B^*) \quad (\text{say}), \quad (44)$$

which can be analysed as follows:

(i) When  $B^* = 0$ , gives:

$$R_1(0) = s - \beta_1 M,$$

which is positive provided condition (33) holds.

(ii) When  $B^* = L$ , gives:

$$R_1(L) = -\beta_1 h_1(L) - \beta_2 L h_2(L) < 0.$$

(iii) The derivative of  $R_1(B^*)$  with respect to  $B^*$  gives:

$$R'_1(B^*) = -\frac{s}{L} - \beta_1 h'_1(B^*) - \beta_2 [B^* h'_2(B^*) + h_2(B^*)] < 0.$$

Hence, these considerations (i-iii) lead to the conclusion that equation (44) has a unique positive root in  $(0, L)$ .

(b)  $B^* < 0$  as  $W^* \rightarrow \infty$ .

(c) If the derivative of  $W^*$  with respect to  $B^*$  from isocline (42) is denoted by  $\left(\frac{dW^*}{dB^*}\right)_1$ , then  $\left(\frac{dW^*}{dB^*}\right)_1 < 0$ .

Similarly, from the isocline (43), the following inferences can be made:

(a) When  $W = 0$ ,  $B^*$  is defined by equation (45):

$$H_2(B^*, 0) = r(B^*) - \nu_1 h_1(B^*) - \nu_2 h_2(B^*) = R_2(B^*) \quad (\text{say}), \quad (45)$$

which is analysed as follows:

(i) When  $B^* = 0$  gives:

$$R_2(0) = -\nu_1 M - \frac{M\gamma\nu_2}{\gamma_1} < 0.$$

(ii) When  $B^* = L$ , gives:

$$R_2(L) = r(L) - \nu_1 h_1(L) - \nu_2 h_2(L),$$

which is positive provided:

$$r(L) > \nu_1 h_1(L) + \nu_2 h_2(L).$$

(iii) The derivative of  $R_2(B^*)$  with respect to  $B^*$  gives:

$$R'_2(B^*) = r'(B^*) - \nu_1 h'_1(B^*) - \nu_2 h'_2(B^*),$$

which is positive, provided:

$$r'(B^*) > \nu_1 h'_1(B^*) + \nu_2 h'_2(B^*).$$

(b)  $W^* > 0$  as  $B^* \rightarrow \infty$ .

(c) If the derivative of  $W^*$  with respect to  $B^*$  from isocline (43) is denoted by  $\left(\frac{dW^*}{dB^*}\right)_2$ , then  $\left(\frac{dW^*}{dB^*}\right)_2 > 0$ .

It is worth noting that for uniqueness of  $(B^*, W^*)$  in the regions of  $(0, L)$  and  $(0, K(L))$ ,

$$\left(\frac{dW^*}{dB^*}\right)_1 < 0,$$

and

$$\left(\frac{dW^*}{dB^*}\right)_2 > 0.$$

The expressions for  $\left(\frac{dW^*}{dB^*}\right)_1$  and  $\left(\frac{dW^*}{dB^*}\right)_2$  are determined as follows:

For  $\left(\frac{dW^*}{dB^*}\right)_1$ , taking equation (42), its total derivative is given as:

$$H'_1(B^*, W^*) = \frac{\partial H_1}{\partial B^*} dB^* + \frac{\partial H_1}{\partial W^*} dW^* = 0 \implies \left(\frac{dW^*}{dB^*}\right)_1 = -\frac{\frac{\partial H_1}{\partial B^*}}{\frac{\partial H_1}{\partial W^*}}$$

$$\left(\frac{dW^*}{dB^*}\right)_1 = -\left(\frac{\frac{s}{L} + \beta_1 \frac{\partial h_1}{\partial B^*} + \beta_2 \frac{\partial h_2}{\partial B^*} + \beta_2 h_2}{\beta_0 + \beta_1 \frac{\partial h_1}{\partial W^*} + \beta_2 \frac{\partial h_2}{\partial W^*}}\right) \quad (46)$$

which should be negative for uniqueness  $(B^*, W^*)$ .

Similarly, the total derivative of equation (43) is as follows:

$$H'_2(B^*, W^*) = \frac{\partial H_2}{\partial B^*} dB^* + \frac{\partial H_2}{\partial W^*} dW^* = 0 \implies \left(\frac{dW^*}{dB^*}\right)_2 = -\frac{\frac{\partial H_2}{\partial B^*}}{\frac{\partial H_2}{\partial W^*}}$$

$$\left(\frac{dW^*}{dB^*}\right)_2 = \frac{r'(B^*) - \left(\frac{K(B^*)r'(B^*) - r(B^*)K'(B^*)}{(K(B^*))^2}\right)W^* - \nu_1 \frac{\partial h_2}{\partial B^*} - \nu_2 \frac{\partial h_2}{\partial B^*}}{\frac{r(B^*)}{K(B^*)} + \nu_1 \frac{\partial h_2}{\partial W^*} + \nu_2 \frac{\partial h_2}{\partial W^*}} \quad (47)$$

which is positive for uniqueness of  $(B^*, W^*)$ . The conditions  $(dW^*/dB^*)_1 < 0$  and  $(dW^*/dB^*)_2 > 0$  were verified by using the following counter example; consider if the forest

biomass  $B$  is not affected by human population  $N$  and human activities  $H$ , that is,  $\beta_1, \beta_2 = 0$ , then equation (46) is reduced to:

$$\left(\frac{dW^*}{dB^*}\right)_1 = -\frac{s}{\beta_0 L} < 0.$$

Similarly, when forest-dependent wildlife population ( $W$ ) is not affected by human population ( $N$ ) and human activities ( $H$ ), that is,  $\nu_1 = 0, \nu_2 = 0$ , equation (47) is reduced to:

$$\left(\frac{dW^*}{dB^*}\right)_2 = K_1 > 0.$$

With this example, the conditions are satisfied, this completes the proof of the existence of positive and unique roots  $(B^*, W^*)$ . Thus, once the values of  $B^*$  and  $W^*$  are known, the values of  $N^*$  and  $H^*$  can be evaluated from equations (39) and (41), respectively.

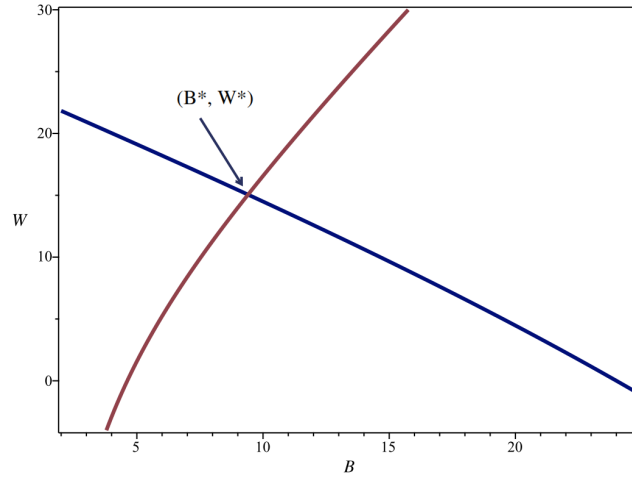


**Figure 18:** The isoclines (42 and 43) plot shows an intersection at a unique point  $(B^*, W^*)$  in the interior of the first quadrant

#### Remark 4.4

Under certain conditions at equilibrium point  $E^*$ ,  $dB/dH < 0$  and  $dW/dH < 0$ . This implies that, the density of forest biomass and forest-dependent wildlife populations decrease as human activities increases.

*Proof.* Consider equations (48) and (49), required to show that  $dB/dH < 0$  and  $dW/dH < 0$  at equilibrium point  $E^*$

$$s \left(1 - \frac{B}{L}\right) - \beta_0 W - \beta_1 h_1(B, W, H) - \beta_2 B h_2(B, W, H) = 0 = H_1(B, W, H), \quad (48)$$

$$r(B) \left( 1 - \frac{W}{K(B)} \right) - \nu_1 h_1(B, W, H) - \nu_2 h_2(B, W, H) = 0 = H_2(B, W, H). \quad (49)$$

Differentiating equation (48) with respect to  $H$  and applying the chain rule gives:

$$Z_1 \frac{dB}{dH} = F_1 \frac{dW}{dH} + G_1, \quad (50)$$

where,

$$\begin{aligned} Z_1 &= -\frac{s}{L} - \beta_1 \frac{\partial h_1}{\partial B} - \beta_2 B \frac{\partial h_2}{\partial B} - \beta_2 h_2, \\ F_1 &= \beta_0 + \beta_1 \frac{\partial h_1}{\partial W} + \beta_2 B \frac{\partial h_2}{\partial W}, \\ G_1 &= \beta_1 \frac{\partial h_1}{\partial H} + \beta_2 B \frac{\partial h_2}{\partial H}. \end{aligned}$$

On differentiating equation (49) with respect to  $H$  and using chain rule gives:

$$Z_2 \frac{dB}{dH} = F_2 \frac{dW}{dH} + G_2 \quad (51)$$

where,

$$\begin{aligned} Z_2 &= r'(B) - \left( \frac{r'(B)K(B) - r(B)K'(B)}{(K(B))^2} \right) W - \nu_1 \frac{\partial h_1}{\partial B} - \nu_2 \frac{\partial h_2}{\partial B}, \\ F_2 &= \frac{r(B)}{K(B)} - \nu_1 \frac{\partial h_1}{\partial W} - \nu_2 \frac{\partial h_2}{\partial W}, \\ G_2 &= \frac{dW}{dH} + \nu_1 \frac{\partial h_1}{\partial H} + \nu_2 \frac{\partial h_2}{\partial H}. \end{aligned}$$

Taking equations (50) and (51), the following equations are established:

$$\begin{aligned} \frac{dW}{dH} &= \frac{Z_2 G_1 - Z_1 G_2}{Z_1 F_2 - Z_2 F_1}, \\ \frac{dB}{dH} &= \frac{F_2 G_1 - F_1 G_2}{Z_1 F_2 - Z_2 F_1}, \end{aligned}$$

where  $Z_1 F_2 \neq Z_2 F_1$ , and from equation (43), the following inequalities are established:

$$\frac{\partial h_1}{\partial W} > 0, \quad \frac{\partial h_2}{\partial W} > 0, \quad \frac{\partial h_1}{\partial B} < 0, \quad \frac{\partial h_2}{\partial B} < 0, \quad \frac{\partial h_1}{\partial H} \geq 0, \text{ and } \frac{\partial h_2}{\partial H} \geq 0.$$

Eventually, the following inequalities are obtained:

$$Z_1 < 0, \quad Z_2 > 0, \quad F_1 > 0, \quad F_2 > 0, \quad G_1 \geq 0 \text{ and } G_2 \geq 0.$$

From equation (46), it can be noted that;

$$\left( \frac{dW}{dB} \right)_1 = \frac{Z_1}{F_1} \implies Z_1 = F_1 \left( \frac{dW}{dB} \right)_1.$$

For uniqueness of point  $(E^*)$ , equation (46) should be negative, that is  $(dW/dB)_1 < 0$ . Since  $F_1 > 0$ , thus,  $Z_1 < 0$ . In this case  $Z_1 F_2 - Z_2 F_1$  is always negative and  $Z_2 G_1 - E_1 G_2$  is always positive. Therefore,

$$\frac{dB}{dH} < 0,$$

provided:

$$F_2 G_1 - F_1 G_2 > 0.$$

This implies that density of forest biomass decreases as human activities increase. Also it can be noted that

$$\frac{dW}{dH} < 0, \text{ as } Z_2 G_1 - Z_1 G_2 > 0.$$

This signifies that when human activities grow, the density of forest-dependent wildlife population decreases. □

#### 4.2.4 Stability analysis

Stability analysis is necessary to examine the system's dynamics. For local asymptotic stability, solutions must approach an equilibrium point under initial conditions that are close to the equilibrium point, whereas for global asymptotic stability, solutions must approach an equilibrium point under all initial conditions [53, 41].

Following the inferences in Table 6, the signs of the eigenvalues of the Jacobian matrix evaluated at the respective equilibrium point was used to determine the local stability behaviour of the system. Thus, the general Jacobian matrix ( $J$ ) of the system (15) is given as:

$$J = \begin{pmatrix} j_{11} & -\beta_0 B & -\beta_1 B & -\beta_2 B^2 \\ j_{21} & j_{22} & -\nu_1 W & -\nu_2 W \\ \lambda \beta_1 N & -\sigma N & j_{33} & 0 \\ 0 & 0 & \gamma & -\gamma_1 \end{pmatrix},$$

such that,

$$j_{11} = s \left( 1 - \frac{B}{L} \right) - \frac{sB}{L} - \beta_0 W - \beta_1 N - 2\beta_2 BH,$$

$$j_{21} = r'(B)W \left( 1 - \frac{W}{K(B)} \right) + \frac{r(B)W^2 K'(B)}{(K(B))^2},$$

$$j_{22} = r(B) \left( 1 - \frac{W}{K(B)} \right) - \frac{r(B)W}{K(B)} - \nu_1 N - \nu_2 H,$$

$$j_{33} = \theta \left( 1 - \frac{N}{M} \right) - \frac{\theta N}{M} + \lambda \beta_1 B - \sigma W,$$

To predict the stability behaviour of model system (15) at  $E_0$ , the corresponding Jacobian matrix evaluated at  $E_0$  is given as:

$$J(E_0) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \theta & 0 \\ 0 & 0 & \gamma & -\gamma_1 \end{pmatrix}$$

It can be observed that matrix  $J(E_0)$  has two positive eigenvalues,  $s, \theta$ , simple zero eigenvalue as well as one negative eigenvalue,  $-\gamma_1$ . This suggests that at  $E_0$  the system exhibits unstable manifold locally in the direction of  $B - N$  space and it is locally stable in the direction of  $H$ . The zero eigenvalue suggests that in the direction of  $W$ , the system neither converges towards nor diverges from the equilibrium point. This direction is referred to as the centre manifold, indicating that the system might be semi-stable at the equilibrium point, that is, exhibiting stability in some directions but instability in others.

Similarly at  $E_1$ , the corresponding Jacobian matrix is given as:

$$J(E_1) = \begin{pmatrix} -s & -\beta_0 L & -\beta_1 L & -\beta_2 L^2 \\ 0 & r(L) & 0 & 0 \\ 0 & 0 & L\lambda\beta_1 + \theta & 0 \\ 0 & 0 & \gamma & -\gamma_1 \end{pmatrix}$$

The matrix  $J(E_1)$  has two positive eigenvalues  $(r(L), \theta + \beta_1 \lambda L)$  and two negative eigenvalues  $(-s, -\gamma_1)$ . This establishes that the system exhibits stable behaviour locally in the direction of  $B - H$  plane and unstable behaviour in the direction of  $W - N$  plane.

Further, at  $E_2$ , the corresponding Jacobian matrix is given as:

$$J(E_2) = \begin{pmatrix} -b_1 & -\beta_0 B_2 & -\beta_1 B_2 & -\beta_2 B_2^2 \\ b_2 & -b_3 & -\nu_1 W_2 & -\nu_2 W_2 \\ 0 & 0 & \lambda \beta_1 B_2 - \sigma W_2 + \theta & 0 \\ 0 & 0 & \gamma & -\gamma_1 \end{pmatrix}$$

where,

$$\begin{aligned} b_1 &= \frac{sB_2}{L}, \\ b_2 &= r'(B_2)W_2 \left(1 - \frac{W_2}{K(B_2)}\right) + \frac{r(B_2)K'(B_2)W_2^2}{(K(B_2))^2}, \\ b_3 &= \frac{r(B_2)W_2}{K(B_2)}. \end{aligned}$$

The characteristic equation of matrix  $J(E_2)$  is given as:

$$(\lambda + \gamma_1)(\lambda + z_1)(\lambda^2 + z_2\lambda + z_3) = 0,$$

$$z_1 = \sigma W_2 - \lambda \beta_1 B_2 - \theta,$$

$$z_2 = b_1 + b_3, \quad z_3 = b_1 b_3 + b_2 \beta_0 B_2.$$

Given that all of the values  $z_2$  and  $z_3$  are positive, when  $z_1$  is also positive, that is,  $\sigma W_2 > \lambda \beta_1 B_2 + \theta$  the application of the Routh-Hurwitz criteria indicates that the Jacobian matrix ( $J_2$ ) will have eigenvalues that are either negative or have negative real parts. As a result, the system represented by  $E_2$  can be considered stable. Conversely, if  $z_1$  is negative, that is,  $\sigma W_2 < \lambda \beta_1 B_2 + \theta$  the matrix  $J(E_2)$  will have at least one eigenvalue that is positive, implying that  $E_2$  will always be unstable.

Similarly at  $E_3$ , the corresponding Jacobian matrix is given as:

$$J(E_3) = \begin{pmatrix} s - \beta_1 M & 0 & 0 & 0 \\ 0 & -\left(\frac{\nu_1 \gamma_1 M + \nu_2 \gamma M}{\gamma_1}\right) & 0 & 0 \\ \lambda \beta_1 M & -\sigma M & -\theta & 0 \\ 0 & 0 & \gamma & -\gamma_1 \end{pmatrix}$$

It can be seen that the Jacobian matrix  $J(E_3)$  has one eigenvalue which is positive always whenever condition (33) holds, the remaining eigenvalues are negative always. Therefore, this establishes that at  $E_3$  the system exhibits unstable manifold locally in the direction of  $B$  if and only if  $E_4$  or  $E^*$  exists, that is inequality (33) holds.

At  $E_4$  the corresponding Jacobian matrix is given as:

$$J(E_4) = \begin{pmatrix} -d_1 & -\beta_0 B_4 & -\beta_1 B_4 & -\beta_2 B_4^2 \\ 0 & r(B_4) - \nu_1 N_4 - \nu_2 H_4 & 0 & 0 \\ \lambda \beta_1 N_4 & -\sigma N_4 & -d_2 & 0 \\ 0 & 0 & \gamma & -\gamma_1 \end{pmatrix},$$

where,

$$d_1 = \frac{sB_4}{L} + \beta_2 B_4 H_4,$$

$$d_2 = \frac{\theta N_4}{M}.$$

The characteristic equation of the matrix  $J(E_4)$  is given as:

$$(\lambda + D_3) (\lambda^3 + D_0 \lambda^2 + D_1 \lambda + D_2) = 0,$$

where,

$$D_0 = d_1 + d_2 + \gamma_1,$$

$$D_1 = \gamma_1(d_1 + d_2) + d_1 d_2 + \lambda \beta_1^2 B_4^2,$$

$$D_2 = \gamma_1(d_1 d_2 + \lambda \beta_1^2 B_4^2) + \lambda \gamma \beta_1 \beta_2 B_4^3,$$

$$D_3 = \nu_1 N_4 + \nu_2 H_4 - r(B_4).$$

Given that  $D_0$ ,  $D_1$ , and  $D_2$  are all positive, when  $D_3$  is also positive, that is,  $\nu_1 N_4 + \nu_2 H_4 > r(B_4)$  the application of the Routh-Hurwitz criteria gives:

$$D_0 D_1 > D_2,$$

suggesting that the Jacobian matrix  $J(E_4)$  will possess eigenvalues that are either negative or have negative real components. Consequently, it can be deduced that the system denoted by  $E_4$  is stable. Conversely, if  $D_3$  is negative, that is,  $\nu_1 N_4 + \nu_2 H_4 < r(B_4)$ , the matrix  $J(E_4)$  will inevitably exhibits at least one positive eigenvalue, leading to the conclusion that  $E_4$  is always unstable. Theorem 4.3 provides the general stability analysis of the boundary equilibrium points  $E_0$ ,  $E_2$ ,  $E_3$ , and  $E_4$ .

**Theorem 4.3.** *Stability analysis of the boundary equilibrium points  $E_0$ ,  $E_1$ ,  $E_2$ ,  $E_3$ , and  $E_4$ :*

- (i) *Equilibrium point  $E_0$  is always unstable;*
- (ii) *Equilibrium point  $E_1$  is always unstable;*
- (iii) *If  $z_1 > 0$ , then equilibrium point  $E_2$  is always stable;*
- (iv) *Equilibrium point  $E_3$  is stable provided that  $E^*$  does not exists ( $\beta_1 > s/M$ ), meaning that a transcritical bifurcation may take place between  $E_3$  and  $E^*$ ;*
- (iv) *If  $D_3 > 0$  and  $D_0 D_1 > D_2$ , then equilibrium point  $E_4$  is stable always.*

To determine the local stability of the system (15) at  $E^*$ , Lyapunov direct method [73] was used instead of directly finding the sign of the eigenvalues of the Jacobian matrix due to the complexities involved. This was accomplished by linearising the system about  $E^*$  through the use of the following transformations:

$B = B^* + b$ ,  $W = W^* + w$ ,  $N = N^* + n$ ,  $H = H^* + h$ , where,  $b$ ,  $w$ ,  $n$  and  $h$  are small perturbations around the equilibrium  $E^*$ . The following linearised system is obtained:

$$\begin{cases} \frac{db}{dt} = -\left(\frac{sB^*}{L} + \beta_2 B^* H^*\right)b - \beta_0 B^* w - \beta_1 B^* n - \beta_2 B^{*2} h, \\ \frac{dw}{dt} = \left(r'(B^*)W^* \left(1 - \frac{W^*}{K(B^*)}\right) + \frac{r(B^*)W^{*2}K'(B^*)}{K(B^*)^2}\right)b \\ \quad - \left(\frac{r(B^*)W^*}{K(B^*)}\right)w - \nu_1 W^* n - \nu_2 W^* h, \\ \frac{dn}{dt} = \lambda\beta_1 N^* b - \sigma N^* w - \frac{\theta N^*}{M}n, \\ \frac{dh}{dt} = \gamma n - \gamma_1 h. \end{cases}$$

On utilising the linearised system, Theorem 4.4 establishes conditions that characterise the local asymptotic stability of the system (15) at the equilibrium point  $E^*$ .

**Theorem 4.4.** *The model system (15) is locally asymptotically stable around the equilibrium point  $E^*$ , provided conditions (52)-(56) hold.*

$$\left(\frac{W^*}{K(B^*)} - \frac{K'(B^*)B^*W^*}{(K(B^*))^2}\right)^2 < \frac{2}{3} \left(\frac{B^*(s + \beta_2 L H^*)}{\beta_0 L K(B^*)}\right), \quad (52)$$

$$\left(\frac{\nu_1}{\eta} + \frac{\sigma}{\lambda}\right)^2 < \frac{2}{3} \left(\frac{\beta_0 B^*}{K(B^*)}\right) \left(\frac{\theta}{\lambda M}\right), \quad (53)$$

$$\gamma^2 < \frac{2\theta\gamma_1}{3\lambda M}, \quad (54)$$

$$\nu_2^2 < \left(4\frac{\gamma_1\eta r(B^*)}{9K(B^*)}\right), \quad (55)$$

$$\beta_2^2 < \left(\frac{2\gamma_1}{3}\right) \left(\frac{s + \beta_2 H^* L}{B^{*2} L}\right). \quad (56)$$

*Proof.* Following Theorem 3.1, consider the positive definite function (57),

$$V = \frac{1}{2} \left( \frac{1}{B^*} b^2 + \frac{k_1}{W^*} w^2 + \frac{k_2}{N^*} n^2 + k_3 \hbar^2 \right), \quad (57)$$

where  $k_1$ ,  $k_2$  and  $k_3$  are positive constants. Upon differentiating function (57) with respect to  $t$  along the solutions of linearised system gives:

$$\begin{aligned} \frac{dV}{dt} = & - \left( \frac{s}{L} + \beta_2 H^* \right) b^2 - \beta_0 b w - \beta_1 b n - \beta_2 B^* b \hbar - k_1 \left( \frac{r(B^*)}{K(B^*)} \right) w^2 - k_1 \nu_1 w n \\ & - k_1 \nu_2 w \hbar + k_1 \left( r'(B^*) \left( 1 - \frac{W^*}{K(B^*)} \right) + \frac{r(B^*) W^* K'(B^*)}{(K(B^*))^2} \right) b w \\ & + k_2 \lambda \beta_1 b n - k_2 \sigma w n - k_2 \frac{\theta}{M} n^2 + k_3 \gamma n \hbar - \gamma_1 \hbar^2. \end{aligned}$$

With some algebraic manipulations,  $\frac{dV}{dt}$  reduced to:

$$\begin{aligned} \frac{dV}{dt} = & - \left( \frac{s}{L} + \beta_2 H^* \right) b^2 - k_1 \left( \frac{r(B^*)}{K(B^*)} \right) w^2 - k_2 \frac{\theta}{M} n^2 - k_3 \gamma_1 \hbar^2 \\ & + (-\beta_0 + k_1 r'(B^*)) b w - k_1 \left( \frac{r'(B^*) W^*}{K(B^*)} - \frac{r(B^*) W^* K'(B^*)}{(K(B^*))^2} \right) b w \\ & + (-\beta_1 + k_2 \lambda \beta_1) b n - \beta_2 B^* b \hbar - (k_1 \nu_1 + k_2 \sigma N^*) w n - k_1 \nu_2 w \hbar + k_3 \gamma n \hbar. \end{aligned}$$

Taking arbitrary values of  $-\beta_0 + k_1 r'(B^*) = 0 \implies k_1 = \frac{1}{\eta}$ ,  $-\beta_1 + k_2 \lambda \beta_1 = 0 \implies k_2 = \frac{1}{\lambda}$ , and  $k_3 = 1$  which are all positive constants, the time derivative of  $V$  is more simplified to

$$\begin{aligned} \frac{dV}{dt} = & - \left( \frac{s}{L} + \beta_2 H^* \right) b^2 - k_1 \left( \frac{r(B^*)}{K(B^*)} \right) w^2 - k_2 \frac{\theta}{M} n^2 - k_3 \gamma_1 \hbar^2 - \beta_2 B^* b \hbar + k_3 \gamma n \hbar \\ & - k_1 \left( \frac{r'(B^*) W^*}{K(B^*)} - \frac{r(B^*) W^* K'(B^*)}{(K(B^*))^2} \right) b w - (k_1 \nu_1 + k_2 \sigma N^*) w n - k_1 \nu_2 w \hbar. \end{aligned}$$

Further algebraic manipulations yields:

$$\begin{aligned}
\frac{dV}{dt} = & - \left[ \frac{1}{2} \left( \frac{s}{L} + \beta_2 H^* \right) b^2 + k_1 \left( \frac{r'(B^*)W^*}{K(B^*)} - \frac{r(B^*)K'(B^*)W^*}{(K(B^*))^2} \right) bw + \frac{k_1}{3} \left( \frac{r(B^*)}{K(B^*)} \right) w^2 \right] \\
& - \left[ \frac{k_1}{3} \left( \frac{r(B^*)}{K(B^*)} \right) w^2 + (k_1 \nu_1 + k_2 \sigma) wn + \frac{k_2}{2} \left( \frac{\theta}{M} \right) n^2 \right] \\
& - \left[ \frac{k_2}{2} \left( \frac{\theta}{M} \right) n^2 - k_3 \gamma v \hbar + \frac{k_3}{3} \gamma_1 \hbar^2 \right] - \left[ \frac{k_3}{3} \gamma_1 \hbar^2 + k_1 \nu_2 w \hbar + \frac{k_1}{3} \left( \frac{r(B^*)}{K(B^*)} \right) w^2 \right] \\
& - \left[ \frac{1}{2} \left( \frac{s}{L} + \beta_2 H^* \right) b^2 + \beta_2 B^* b \hbar + \frac{k_3}{3} \gamma_1 \hbar^2 \right].
\end{aligned}$$

It can be observed that  $\frac{dV}{dt}$  is negative definite provided conditions (52) to (56) are satisfied.

Thus,  $E^*$  is locally asymptotically stable.  $\square$

The conditions for global stability of the system at  $E^*$  in the region  $\Omega$  were determined by using the Lyapunov direct method [56]. Theorem 4.5 provides conditions for global stability.

**Theorem 4.5.** *The interior equilibrium  $E^*$  if exists is globally stable inside the region of attraction  $\Omega$  whenever conditions (58)-(62) hold.*

$$\left( \frac{LK(L)\lambda_1}{K_0^2} + \frac{W^*}{K(B^*)} \right)^2 < \frac{2}{3} \left( \frac{s + \beta_2 H^* L}{\beta_0 K(B^*)} \right), \quad (58)$$

$$\left( \frac{\nu_1}{\eta} + \frac{\sigma}{\lambda} \right)^2 < \frac{2}{3} \left( \frac{\beta_0 L}{K(B^*)} \right) \left( \frac{\theta}{\lambda M} \right), \quad (59)$$

$$\gamma^2 < \frac{2\theta\gamma_1}{3\lambda M}, \quad (60)$$

$$\beta_2^2 < \frac{2\gamma_1 (s + \beta_2 H^* L)}{3L^3}, \quad (61)$$

$$\nu_2^2 < \frac{4\gamma_1 \eta r(L)}{9K(B^*)}. \quad (62)$$

*Proof.* The positive definite function (63) was considered in the analysis as follows:

$$\begin{aligned}
U = & \left( B - B^* - B^* \ln \frac{B}{B^*} \right) + l_1 \left( W - W^* - W^* \ln \frac{W}{W^*} \right) \\
& + l_2 \left( N - N^* - N^* \ln \frac{N}{N^*} \right) + \frac{l_3}{2} (H - H^*)^2, \quad (63)
\end{aligned}$$

where,  $l_1, l_2$  and  $l_3$  are positive constants. It can be easily checked that,  $U(B^*, W^*, N^*, H^*) = 0$  and  $U(B, W, N, H) > 0$  for all other positive values of  $\{B, W, N, H\} - \{B^*, W^*, N^*, H^*\}$ .

On differentiating (63) with respect to time  $t$ , gives:

$$\frac{dU}{dt} = \left( \frac{B - B^*}{B} \right) \frac{dB}{dt} + l_1 \left( \frac{W - W^*}{W} \right) \frac{dW}{dt} + l_2 \left( \frac{N - N^*}{N} \right) \frac{dN}{dt} + l_3 (H - H^*) \frac{dH}{dt}.$$

Utilising the values from the system (15) gives:

$$\begin{aligned} \frac{dU}{dt} = & (B - B^*) \left[ - \left( \frac{s}{L} + \beta_2 H^* \right) (B - B^*) - \beta_0 (W - W^*) - \beta_1 (N - N^*) - \beta_2 B (H - H^*) \right] \\ & + l_1 (W - W^*) \left[ \beta_0 \eta (B - B^*) - \frac{\beta_0 \eta B}{K(B^*)} (W - W^*) - \nu_1 (N - N^*) - \nu_2 (H - H^*) \right] \\ & + l_1 (W - W^*) \left[ - (\beta_0 \eta B W \Gamma(B)) (B - B^*) - \left( \frac{\beta_0 \eta W^*}{K(B^*)} \right) (B - B^*) \right] \\ & + l_2 (N - N^*) \left[ - \frac{\theta}{M} (N - N^*) + \lambda \beta_1 (B - B^*) - \sigma (W - W^*) \right] \\ & + l_3 (H - H^*) [-\gamma_1 (H - H^*) + \gamma (N - N^*)]. \end{aligned}$$

where,

$$\Gamma(B) = \begin{cases} \frac{\frac{1}{K(B)} - \frac{1}{K(B^*)}}{B - B^*}, & B \neq B^* \\ \frac{-K'(B)}{(K(B^*))^2}, & B = B^* \end{cases}$$

On utilising mean value theorem gives:

$$|\Gamma(B)| = \frac{\lambda_0}{K_0^2},$$

such that:

$$0 < K'(B) \leq \lambda_0.$$

Choosing arbitrary  $l_1 = 1/\eta$ ,  $l_2 = 1/\lambda$  and  $l_3 = 1$ , consequently,  $\frac{dU}{dt}$  is reduced to

$$\begin{aligned} \frac{dU}{dt} = & - \left( \frac{s}{L} + \beta_2 H^* \right) (B - B^*)^2 - l_1 \frac{\beta_0 \eta B}{K(B^*)} (W - W^*)^2 - l_2 \frac{\theta}{M} (N - N^*)^2 \\ & - l_3 \gamma_1 (H - H^*)^2 - l_1 \left[ \left( \frac{\lambda_0 r(B) W}{K_0^2} \right) + \left( \frac{\beta_0 \eta W^*}{K(B^*)} \right) \right] (B - B^*) (W - W^*) \\ & - \beta_2 B (B - B^*) (H - H^*) - (l_1 \nu_1 + l_2 \sigma) (W - W^*) (N - N^*) \\ & + l_3 \gamma (N - N^*) (H - H^*) - l_1 \nu_2 (W - W^*) (H - H^*). \end{aligned}$$

Furthermore, with simple manipulations,  $\frac{dU}{dt}$  can be arranged as the difference of quadratic expressions:

$$\begin{aligned} \frac{dU}{dt} = & - \left[ \frac{1}{2} \left( \frac{s}{L} + \beta_2 H^* \right) (B - B^*)^2 + l_1 \zeta (B - B^*) (W - W^*) + \frac{l_1}{3} \frac{\beta_0 \eta B}{K(B^*)} (W - W^*)^2 \right] \\ & - \left[ \frac{l_1}{3} \frac{\beta_0 \eta B}{K(B^*)} (W - W^*)^2 + (l_1 \nu_1 + l_2 \sigma) (W - W^*) (N - N^*) + \frac{l_2 \theta}{2M} (N - N^*)^2 \right] \\ & - \left[ \frac{l_2 \theta}{2M} (N - N^*)^2 - l_3 \gamma (N - N^*) (H - H^*) + \frac{l_3}{3} \gamma_1 (H - H^*)^2 \right] \\ & - \left[ \frac{l_3}{3} \gamma_1 (H - H^*)^2 + \beta_2 B (B - B^*) (H - H^*) + \frac{1}{2} \left( \frac{s}{L} + \beta_2 H^* \right) (B - B^*)^2 \right] \\ & - \left[ \frac{l_1}{3} \frac{\beta_0 \eta B}{K(B^*)} (W - W^*)^2 + l_1 \nu_2 (W - W^*) (H - H^*) + \frac{l_3}{3} \gamma_1 (H - H^*)^2 \right], \end{aligned}$$

where,

$$\zeta = \left[ \frac{\lambda_0 r(B) W}{K_0^2} + \frac{\beta_0 \eta W^*}{K(B^*)} \right].$$

Utilising Theorem 3.1, the interior point  $E^*$  is globally asymptotically stable provided  $\frac{dU}{dt}$  is negative definite. Thus,  $\frac{dU}{dt} < 0$  if conditions (58) to (62) hold.  $\square$

From Theorems 4.4 and 4.5, it was observed that forest biomass and forest-dependent wildlife population stabilise to its equilibrium levels whose magnitude depends on the following:

- (i) forest-dependent wildlife depletion rate due to human population,  $\nu_1$ ,
- (ii) growth rate of human activities due to human population,  $\gamma$ ,
- (iii) forest depletion rate due to human activities,  $\beta_2$ ,
- (iv) forest-dependent wildlife depletion rate due to human activities,  $\nu_2$ , and
- (v) human activities natural depletion rate,  $\gamma_1$ .

#### 4.2.5 Existence of trans-critical bifurcation

The inferences on stability analysis of equilibrium point  $E_3$  shows that the point is stable whenever  $\beta_1 > \beta_1^* = s/M$  (Theorem 4.3 (iv)). When  $\beta_1$  decreases to  $\beta_1^*$ ,  $E_3$  loses its stability and  $E_4$  emerges (condition 33). Thus, if  $\beta_1$  is bifurcation parameter where an exchange of stability

between  $E_3$  and  $E_4$  takes place at  $\beta_1 = \beta_1^*$ , then, there is a trans-critical bifurcation. Following Castillo-Chavez and Song [12], the general centre manifold theorem was utilised to show the existence of trans-critical bifurcation. Consider the following general system of ordinary differential equations (64) with parameter  $v_0$ ,

$$\frac{dX}{dt} = f(X, v_0), \quad f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R} \text{ and } f \in \mathbb{C}^n(\mathbb{R}^n \times \mathbb{R}). \quad (64)$$

Without losing the generality, it is assumed that for all values of the parameter  $v_0$ , 0 is an equilibrium for the system (64), that is:

$$f(0, v_0) \equiv 0 \quad \forall v_0.$$

**Theorem 4.6** (Castillo-Chavez and Song [12]). *Assume the following:*

- (a) *Matrix  $J = D_X f(0, 0) = \frac{\partial f_i}{\partial X_j}(0, 0)$  is the linearisation matrix of the system (64) around the equilibrium 0 with  $v_0$  evaluated at 0. Zero is the simple eigenvalue of  $J$  and all other eigenvalues of  $J$  have negative real parts;*
- (b) *Matrix  $J$  has a non-negative right eigenvector  $w$  and a left eigenvector  $v$  corresponding to the zero eigenvalue.*

*Further, assume  $f_k$  is the  $k^{th}$  component of a function  $f$  and*

$$S = \sum_{k,i,j=1}^n v_k w_i w_j \frac{\partial^2 f_k}{\partial X_i \partial X_j}(0, 0),$$

$$\tau = \sum_{k,i=1}^n v_k w_i \frac{\partial^2 f_k}{\partial X_i \partial v_0}(0, 0),$$

*then, the local dynamics of system (64) around 0 are totally determined by parameters  $S$  and  $\tau$  as follows:*

- (i)  *$S > 0$ ,  $\tau > 0$ . When  $\beta_0 < 0$  with  $|v_0| \ll 1$ , 0 is locally asymptotically stable, and there exists a positive unstable equilibrium; when  $0 < v_0 \ll 1$ , 0 is unstable and there exists a negative and locally asymptotically stable equilibrium;*
- (ii)  *$S < 0$ ,  $\tau < 0$ . When  $v_0 < 0$  with  $|v_0| \ll 1$ , 0 is unstable, when  $0 < v_0 \ll 1$ , 0 is locally asymptotically stable and there exists a positive unstable equilibrium;*

(iii)  $S > 0$ ,  $\tau < 0$ . When  $v_0 < 0$  with  $|v_0| \ll 1$ , 0 is unstable, and there exists a locally asymptotically stable negative equilibrium; when  $0 < v_0 \ll 1$ , 0 is stable, and a positive unstable equilibrium emerges;

(iv)  $S < 0$ ,  $\tau > 0$ . When  $v_0$  changes from negative to positive, 0 changes its stability from stable to unstable. Correspondingly a negative unstable equilibrium becomes positive and locally asymptotically stable.

Upon utilising Theorem 4.6, the matrix of the linearised system (15) around the equilibrium point  $E_3$  with  $\beta_1$  evaluated at  $\beta_1^* = s/M$  is given as:

$$J(E_3, \beta_1^*) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\left(\frac{\nu_1 \gamma_1 M + \nu_2 \gamma M}{\gamma_1}\right) & 0 & 0 \\ \lambda s & -\sigma M & -\theta & 0 \\ 0 & 0 & \gamma & -\gamma_1 \end{pmatrix}$$

It can be noted that  $J(E_3, \beta_1^*)$  has a simple zero eigenvalue and three negative eigenvalues:

$$\left[ -\left(\frac{\nu_1 \gamma_1 M + \nu_2 \gamma M}{\gamma_1}\right), -\theta, -\gamma_1 \right].$$

This property allows the use of centre manifold theory to study the dynamics of the system (15) near  $\beta_1^*$  [11]. It follows, the right eigenvector of  $J(E_3, \beta_1^*)$  corresponding to 0 eigenvalue denoted by  $\mathbf{w} = [w_1, w_2, w_3, w_4]^T$  is obtained as follows:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\left(\frac{\nu_1 \gamma_1 M + \nu_2 \gamma M}{\gamma_1}\right) & 0 & 0 \\ \lambda s & -\sigma M & -\theta & 0 \\ 0 & 0 & \gamma & -\gamma_1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

With some algebraic manipulations and using the concept of liner combination, eigenvector  $\mathbf{w}$  is given as:

$$\mathbf{w} = \left[ \frac{\theta \gamma_1}{\lambda \gamma s}, 0, \frac{\gamma_1}{\gamma}, 1 \right]^T,$$

consequently, the left eigenvector  $\mathbf{v}$  is given by:

$$\mathbf{v} = \left[ \frac{\lambda \gamma s}{\gamma_1 \theta}, 0, 0, 0 \right],$$

such that  $v \cdot w = 1$ . Furthermore, if  $X = (B, W, N, H)$  and  $f_k = (\dot{B}, \dot{W}, \dot{N}, \dot{H})$ , then the coefficients of  $\mathcal{S}$  and  $\tau$  were determined as follows:

$$\begin{aligned}
\mathcal{S} &= \sum_{k,i,j=1}^4 v_k w_i w_j \frac{\partial^2 f_k}{\partial X_i \partial X_j} (E_3, \beta_1^*), \\
&= v_1 w_1^2 \frac{\partial^2 f_1}{\partial B^2} (E_3, \beta_1^*) + 2v_1 w_1 w_3 \frac{\partial^2 f_1}{\partial B \partial N} (E_3, \beta_1^*), \\
&= -2v_1 \left[ w_1 w_3 \frac{s}{M} + w_1^2 \left( \frac{s}{L} + \frac{\beta_2 M \gamma}{\gamma_1} \right) \right], \\
\mathcal{S} &< 0. \\
\tau &= \sum_{k,i=1}^4 v_k w_i \frac{\partial^2 f_k}{\partial X_i \partial \beta_1} (E_3, \beta_1^*), \\
&= v_1 w_1 \frac{\partial^2 f_1}{\partial B \partial \beta_1}, \\
\tau &= -M < 0.
\end{aligned}$$

Following Theorem 4.6, since  $\mathcal{S} < 0$  and  $\tau < 0$ , when  $\beta_1 < \beta_1^*$  the boundary equilibrium  $E_3$  is unstable and there exists a positive stable interior equilibrium point  $E_4$ . When  $\beta_1 > \beta_1^*$ ,  $E_3$  becomes stable while positive interior equilibrium loses its feasibility and negative unstable equilibrium point  $E_4$  emerges. Thus, there exists a trans-critical bifurcation between  $E_3$  and  $E_4$  with bifurcation parameter  $\beta_1$ .

#### 4.2.6 The Volterra-Lyapunov matrix theory for global stability analysis of interior equilibrium point $E^*$

Consider the positive definite function (65),

$$\mathcal{U} = l_1 (B - B^*)^2 + l_2 (W - W^*)^2 + l_3 (N - N^*)^2 + l_4 (H - H^*)^2, \quad (65)$$

where  $l_1, l_2, l_3$  and  $l_4$  are positive constants. It can be easily checked that  $\mathcal{U}(B^*, W^*, N^*, H^*) = 0$  and  $\mathcal{U}(B, W, N, H) > 0$  for all other positive values of  $\{B, W, N, H\} - \{B^*, W^*, N^*, H^*\}$ .

The time derivative of the function (65) gives:

$$\dot{\mathcal{U}} = 2l_1 (B - B^*) \dot{B} + 2l_2 (W - W^*) \dot{W} + 2l_3 (N - N^*) \dot{N} + 2l_4 (H - H^*) \dot{H}.$$

Substituting the values of  $\dot{B}$ ,  $\dot{W}$ ,  $\dot{N}$  and  $\dot{H}$  from the system (15), the following equation is obtained:

$$\begin{aligned}\dot{\mathbf{U}} = & 2l_1 (B - B^*) \left[ -B \left( \frac{s}{L} + \beta_2 H^* \right) (B - B^*) - \beta_0 B (W - W^*) \right] \\ & + 2l_1 (B - B^*) \left[ -\beta_1 B (N - N^*) - \beta_2 B^2 (H - H^*) \right] \\ & + 2l_2 (W - W^*) \left[ -\frac{r(B^*)W}{K(B^*)} (W - W^*) - \nu_1 W (N - N^*) - \nu_2 W (H - H^*) \right] \\ & + 2l_2 (W - W^*) \left[ W \left( \beta_0 \eta - \beta_0 \eta B W \Gamma(B) - \frac{\beta_0 \eta W}{K(B^*)} \right) (B - B^*) \right] \\ & + 2l_3 (N - N^*) \left[ -\frac{\theta}{M} N (N - N^*) + \lambda \beta_1 N (B - B^*) - \sigma N (W - W^*) \right] \\ & + 2l_4 (H - H^*) \left[ -\gamma_1 (H - H^*) + \gamma (N - N^*) \right].\end{aligned}$$

where,

$$\Gamma(B) = \begin{cases} \frac{\frac{1}{K(B)} - \frac{1}{K(B^*)}}{B - B^*}, & B \neq B^* \\ \frac{-K'(B)}{(K(B^*))^2}, & B = B^* \end{cases}$$

After simple algebraic manipulation gives equation (66):

$$\begin{aligned}\dot{\mathbf{U}} = & -2l_1 B \left( \frac{s}{L} + \beta_2 H^* \right) (B - B^*)^2 - 2l_1 \beta_0 B (B - B^*) (W - W^*) \\ & + 2l_2 W \left( \beta_0 \eta - \beta_0 \eta B W \Gamma(B) - \frac{\beta_0 \eta W}{K(B^*)} \right) (W - W^*) (B - B^*) \\ & - 2l_1 \beta_1 B (B - B^*) (N - N^*) - 2l_1 \beta_2 B^2 (B - B^*) (H - H^*) \\ & - 2l_2 \frac{\beta_0 \eta B^* W}{K(B^*)} (W - W^*)^2 - \nu_1 l_2 W (W - W^*) (N - N^*) \\ & - 2l_2 \nu_2 W (W - W^*) (H - H^*) - 2l_3 \frac{\theta N}{M} (N - N^*)^2 \\ & + 2l_3 \lambda \beta_1 N (N - N^*) (B - B^*) - 2l_3 \sigma N (N - N^*) (W - W^*) \\ & - 2l_4 \gamma_1 (H - H^*)^2 + 2l_4 \gamma (H - H^*) (N - N^*).\end{aligned}\tag{66}$$

Following Chien [16] and utilising Lemma 4.4, the equation (66) is expressed as:

$$\dot{\mathbf{U}} = \mathbf{Y} \left( \mathbf{Q}A + A^T \mathbf{Q}^T \right) \mathbf{Y}^T,$$

where,

$$\mathbf{Y} = (B - B^*, W - W^*, N - N^*, H - H^*),$$

$$\mathbf{Q} = \begin{pmatrix} l_1 & 0 & 0 & 0 \\ 0 & l_2 & 0 & 0 \\ 0 & 0 & l_3 & 0 \\ 0 & 0 & 0 & l_4 \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{pmatrix}, \quad (67)$$

$$\begin{aligned} A_{11} &= -\left(\frac{s}{L} + \beta_2 B^*\right) L, & A_{12} &= -\beta_0 L, & A_{13} &= -\beta_1 L, \\ A_{14} &= -\beta_2 L^2, & A_{21} &= \psi_0 K(L), & A_{22} &= -\frac{\beta_0 \eta B^* K(L)}{K(B^*)}, \\ A_{23} &= -\nu_1 K(L), & A_{24} &= -\nu_2 K(L), & A_{31} &= \lambda \beta_1 N_m, \\ A_{32} &= -\sigma N_m, & A_{33} &= -\frac{\theta}{M} N_m, & A_{43} &= \gamma, \\ A_{44} &= -\gamma_1, \end{aligned}$$

$$\psi_0 = \left( \beta_0 \eta - \frac{\beta_0 \lambda_0 \eta L K(L)}{K_0^2} - \frac{\beta_0 \eta K(L)}{K(B^*)} \right).$$

To prove the global asymptotic stability of  $\mathcal{Y}$ , required to show that the matrix  $A$  defined in equation (67) is Volterra-Lyapunov stable or  $-A$  is diagonal stable. It can be noted that  $-A_{44} > 0$ , hence, the following lemmas follow:

**Lemma 4.5.** *For the matrix  $A$  defined in equation (67), let matrix  $D = -\check{A}$ , then matrix  $D$  is diagonal stable.*

*Proof.* From matrix  $A$ , matrix  $D$  is defined as

$$D = -\check{A} = \begin{pmatrix} \left(\frac{s}{L} + \beta_2 H^*\right) L & \beta_0 L & \beta_1 L \\ -\psi_0 K(L) & \frac{\beta_0 \eta B^* K(L)}{K(B^*)} & \nu_1 K(L) \\ -\lambda \beta_1 N_m & \sigma N_m & \frac{\theta}{M} N_m \end{pmatrix}$$

Further, from matrix  $D$  it is worth noting that  $D_{33} = \frac{\theta N_m}{M} > 0$  allowing utilisation of Lemma 3.3, that matrix  $D$  is diagonally stable if and only if the following conditions hold:

**Condition 1.** *Matrix  $\check{D}$  is diagonal stable if and only if  $-\check{D}$  is Volterra-Lyapunov stable.*

In this case, it is required to show that  $-\check{D}$  is Volterra-Lyapunov stable, therefore, from matrix  $D$ , matrix  $-\check{D}$  is defined as follows:

$$-\check{D} = \begin{pmatrix} -\left(\frac{s}{L} + \beta_2 H^*\right) L & -\beta_0 L \\ \psi_0 K(L) & -\frac{\beta_0 \eta B^* K(L)}{K(B^*)} \end{pmatrix}$$

It is can be noted that  $-\check{D}_{11} < 0$ ,  $-\check{D}_{22} < 0$  and  $\det(-\check{D}) > 0$ , following Lemma 3.2, matrix  $-\check{D}$  is Volterra-Lyapunov stable.

**Condition 2.** Matrix  $\check{D}^{-1}$  is diagonal stable. Thus, required show that matrix  $-\check{D}^{-1}$  is Volterra-Lyapunov stable.

It can be noted that matrix  $-\check{D}^{-1}$  is defined as follows:

$$-\check{D}^{-1} = \frac{1}{\det(D)} \begin{pmatrix} -\check{D}^{-1}_{11} & -\check{D}^{-1}_{12} \\ -\check{D}^{-1}_{21} & -\check{D}^{-1}_{22} \end{pmatrix}$$

where,

$$\begin{aligned} -\check{D}^{-1}_{11} &= -\sigma \nu_1 K(L) N_m - \frac{\beta_0 \eta \theta B^* K(L) L}{K(B^*) M}, \\ -\check{D}^{-1}_{12} &= \sigma \beta_1 L N_m - \frac{\beta_0 \theta L N_m}{M}, \\ -\check{D}^{-1}_{21} &= \frac{\beta_0 \beta_1 \eta B^* L K(L)}{K(B^*)} - \nu_1 \beta_0 L K(L), \\ -\check{D}^{-1}_{22} &= -\lambda \beta_1^2 L N_m - \frac{(s + \beta_2 L H^*) \theta N_m}{M} \end{aligned}$$

Since,  $-\check{D}^{-1}_{11} < 0$ ,  $-\check{D}^{-1}_{22} < 0$ , and  $\det(D) > 0$ , based on Lemma 3.2, matrix  $-\check{D}^{-1}$  is Volterra-Lyapunov stable. Thus, Lemma 3.3 guarantees that matrix  $D$  is diagonal stable.  $\square$

**Lemma 4.6.** For the matrix  $A$  defined in equation (67), let matrix  $\mathcal{P} = -\check{A}^{-1}$ , then matrix  $\mathcal{P}$  is diagonal stable.

*Proof.* Lemma 3.3 is used to prove diagonal stability of matrix  $\mathcal{P}$ , that is,  $\check{\mathcal{P}}$  is diagonal stable,  $\check{\mathcal{P}}^{-1}$  is diagonal stable, and  $\mathcal{P}_{33} > 0$ . Matrix  $A$  defined in equation

(67) gives the following:

$$\mathcal{P} = -\check{A}^{-1} = \frac{1}{\det(-A)} \begin{pmatrix} -\check{A}_{11}^{-1} & -\check{A}_{12}^{-1} & -\check{A}_{13}^{-1} \\ -\check{A}_{21}^{-1} & -\check{A}_{22}^{-1} & -\check{A}_{23}^{-1} \\ -\check{A}_{31}^{-1} & -\check{A}_{32}^{-1} & -\check{A}_{33}^{-1} \end{pmatrix},$$

where,

$$\begin{aligned} -\check{A}_{11}^{-1} &= -A_{22}A_{33}A_{44} + A_{23}A_{32}A_{44} - A_{24}A_{32}A_{43} \\ -\check{A}_{12}^{-1} &= A_{12}A_{33}A_{44} - A_{13}A_{32}A_{44} + A_{14}A_{32}A_{43} \\ -\check{A}_{13}^{-1} &= -A_{12}A_{23}A_{44} + A_{12}A_{24}A_{43} + A_{13}A_{22}A_{44} - A_{14}A_{22}A_{43} \\ -\check{A}_{21}^{-1} &= A_{21}A_{33}A_{44} - A_{23}A_{31}A_{44} + A_{24}A_{31}A_{43} \\ -\check{A}_{22}^{-1} &= -A_{11}A_{33}A_{44} + A_{13}A_{31}A_{44} - A_{14}A_{31}A_{43} \\ -\check{A}_{23}^{-1} &= A_{11}A_{23}A_{44} - A_{11}A_{24}A_{43} - A_{13}A_{21}A_{44} + A_{14}A_{21}A_{43} \\ -\check{A}_{31}^{-1} &= -A_{21}A_{32}A_{44} + A_{22}A_{31}A_{44} \\ -\check{A}_{32}^{-1} &= A_{11}A_{32}A_{44} - A_{12}A_{31}A_{44} \\ -\check{A}_{33}^{-1} &= -A_{11}A_{22}A_{44} + A_{12}A_{21}A_{44} \end{aligned}$$

$$\begin{aligned} \det(-A) &= A_{11}A_{22}A_{33}A_{44} - A_{11}A_{23}A_{32}A_{44} + A_{11}A_{24}A_{32}A_{43} - A_{12}A_{21}A_{33}A_{44} \\ &\quad + A_{12}A_{23}A_{31}A_{44} - A_{12}A_{24}A_{31}A_{43} + A_{13}A_{21}A_{32}A_{44} \\ &\quad - A_{13}A_{22}A_{31}A_{44} - A_{14}A_{21}A_{32}A_{43} + A_{14}A_{22}A_{31}A_{43}. \end{aligned}$$

It can be easily noted that,  $\mathcal{P}_{33} = -\check{A}_{33}^{-1} > 0$  which guarantee to have the following matrix:

$$-\check{\mathcal{P}} = \begin{pmatrix} \check{A}_{11}^{-1} & \check{A}_{12}^{-1} \\ \check{A}_{21}^{-1} & \check{A}_{22}^{-1} \end{pmatrix}. \quad (68)$$

Now, for matrix  $\check{\mathcal{P}}$  be diagonal stable, required to show that  $-\check{\mathcal{P}}$  is Volterra-Lyapunov stable. It is clear that,  $\check{A}_{11}^{-1} < 0$ ,  $\check{A}_{22}^{-1} < 0$  and  $\det(-A) > 0$ . This completes the proof that, matrix  $-\check{\mathcal{P}}$  is Volterra-Lyapunov stable, consequently, matrix  $\check{\mathcal{P}}$  is diagonal stable. Next is to show that matrix  $\check{\mathcal{P}}^{-1}$  is diagonal stable, that is matrix  $-\check{\mathcal{P}}^{-1}$  is Volterra-Lyapunov stable [59]. From Matrix  $\mathcal{P}$ ,  $-\check{\mathcal{P}}^{-1}$  is given as follows;

$$-\check{\mathcal{P}}^{-1} = \frac{1}{\det(\mathcal{P})} \begin{pmatrix} -p_{11} & -p_{12} \\ -p_{21} & -p_{22} \end{pmatrix}, \quad (69)$$

where,

$$\begin{aligned}
-p_{11} &= \check{A}_{23}^{-1} \check{A}_{32}^{-1} - \check{A}_{22}^{-1} \check{A}_{33}^{-1}, & -p_{12} &= \check{A}_{12}^{-1} \check{A}_{33}^{-1} - \check{A}_{13}^{-1} \check{A}_{32}^{-1}, \\
-p_{21} &= \check{A}_{21}^{-1} \check{A}_{23}^{-1} - \check{A}_{23}^{-1} \check{A}_{31}^{-1}, & -p_{22} &= n \check{A}_{13}^{-1} \check{A}_{31}^{-1} - \check{A}_{11}^{-1} \check{A}_{33}^{-1}, \\
\det(\mathcal{P}) &= \left( \check{A}_{11}^{-1} \check{A}_{22}^{-1} \check{A}_{33}^{-1} + \check{A}_{12}^{-1} \check{A}_{23}^{-1} \check{A}_{31}^{-1} + \check{A}_{13}^{-1} \check{A}_{21}^{-1} \check{A}_{33}^{-1} \right) \\
&\quad - \left( \check{A}_{11}^{-1} \check{A}_{23}^{-1} \check{A}_{32}^{-1} + \check{A}_{12}^{-1} \check{A}_{21}^{-1} \check{A}_{33}^{-1} + \check{A}_{13}^{-1} \check{A}_{22}^{-1} \check{A}_{31}^{-1} \right).
\end{aligned}$$

With proper substitutions, it can be observed that  $-p_{11} < 0$ ,  $-p_{22} < 0$  and  $\det(\mathcal{P}) > 0$ . With these conditions, based on Lemma (3.2), it is true that matrix  $-\check{\mathcal{P}}^{-1}$  is Volterra-Lyapunov stable. Hence, matrix  $\check{\mathcal{P}}^{-1}$  is diagonal stable. Therefore, it has been shown that  $\check{\mathcal{P}}$  is diagonal stable,  $\check{\mathcal{P}}^{-1}$  is diagonal stable, and  $\mathcal{P}_{33} > 0$ , consequently, this ends the proof that matrix  $\mathcal{P}$  is diagonal stable. Thus, Theorem 4.7 follows.  $\square$

**Theorem 4.7.** *Matrix  $A$  defined in equation (67) is Volterra-Lyapunov stable.*

*Proof.* Since  $-A_{44} > 0$ , taking into consideration Lemma 4.5 and Lemma 4.6, it is fairly to conclude that there exists a positive diagonal matrix  $\mathcal{Q}$  which guarantee that matrix  $-A$  is diagonally stable,

$$\mathcal{Q}(-A) + (-A)^T \mathcal{Q}^T > 0.$$

Thus, matrix  $A$  is Volterra-Lyapunov stable,

$$\mathcal{Q}A + A^T \mathcal{Q}^T < 0.$$

$\square$

Since, matrix  $\mathcal{Q}$  is a constant matrix, then Theorem 4.8 follows.

**Theorem 4.8.** *The equilibrium point  $E^*$  of the model system (15) is globally asymptotically stable.*

*Proof.* Based on Lemmas 4.5, 4.6 and Theorem 4.7,  $d\mathcal{U}/dt < 0$  when  $\mathcal{Y} \neq \mathcal{Y}^*$  such that  $\mathcal{Y}$  is not on a set of measure zero. This proves that the equilibrium point  $E^*$  of the model system (15) is globally asymptotically stable.  $\square$

After understanding the asymptotic behaviour and long-term qualitative results of the model system, numerical simulations were performed to obtain the numerical solutions of the system and predict the variations of the dynamics over time for different parameter values. This process relies on the numerical values of parameters, and to estimate them, a Bayesian approach was employed through coding in MATLAB R2022a software.

#### 4.2.7 Markov Chain Monte Carlo (MCMC) sampling for parameter identifiability

The parameter values for numerical simulations were estimated by utilising the Adaptive Metropolis MCMC algorithm. Due to the lack of real data, the model (15) was simulated using literature and assumed values (Table 11) to understand the evolution of the model variables over time.

**Table 11:** Model parameter, values and source

| Parameter | Value | Source       | Parameter  | Value    | Source   |
|-----------|-------|--------------|------------|----------|----------|
| $L$       | 100   | [76, 56, 89] | $\theta$   | 0.5, 0.1 | [92, 76] |
| $M$       | 100   | [56, 89]     | $\beta_2$  | 0.0004   | Assumed  |
| $s$       | 0.8   | [75, 56, 92] | $\gamma_1$ | 0.01     | [92]     |
| $\beta_0$ | 0.05  | [49]         | $\gamma$   | 0.004    | Assumed  |
| $\beta_1$ | 0.003 | [92]         | $\eta$     | 0.9      | [49]     |
| $\lambda$ | 0.05  | [92]         | $\nu_1$    | 0.002    | [92]     |
| $\sigma$  | 0.001 | [92]         | $\nu_2$    | 0.0001   | [92]     |
| $K_0$     | 10    | Assumed      | $K_1$      | 3        | [49]     |

Thereafter, independent and identically distributed Gaussian noise was added to corrupt the data, and the resulting corrupted data values were treated as the real data. To obtain the initial values for the MCMC algorithm, the Least Square method was utilised, taking advantage of the MATLAB built-in function `fminsearch`, which uses the Nelder-Mead simplex algorithm [54] to find the local minimizers of the sum of the squares of residuals. During the MCMC implementation, since there were 16 parameters to be estimated, a sample of  $10^5 \times 16$  parameters was generated in the simulations, and the method was adapted 50 times to improve convergence. The initial covariance matrix,  $\Sigma_0$ , was chosen for the MCMC algorithm.

$$\Sigma_0 = \frac{10^{-5} \times \mathbf{I}_{16}}{\sqrt{16}},$$

such that  $\mathbf{I}_{16}$  is the  $16 \times 16$  identity matrix.

The summary statistics of the MCMC samples are presented in Table 12. It is observed that almost all estimated parameter values converge to their initial values, and the posterior MCMC means fall within the confidence intervals (CI), with means and medians being nearly equal. The MCMC errors and standard deviations were reasonably minimal, suggesting that the approach performed well. The small autocorrelations (tau) highlight the significance of sampling each subsequent posterior separately and the effectiveness of the convergence procedure. The skewness and kurtosis values in the Gaussian distribution are approximately 0 and 3, respectively, while all Geweke values are close to 1. As a consequence, it can be shown that the values of skewness and kurtosis for all 16 parameters accurately capture the characteristics of the Gaussian distribution.

**Table 12:** Statistics of MCMC (Number of simulations = 100 000)

| Par.       | LS est. | Mean   | Median   | CI                  | std     | MCerr. | tau    | Geweke  | Kurt.  | Skew.   |
|------------|---------|--------|----------|---------------------|---------|--------|--------|---------|--------|---------|
| $L$        | 100.93  | 100.89 | 100.9137 | [100.8837 100.8939] | 0.8239  | 0.0288 | 140.67 | 0.9995  | 3.1419 | 0.0282  |
| $M$        | 99.97   | 100.2  | 100.0865 | [100.1943 100.2019] | 0.6135  | 0.0299 | 165.32 | 0.9960  | 3.7165 | 0.1121  |
| $s$        | 0.79    | 0.7961 | 0.7963   | [0.7960 0.7961]     | 0.0058  | 0.0002 | 118.62 | 0.9964  | 3.0287 | -0.0497 |
| $\beta_0$  | 0.05    | 0.0497 | 0.0498   | [0.0498 0.0498]     | 0.0004  | 1.6e-5 | 102.95 | 0.9958  | 3.0267 | -0.0615 |
| $\beta_1$  | 0.003   | 0.0031 | 0.0030   | [0.0030 0.0032]     | 3.6e-5  | 9.3e-7 | 82.611 | 0.9979  | 3.1165 | -0.0308 |
| $\lambda$  | 0.05    | 0.0549 | 0.0534   | [0.0548 0.0551]     | 0.0215  | 0.0005 | 112.92 | 0.9360  | 3.1206 | 0.1772  |
| $\sigma$   | 0.001   | 0.0011 | 0.0010   | [0.0010 0.0010]     | 7.4e-5  | 3.7e-6 | 143.44 | 0.9454  | 3.3648 | 0.1497  |
| $K_0$      | 9.96    | 10.076 | 10.0035  | [10.0747 10.0779]   | 0.2572  | 0.0194 | 257.33 | 0.9759  | 3.6086 | 0.6480  |
| $K_1$      | 1.99    | 1.9924 | 1.9970   | [1.9921 1.9927]     | 0.0488  | 0.0025 | 132.31 | 0.9918  | 3.4379 | -0.0002 |
| $\theta$   | 0.1     | 0.1005 | 0.1004   | [0.1005 0.1005]     | 0.0010  | 5.3e-5 | 156.42 | 0.9922  | 3.4437 | 0.1587  |
| $\beta_2$  | 4e-4    | 3.6e-3 | 0.0004   | [0.0004 0.0004]     | 8.1e-5  | 2.9e-6 | 138.22 | 0.9141  | 2.9374 | -0.0009 |
| $\gamma_1$ | 0.05    | 0.0535 | 0.0532   | [0.0535 0.0536]     | 0.0032  | 0.0002 | 269.27 | 0.9134  | 3.6091 | 0.6544  |
| $\gamma$   | 0.004   | 0.0041 | 0.0041   | [0.0041 0.0041]     | 0.0002  | 9.8e-6 | 135.99 | 0.9625  | 3.5183 | 0.4813  |
| $\eta$     | 0.89    | 0.9095 | 0.9083   | [0.9094 0.9096]     | 0.01597 | 0.0009 | 131.15 | 0.9869  | 3.1966 | 0.1929  |
| $\nu_1$    | 0.002   | 0.0019 | 0.0020   | [0.0020 0.0020]     | 3.2e-5  | 6.2e-7 | 76.395 | 0.9983  | 3.4043 | -0.3153 |
| $\nu_2$    | 0.0001  | 5.2e-4 | 0.0004   | [0.0005 0.0005]     | 0.0004  | 2.7e-5 | 136.96 | 0.87802 | 3.0647 | 1.2739  |

Par = Parameter, Ls est = Least square estimates, CI = Credible Interval, std = Standard deviation, MCerr = MCMC error,

Kurt = Kurtosis and Skew = Skewness.

The accurate estimation of parameter values using Markov Chain Monte Carlo (MCMC) hinges on the convergence of the chains. To evaluate the convergence of MCMC chains, Geweke's diagnostic test was conducted through the application of the Geweke criterion. This criterion enabled a comparison between the mean values at the initial stage of the chain ( $a_0\%$ ) and the mean values at the later stage of the chain ( $b_0\%$ ), employing a difference-of-means test [62]. The difference-of-means test involves a straightforward calculation of the  $z$ -ratio, but with an adjustment for the standard error. This error is estimated by utilising the spectral density

at zero to account for the inherent autocorrelation within the chain [43]. The detection of a significant discrepancy between the means signifies a lack of convergence within the chain [62]. This computational approach adheres to convention by adopting the recommended values of  $a_0 = 10\%$  and  $b_0 = 50\%$  by Geweke [9].

Upon employing the `geweke` function from the MATLAB toolbox [67], vectors  $z$  and  $c$  are obtained:

$$\begin{aligned} z &= [-0.06, 0.12, 0.16, 0.21, -0.01, 0.66, 2.06, -0.63, -0.03, 0.24, 1.47, -2.31, -1.10, \\ &\quad -0.51, -0.05, -15.35] \times 10^{-2}, \\ c &= [0.9995, 0.9991, 0.9987, 0.9983, 0.9999, 0.9475, 0.9836, 0.9949, 0.9998, 0.9981 \\ &\quad 0.9883, 0.9815, 0.9912, 0.9959, 0.9996, 0.8780]. \end{aligned}$$

The interpretation of these vectors is as follows: If the chain converges, then as the number of simulations ( $\kappa$ ) approaches infinity,  $z_\kappa$  tends towards a standard normal distribution  $\mathcal{N}(0, 1)$ , while keeping the values of  $a_0$  and  $b_0$  fixed. The graphical representation of Geweke's diagnostics using the  $z$  vector to demonstrate its adherence to  $\mathcal{N}(0, 1)$  is shown in Fig. 19. It is evident that the diagnostic  $z$  closely aligns with  $\mathcal{N}(0, 1)$  for all parameters, indicating convergence across all parameters.

The vector  $c$  serves as a simple test (`geweke`) for the null hypothesis of chain convergence, computed as:

$$c = 2 \times \left( 1 - \text{nordf}(|z|) \right).$$

The `nordf` is the standard normal (Gaussian) cumulative distribution. For an ideally convergent chain, all components of the vector  $c$  should equate to 1 [62]. Notably, all components of  $c$  cluster around 1, corroborating the convergence of the chain across all parameters.

Other graphical convergence tests, including sample autocorrelation functions, tracer plots, scatter plots, and marginal density distributions were also performed. The tracer plot visually demonstrates how well the generated sample of posteriors mixes or revolves within the parameter space. In Fig. 20a, the tracer plots of all 16 parameters are shown, and it can be observed that there are no obvious spikes, indicating that the generated chain is stationary for several initializations. This suggests that the samples have good mixing and effectively cover the intended distribution.

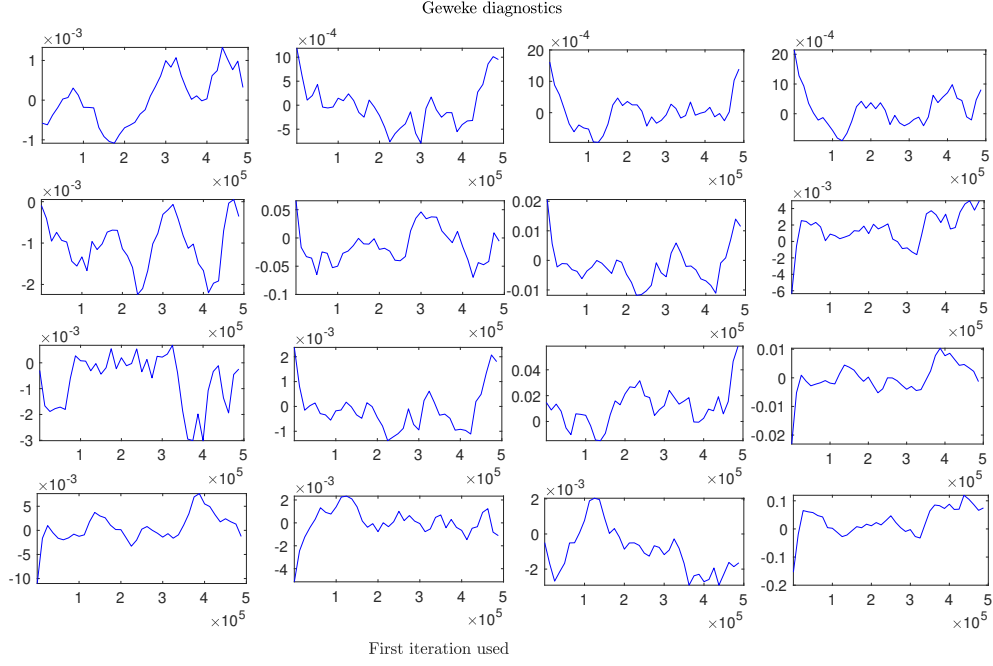
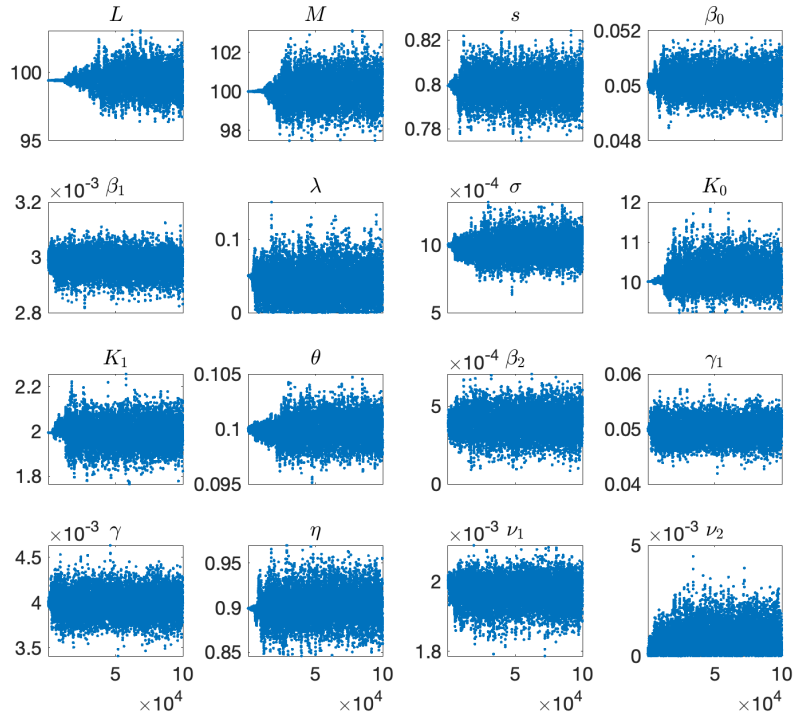


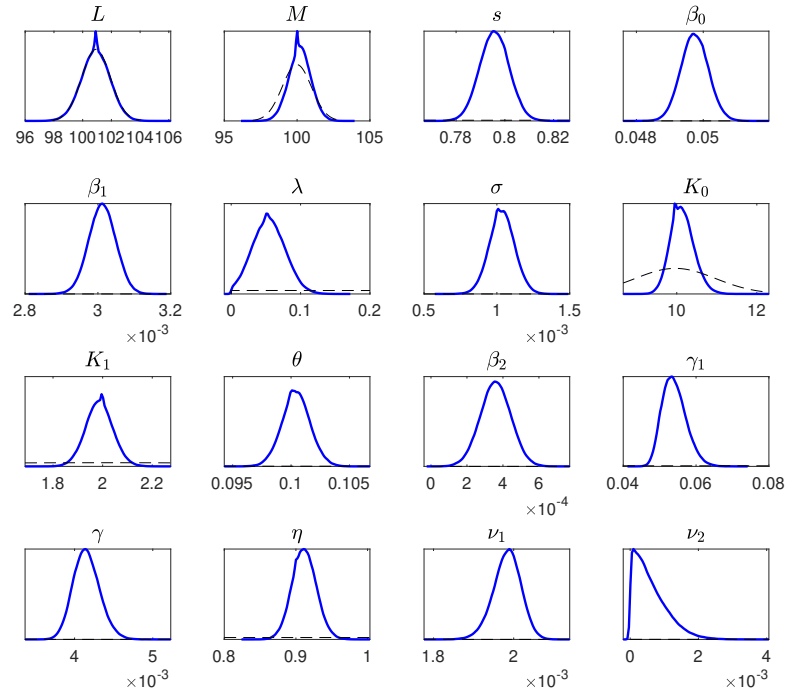
**Figure 19:** Geweke’s diagnostics. The first row:  $L$ ,  $M$ ,  $s$  and  $\beta_0$ , the second row:  $\beta_1$ ,  $\lambda$ ,  $\sigma$  and  $K_0$ , the third row:  $K_1$ ,  $\theta$ ,  $\beta_2$  and  $\gamma_1$ , fourth row:  $\gamma$ ,  $\eta$ ,  $\nu_1$  and  $\nu_2$

For a well-mixed sample, the marginal distribution of all estimated parameters should follow a Gaussian distribution. The plots of the marginal distributions of all 16 parameters are shown in Fig. 20b. Apart from  $\nu_2$ , almost all parameters follow a Gaussian distribution, with MCMC means as the distributions’ means. The poor convergence of  $\nu_2$  could be attributed to various factors, such as the complexity of the model, limited prior knowledge regarding the mean values of the parameter, or shortcomings of the MCMC method in effectively sampling this particular parameter from the proposed distribution.

Scatter plots were used to display the correlation between the sampled parameters. For demonstration purposes, the plots of the first six parameters are shown in Fig. 21, as there are a total of 16 parameters, which would require 128 plots and be too numerous for practical presentation. It is essential to clarify that the selection of the first six parameters was not based on any specific scientific reason. The marginal posteriors for the sampled parameters are shown to follow a Gaussian distribution, indicating that the parameters are well mixed, and cross-correlations are of insignificant magnitude. However, a positive correlation is observed between  $s$  and  $\beta_0$ . This correlation is expected, as the growth of the forest-dependent wildlife population entirely depends on the density of forest biomass.



(a)



(b)

**Figure 20:** (a) Trends of MCMC samples demonstrating the quality of the mixing (b) The sampled parameters' marginal density distribution

The sample Autocorrelation Function (ACF) of all 16 parameters, shown in Fig. 22, was also used for the convergence test. From this figure, it can be observed that all ACFs decay toward zero and stabilize around zero as the number of lags increases. This suggests that the mixing of the samples is good and further confirms the successful convergence of the MCMC samples.

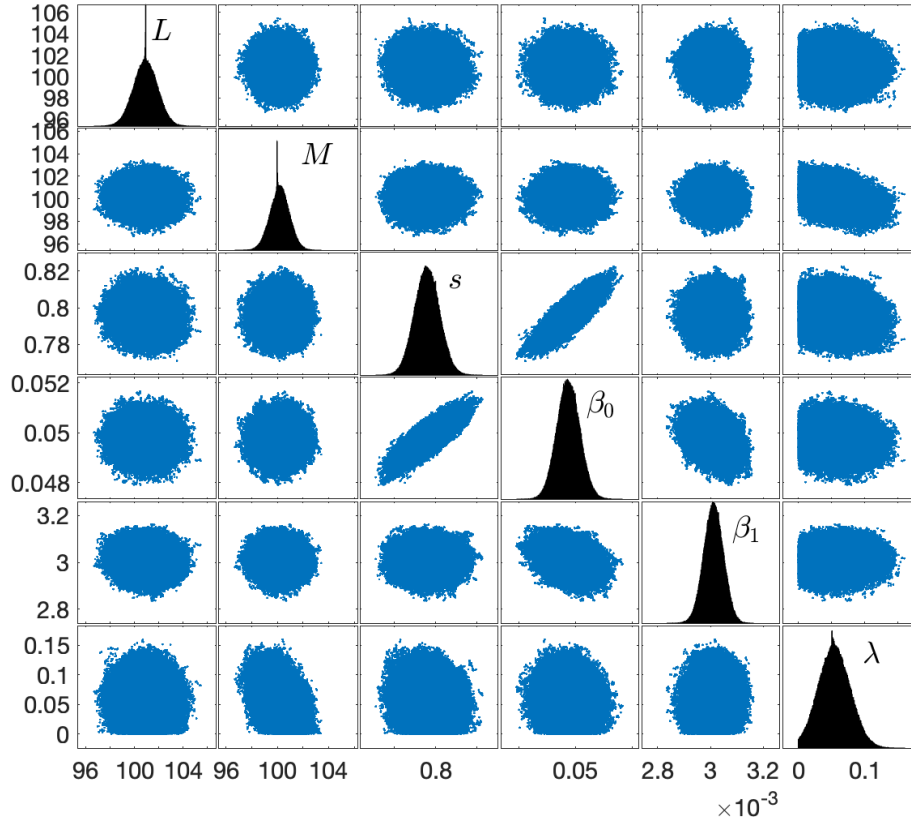


**Figure 21:** The scatter plots show cross-correlations and histogram on diagonal of the combined chains for the first six parameters to visualize the 1-D and 2-D marginal posterior distributions

The histograms shown in Fig. 23 serve to encapsulate data density and are frequently employed for density estimation, specifically in the estimation of the probability density function for the underlying variable. These histograms illustrate the distributions of model parameters. The cumulative area beneath each histogram, representing probability density, is normalized to unity. In instances where the intervals on the x-axis possess a uniform length of 1, the histogram essentially equates to a relative frequency plot. The discernible Gaussian nature of the distributions for almost all parameters signifies favourable mixing.

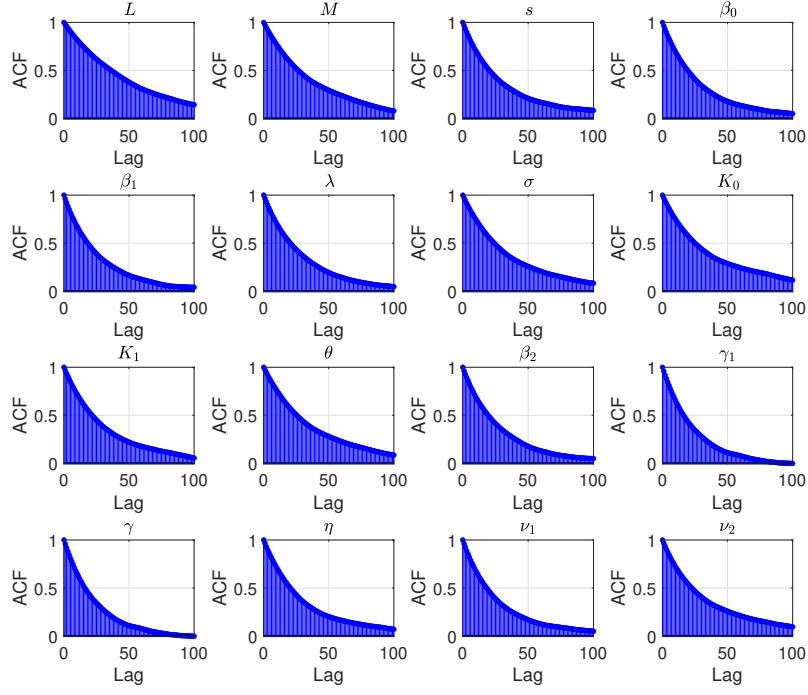


**Figure 22:** Sample autocorrelation plots for the 16 MCMC sample parameters. The ACF coefficients are represented on the  $x$  axis, while lags are represented on the  $y$  axis

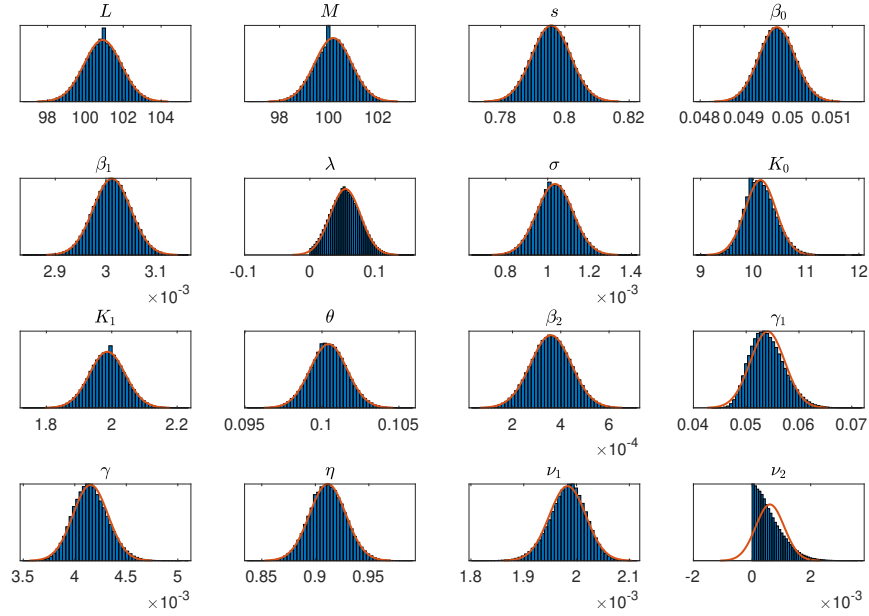


**Figure 23:** Histograms illustrating the probability density function estimation for model parameters, showcasing Gaussian distributions indicative of effective mixing

#### 4.2.8 Numerical simulation

The numerical solutions were obtained using the arbitrarily chosen initial values of the variables as follows:  $B(0) = 2$ ;  $W(0) = 2$ ;  $N(0) = 10$ ;  $H(0) = 1$ . Additionally, the estimated parameter values in Table 12 were utilised. It can be observed that with these numerical values, the conditions for local stability (Theorem 4.4) and global stability (Theorem 4.5) are all satisfied. This suggests that the interior equilibrium  $E^*$  is both locally and globally asymptotically stable for the estimated parameter values. The corresponding values of the interior equilibrium point for the model (15) are  $B^* = 6.68$ ,  $W^* = 9.02$ ,  $N^* = 91.92$ ,  $H^* = 7.34$ .

On utilising these parametric values, the eigenvalues of the Jacobian matrix corresponding to  $E^*$  are as follows:

$$-0.9774, -0.0099, -0.0405 + 0.3281i, -0.0405 - 0.3281i.$$

It can be observed that the eigenvalues are either negative or have a negative real part, the negative sign ensures the local stability of interior equilibrium point  $E^*$ , therefore, Fig. 24 shows the trajectories of the simulated solutions of the model system (15).

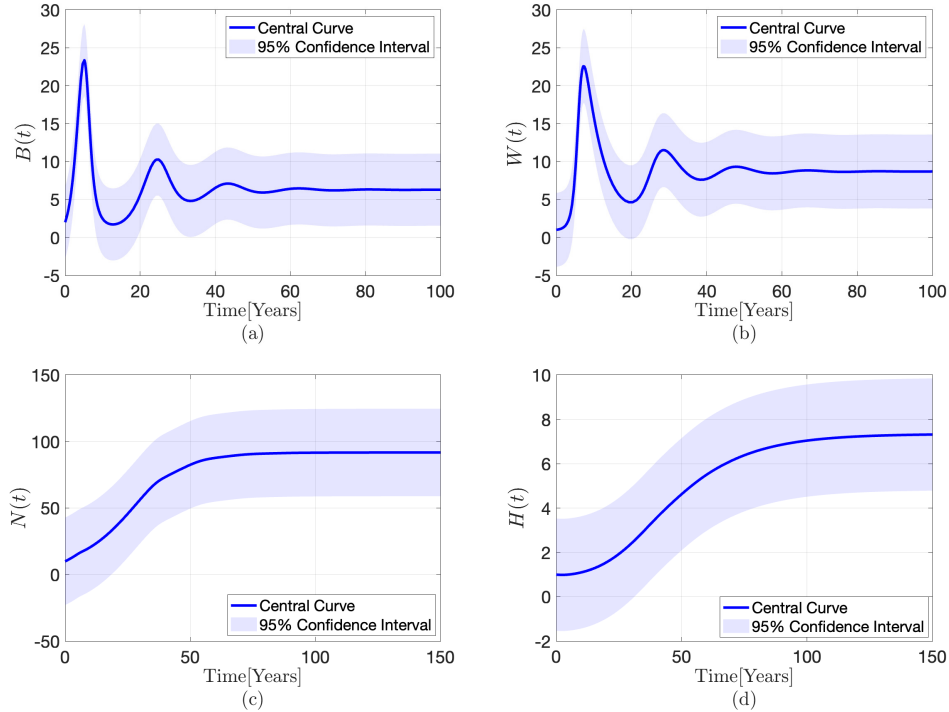


**Figure 24:** Solutions of the model system (15) using the parameter values in Table 12; (a) Forest biomass, (b) Wildlife population, (c) Human population and (d) Human activities

The parameter values in Table 12 were used to demonstrate the non-linear stability of the equi-

librium point  $E^*$  using different initial conditions, as shown in Figs. 25a and 25b. It can be observed that all the solution trajectories initiating inside the region of attraction are approaching the equilibrium values  $(B^*, W^*)$  and  $(B^*, W^*, N^*)$  in the respective planes  $B - W$  and  $B - W - N$ . This indicates that the equilibrium  $E^*$  is globally asymptotically stable in these planes. A similar approach can be used to demonstrate the nonlinear stability in other planes.

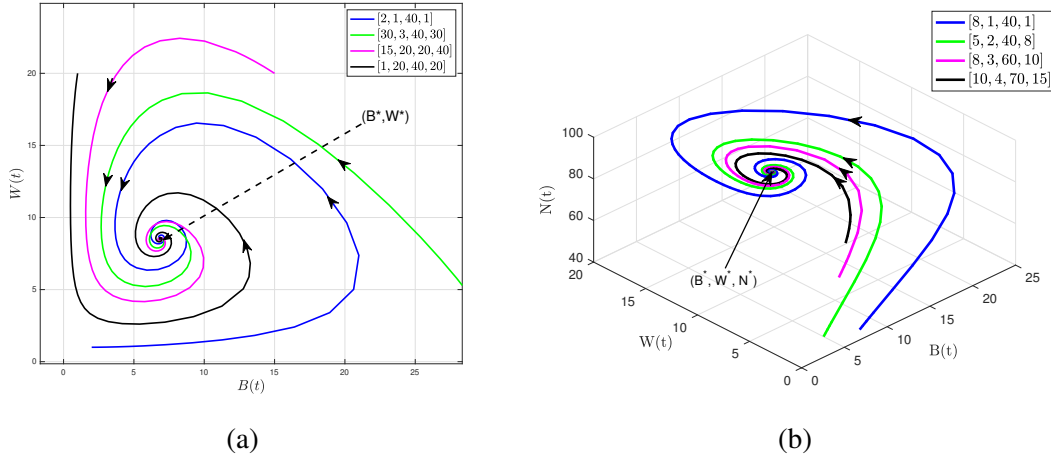


**Figure 25:** Non-linear stability of the equilibrium point  $E^*$  in (a)  $B - W$  plane, (b)  $B - W - N$  plane

The critical value of the bifurcation parameter  $\beta_1$  for the trans-critical bifurcation was found to be  $\beta_1^* = 0.008$ . The plot for the trans-critical bifurcation is shown in Fig. 26. It can be observed that when the value of  $\beta_1$  crosses the critical value, a change in feasibility and stability occurs between  $E_3$  and  $E^*$ . At the critical point, these equilibrium points undergo a bifurcation, leading to significant changes in the system's dynamics.

The influence of some parameters, such as  $\beta_1$ ,  $\beta_2$ ,  $\gamma$ , and  $\nu_2$ , on the system dynamics was explored. These parameters were selected based on the conditions stated in Theorems 4.4 and 4.5. For instance, Fig. 27 presents the variations of forest biomass, forest-dependent wildlife population, human population, and human activities with time for different values of the forest depletion rate due to human population ( $\beta_1$ ). From the figure, it can be observed that as  $\beta_1$  increases, the forest biomass and forest-dependent wildlife population decrease and eventually become extinct. Conversely, changes in  $\beta_1$  have insignificant effects on the dynamics of human population and human activities.

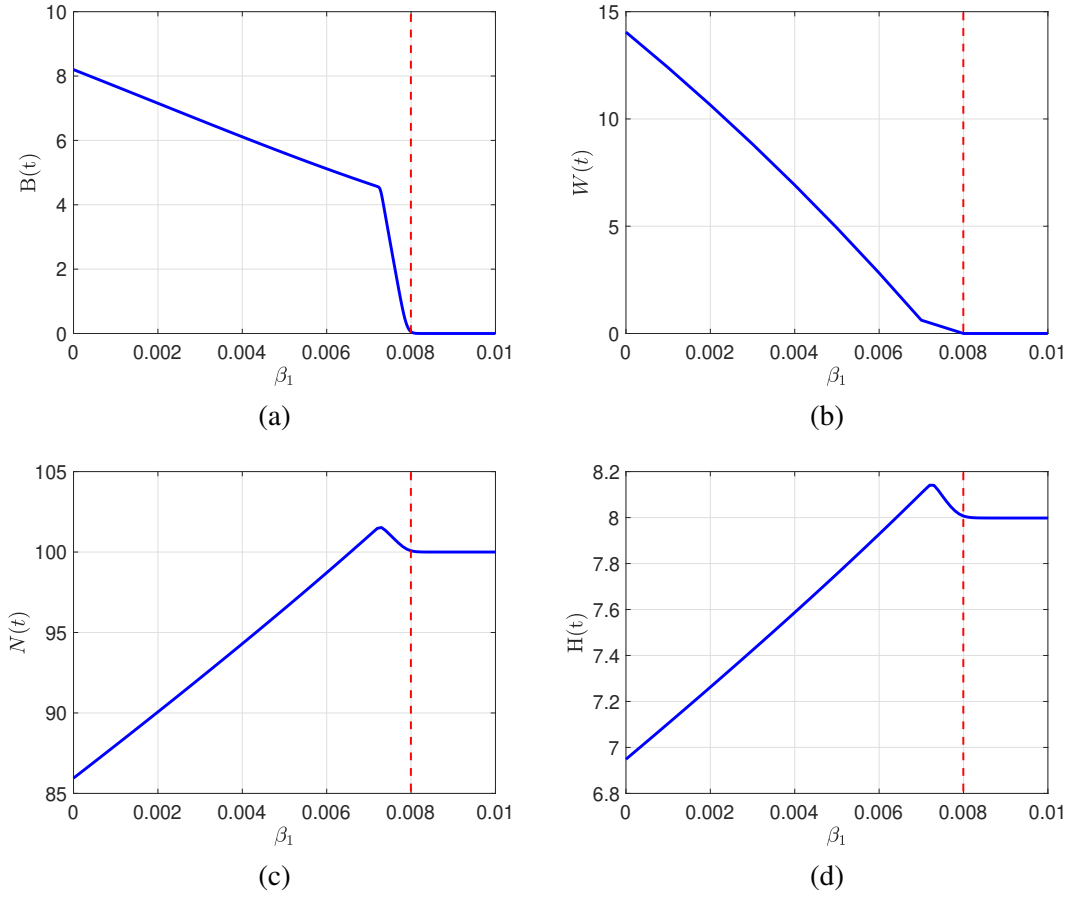


**Figure 26:** Trans-critical bifurcation between  $E_3$  and  $E^*$  with respect to the depletion rate of forest biomass due to human population ( $\beta_1$ ), other parameter values were kept fixed as in Table 12 and taking  $s = 0.8$

This suggests that as human encroachment into forest areas increases, the density of the forest decreases, ultimately affecting the population of wildlife species dependent on the forest, leading to increased intraspecific competition among the species.

The variations of forest biomass, forest-dependent wildlife population, human population, and human activities over time, for different values of the forest depletion rate due to human activities ( $\beta_2$ ), are shown in Fig. 28. From this figure, it is observed that as the value of this parameter increases, there is a corresponding decrease in the densities of forest biomass and wildlife populations. In contrast, the dynamics of human population and human activities show relatively insignificant effects compared to Fig. 27, where pronounced effects were observed in all variables.

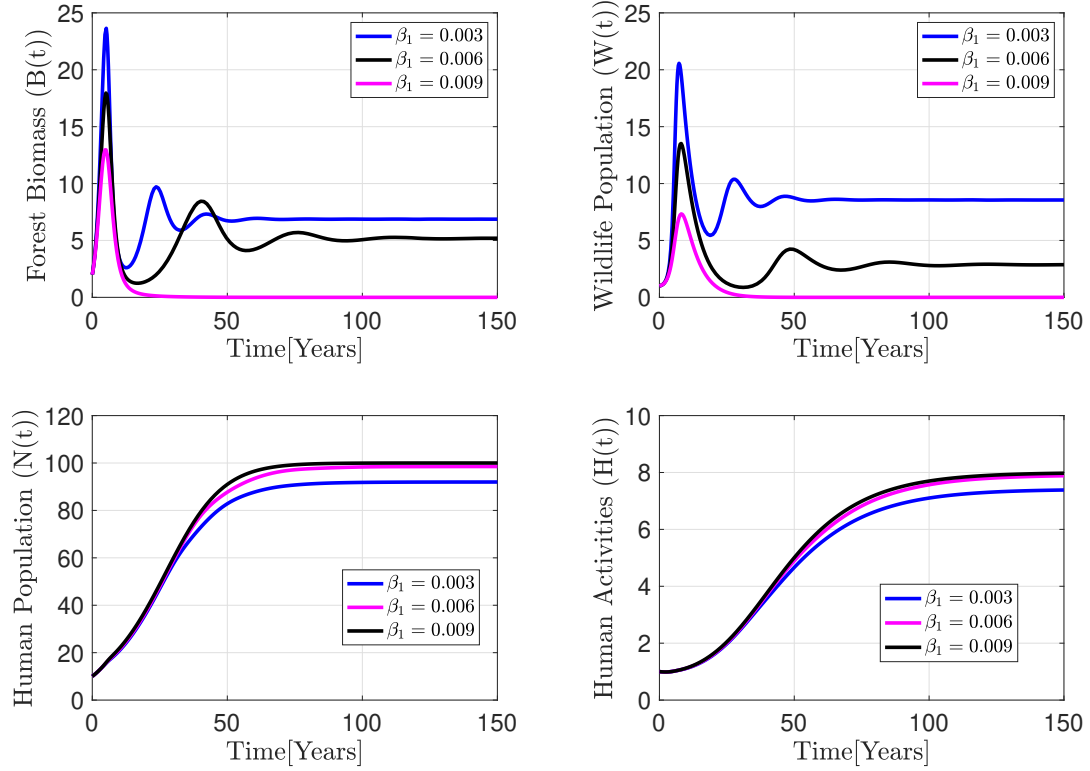


**Figure 27:** Variations of  $B$ ,  $W$ ,  $N$ , and  $H$  with respect to time ( $t$ ) for different values of  $\beta_1$

This observation suggests that as human activities increase, more forest resources are utilised, thereby reducing its carrying capacity and leading to increased intraspecific competition among wildlife species. Consequently, the decline in forest biomass and wildlife populations becomes more evident.

The impacts of the growth rate of human activities due to the human population ( $\gamma$ ) on system stability have also been studied, and the results are shown in Fig. 29. From this figure, it is observed that an increase in  $\gamma$  destabilises the system as it intensifies human activities, leading to a decrease in the densities of forest biomass and wildlife populations. The decrease is more pronounced in the wildlife population, suggesting that wildlife species are affected by human activities both directly and indirectly. Furthermore, the findings suggest that human population growth is a key driver of deforestation, leading to the depletion of forest resources. Moreover, it indicates that human activities are responsible for the decline and potential extinction of wildlife species, highlighting the critical role of sustainable human practices and conservation efforts to maintain ecological balance and biodiversity.

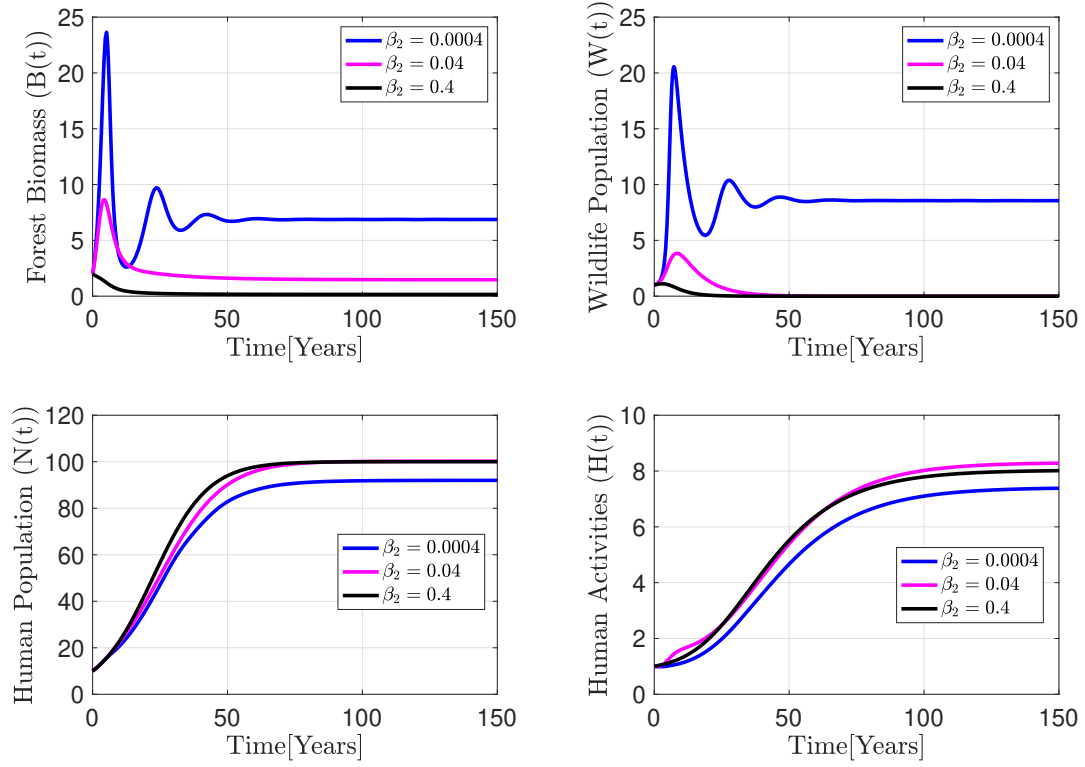


**Figure 28:** Variations of  $B$ ,  $W$ ,  $N$ , and  $H$  with respect to time ( $t$ ) for different values of  $\beta_2$

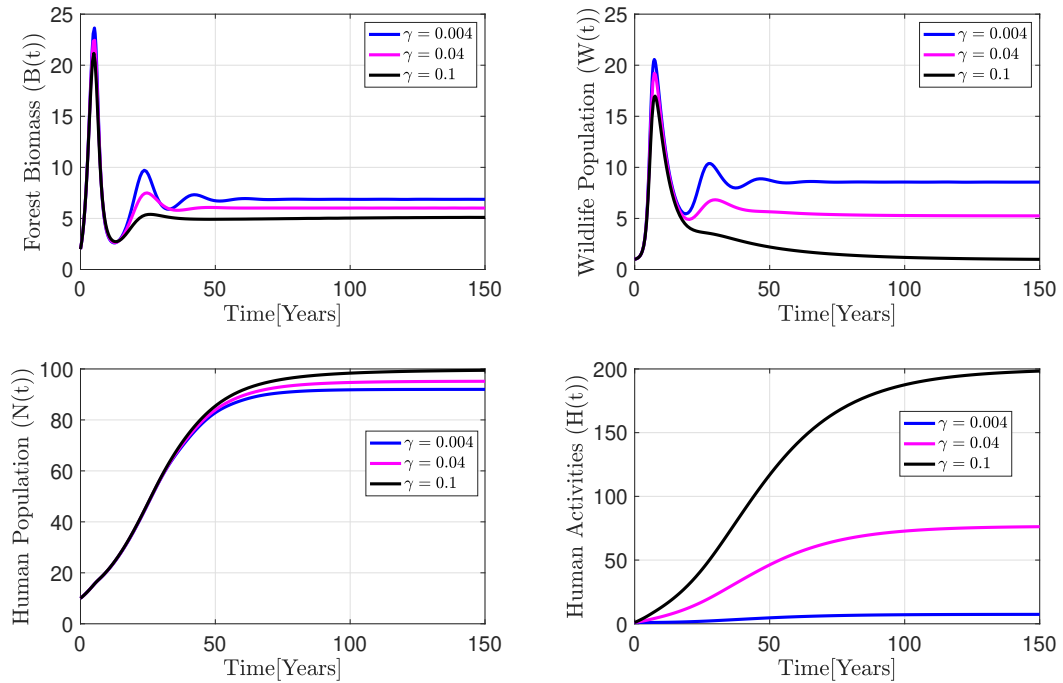


**Figure 29:** Variations of  $B$ ,  $W$ ,  $N$ , and  $H$  with respect to time ( $t$ ) for different values of  $\gamma$

The changes in the system dynamics over time for different values of the wildlife population depletion rate due to human activities ( $\nu_2$ ) were investigated, and the results are presented in Fig. 30. From this figure, it is observed that as the value of  $\nu_2$  increases, the equilibrium levels of forest biomass and human activities also increase, while the density of wildlife populations decreases. This observation suggests that human activities are primarily responsible for the depletion of wildlife populations, which, in turn impacts the equilibrium levels of forest biomass and human activities. The results underscore the importance of sustainable practices and conservation efforts to safeguard wildlife populations and maintain ecological balance in the region.

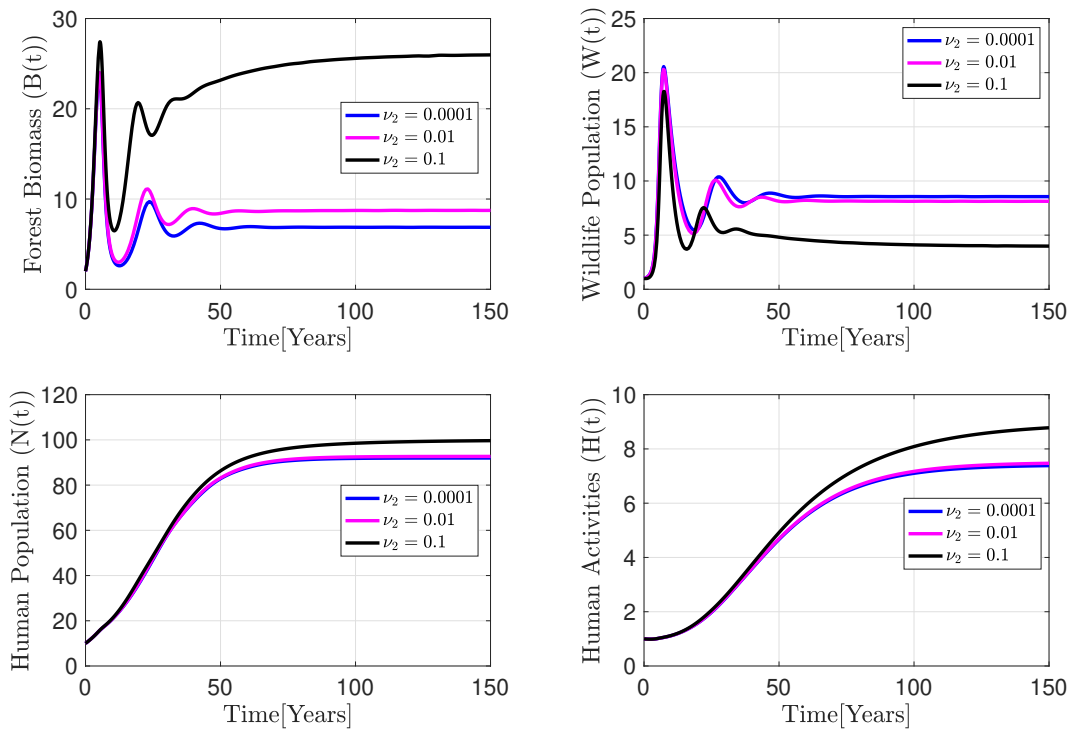


**Figure 30:** Variations of  $B$ ,  $W$ ,  $N$ , and  $H$  with respect to time ( $t$ ) for different values of  $\nu_2$

The dynamics of the system were further explored by analysing the impact of parameter variations, specifically focusing on  $\beta_2$ ,  $\gamma$ , and  $\nu_2$  as were identified to be important parameters in the stability of the interior equilibrium point. The corresponding results shown in Figs. 31, 32, and 33, respectively, shed light on the system's behaviour under different parameter settings. Notably, the analysis reveals that maintaining system stability requires these parameters to remain below certain thresholds. Specifically, the depletion rate of forest biomass due to human activities should not exceed 0.03, while the growth rate of human activities should be limited

to 0.2. Additionally, the depletion rate of wildlife population resulting from human activities should be constrained to 0.3, assuming all other parameters remain unchanged. Consequently, the determined threshold values for  $\beta_2$ ,  $\gamma$ , and  $\nu_2$  are identified as 0.03, 0.2, and 0.3, respectively. These findings provide valuable insights into the sustainable management of the system, highlighting the critical role of controlling these parameters within specified limits.

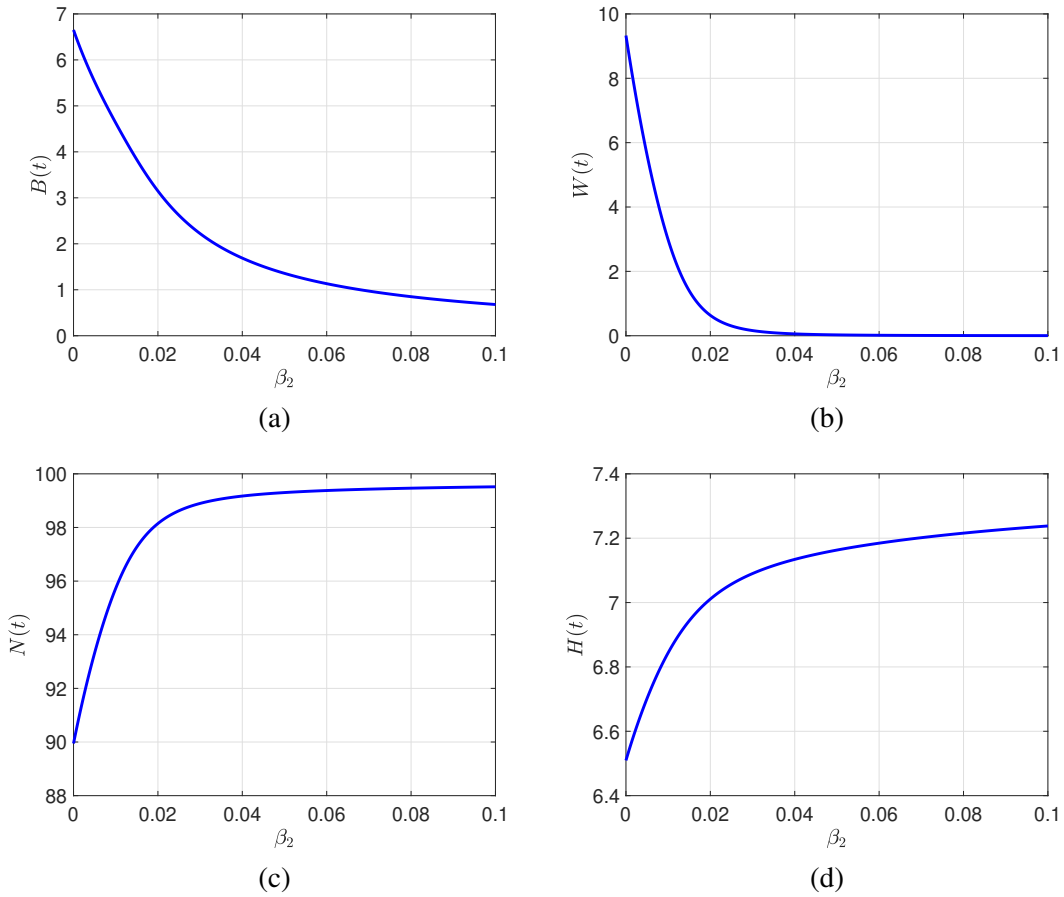


**Figure 31:** Evolution of solution trajectories of (a) forest biomass ( $B$ ), (b) forest-dependent wildlife population ( $W$ ), (c) human population ( $N$ ) and (d) human activities ( $H$ ) with different values of  $\beta_2$ , other parameters were kept fixed

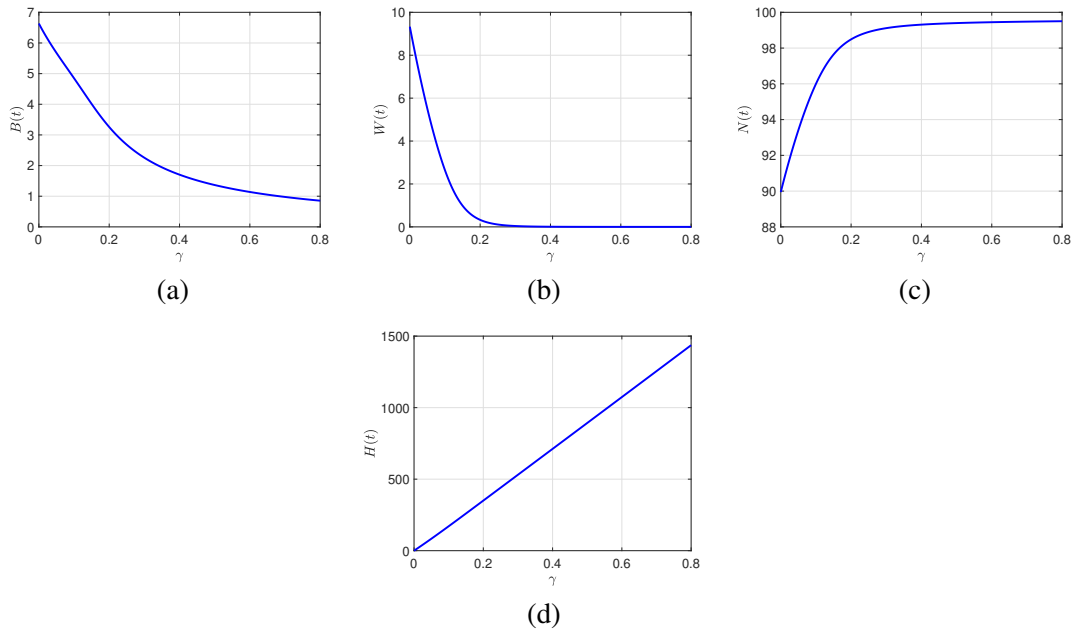


**Figure 32:** Evolution of solution trajectories of (a) forest biomass ( $B$ ), (b) forest-dependent wildlife population ( $W$ ), (c) human population ( $N$ ) and (d) human activities ( $H$ ) with different values of  $\gamma$ , other parameters were kept fixed

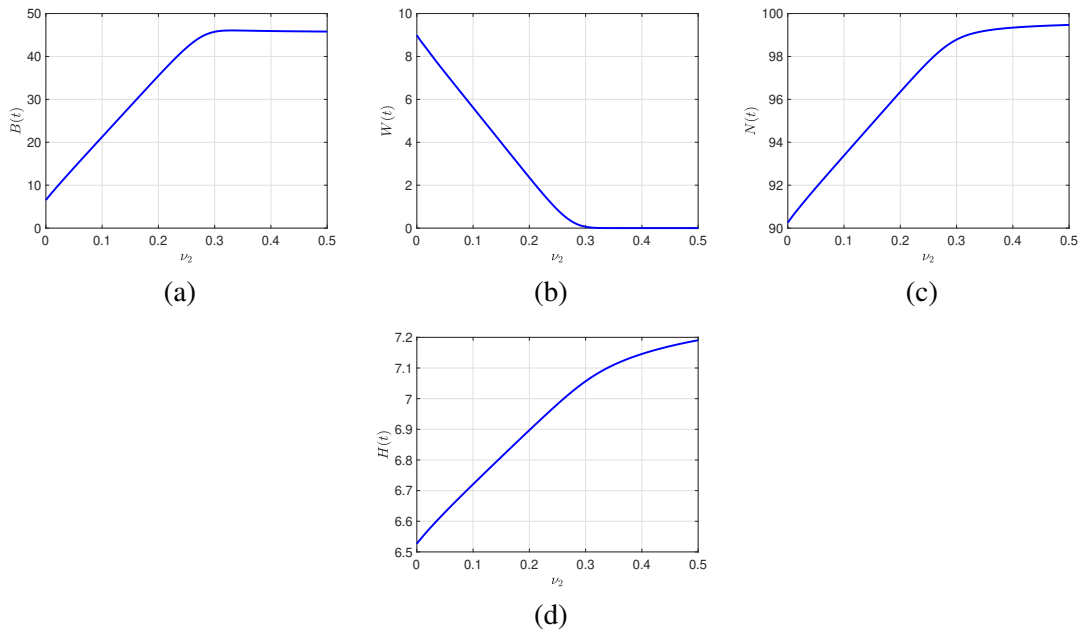


**Figure 33:** Evolution of solution trajectories of (a) forest biomass ( $B$ ), (b) forest-dependent wildlife population ( $W$ ), (c) human population ( $N$ ) and (d) human activities ( $H$ ) with different values of  $\nu_2$ , other parameters were kept fixed

### 4.3 Fuzzy Modelling on Depletion of Forest Biomass and Forest-Dependent Wildlife Population

This model is formulated by assuming that the parameters used in the model system (15) are triangular fuzzy numbers [20]. This extension allowed to account for the inherent imprecision that often exists in real-world scenarios [20]. Therefore, suppose  $\Delta_i$  is the parameter responsible for controlling the degree of fuzziness in a fuzzy set, it follows, if  $\tilde{p}_i$  is the fuzzy parameter in the set of parameters (Table 1) and  $p_i$  is its corresponding crisp parameter, then the parameter  $\tilde{p}_i$  is given by  $\tilde{p}_i = (p_i - \Delta_i, p_i, p_i + \Delta_{i+1})$ , such that,  $0 < \Delta_i < p_i$  and  $\Delta_{i+1} > 0$ . Thus, on utilising Hukuhara derivative concept [94], the system of fuzzy differential equations (70) describes the model in fuzzy environment.

$$\begin{cases} \frac{d\tilde{B}}{dt} = \tilde{s}B \left(1 - \frac{B}{L}\right) - \tilde{\beta}_0 BW - \tilde{\beta}_1 BN - \tilde{\beta}_2 B^2 H, \\ \frac{d\tilde{W}}{dt} = \tilde{\beta}_0 \tilde{\eta} BW \left(1 - \frac{W}{K(B)}\right) - \tilde{\nu}_1 WN - \tilde{\nu}_2 WH, \\ \frac{d\tilde{N}}{dt} = \tilde{\theta}N \left(1 - \frac{N}{M}\right) + \tilde{\lambda} \tilde{\beta}_1 BN - \tilde{\sigma} NW, \\ \frac{d\tilde{H}}{dt} = \tilde{\gamma}N - \tilde{\gamma}_1 H, \end{cases} \quad (70)$$

with initial conditions:

$$X(t) = (B(t), W(t), N(t), H(t)) : X(0) \geq 0,$$

where,

$$\tilde{s} = (s - \Delta_1, s, s + \Delta_2), \quad 0 < s < \Delta_1, \Delta_2 > 0,$$

$$\tilde{\beta}_0 = (\beta_0 - \Delta_3, \beta_0, \beta_0 + \Delta_4), \quad 0 < \beta_0 < \Delta_3, \Delta_4 > 0,$$

$$\tilde{\beta}_1 = (\beta_1 - \Delta_5, \beta_1, \beta_1 + \Delta_6), \quad 0 < \beta_1 < \Delta_5, \Delta_6 > 0,$$

$$\tilde{\beta}_2 = (\beta_2 - \Delta_7, \beta_2, \beta_2 + \Delta_8), \quad 0 < \beta_2 < \Delta_7, \Delta_8 > 0,$$

$$\tilde{\eta} = (\eta - \Delta_9, \eta, \eta + \Delta_{10}), \quad 0 < \eta < \Delta_9, \Delta_{10} > 0,$$

$$\tilde{\nu}_1 = (\nu_1 - \Delta_{11}, \nu_1, \nu_1 + \Delta_{12}), \quad 0 < \nu_1 < \Delta_{11}, \Delta_{12} > 0,$$

$$\tilde{\nu}_2 = (\nu_2 - \Delta_{13}, \nu_2, \nu_2 + \Delta_{14}), \quad 0 < \nu_2 < \Delta_{13}, \Delta_{14} > 0,$$

$$\tilde{\theta} = (\theta - \Delta_{15}, \theta, \theta + \Delta_{16}), \quad 0 < \theta < \Delta_{15}, \Delta_{16} > 0,$$

$$\tilde{\lambda} = (\lambda - \Delta_{17}, \lambda, \lambda + \Delta_{18}), \quad 0 < \lambda < \Delta_{17}, \Delta_{18} > 0,$$

$$\tilde{\sigma} = (\sigma - \Delta_{19}, \sigma, \sigma + \Delta_{20}), \quad 0 < \sigma < \Delta_{19}, \Delta_{20} > 0,$$

$$\tilde{\gamma} = (\gamma - \Delta_{21}, \gamma, \gamma + \Delta_{22}), \quad 0 < \gamma < \Delta_{21}, \Delta_{22} > 0,$$

$$\tilde{\gamma}_1 = (\gamma_1 - \Delta_{23}, \gamma_1, \gamma_1 + \Delta_{24}), \quad 0 < \gamma_1 < \Delta_{23}, \Delta_{24} > 0.$$

To analyse the model system (70), the fuzzy parameters were transformed into intervals by taking the  $\alpha$ -cut on both sides of the model equation (definition 3.3). The values of  $\alpha$  are within the interval  $\alpha \in [0, 1]$ , where a value of 1 indicates full membership and a value of 0 indicates no membership. This transformation offers intervals that represent the range of possible values for each parameter, considering their fuzzy nature. Thus, this process results in the system of equations (71):

$$\begin{cases} \frac{d\tilde{B}}{dt}(\alpha) = (\tilde{s})(\alpha) B \left(1 - \frac{B}{L}\right) - (\tilde{\beta}_0)(\alpha) BW - (\tilde{\beta}_1)(\alpha) BN - (\tilde{\beta}_2)(\alpha) B^2H, \\ \frac{d\tilde{W}}{dt}(\alpha) = (\tilde{\beta}_0 \otimes \tilde{\eta})(\alpha) BW \left(1 - \frac{W}{K(B)}\right) - (\tilde{\nu}_1)(\alpha) WN - (\tilde{\nu}_2)(\alpha) WH, \\ \frac{d\tilde{N}}{dt}(\alpha) = (\tilde{\theta})(\alpha) N \left(1 - \frac{N}{M}\right) + (\tilde{\lambda} \otimes \tilde{\beta}_1)(\alpha) BN - (\tilde{\sigma})(\alpha) NW, \\ \frac{d\tilde{H}}{dt}(\alpha) = (\tilde{\gamma})(\alpha) N - (\tilde{\gamma}_1)(\alpha) H. \end{cases} \quad (71)$$

More precisely, system (71) can be written as:

$$\begin{aligned} \left[ \left( \frac{dB}{dt} \right)_l(\alpha), \left( \frac{dB}{dt} \right)_r(\alpha) \right] &= [s_l(\alpha), s_r(\alpha)] B \left(1 - \frac{B}{L}\right) - [(\beta_0)_l(\alpha), (\beta_0)_r(\alpha)] BW \\ &\quad - [(\beta_1)_l(\alpha), (\beta_1)_r(\alpha)] BN - [(\beta_2)_l(\alpha), (\beta_2)_r(\alpha)] B^2H, \\ \left[ \left( \frac{dW}{dt} \right)_l(\alpha), \left( \frac{dW}{dt} \right)_r(\alpha) \right] &= [(\beta_0)_l \eta_l(\alpha), (\beta_0)_r \eta_r(\alpha)] BW \left(1 - \frac{W}{K(B)}\right) \\ &\quad - [(\nu_1)_l(\alpha), (\nu_1)_r(\alpha)] WN - [(\nu_2)_l(\alpha), (\nu_2)_r(\alpha)] WH, \\ \left[ \left( \frac{dN}{dt} \right)_l(\alpha), \left( \frac{dN}{dt} \right)_r(\alpha) \right] &= [\theta_l(\alpha), \theta_r(\alpha)] N \left(1 - \frac{N}{M}\right) - [\sigma_l(\alpha), \sigma_r(\alpha)] NW \\ &\quad + [\lambda_l(\beta_1)_l(\alpha), \lambda_r(\beta_1)_r(\alpha)] BN, \\ \left[ \left( \frac{dH}{dt} \right)_l(\alpha), \left( \frac{dH}{dt} \right)_r(\alpha) \right] &= [\gamma_l(\alpha), \gamma_r(\alpha)] N - [(\gamma_1)_l(\alpha), (\gamma_1)_r(\alpha)] H. \end{aligned}$$

Subsequently, the principles of interval mathematics [39] were applied to convert the system of differential equations into a pair of distinct differential equations that capture the possible

behaviours of the system within the defined intervals. This approach helps to account the uncertainties in the parameters and better understand the system's dynamics under different scenarios.

This computations give the following set of equations:

$$\begin{aligned}
\left(\frac{dB}{dt}\right)_l(\alpha) &= s_l(\alpha) B \left(1 - \frac{B}{L}\right) - (\beta_0)_l(\alpha) BW - (\beta_1)_l(\alpha) BN - (\beta_2)_l(\alpha) B^2 H, \\
\left(\frac{dB}{dt}\right)_r(\alpha) &= s_r(\alpha) B \left(1 - \frac{B}{L}\right) - (\beta_0)_r(\alpha) BW - (\beta_1)_r(\alpha) BN - (\beta_2)_r(\alpha) B^2 H, \\
\left(\frac{dW}{dt}\right)_l(\alpha) &= (\beta_0)_l \eta_l(\alpha) BW \left(1 - \frac{W}{K(B)}\right) - (\nu_1)_l(\alpha) WN - (\nu_2)_l(\alpha) WH, \\
\left(\frac{dW}{dt}\right)_r(\alpha) &= (\beta_0)_r \eta_r(\alpha) BW \left(1 - \frac{W}{K(B)}\right) - (\nu_1)_r(\alpha) WN - (\nu_2)_r(\alpha) WH, \\
\left(\frac{dN}{dt}\right)_l(\alpha) &= \theta_l(\alpha) N \left(1 - \frac{N}{M}\right) + \lambda_l(\beta_1)_l(\alpha) BN - \sigma_l(\alpha) NW, \\
\left(\frac{dN}{dt}\right)_r(\alpha) &= \theta_r(\alpha) N \left(1 - \frac{N}{M}\right) + \lambda_r(\beta_1)_r(\alpha) BN - \sigma_r(\alpha) NW, \\
\left(\frac{dH}{dt}\right)_l(\alpha) &= \gamma_l(\alpha) N - (\gamma_1)_l(\alpha) H, \\
\left(\frac{dH}{dt}\right)_r(\alpha) &= \gamma_r(\alpha) N - (\gamma_1)_r(\alpha) H,
\end{aligned}$$

where,

$$\begin{aligned}
s_l(\alpha) &= s - \Delta_1 + \alpha \Delta_1, & s_r(\alpha) &= s - \Delta_2 + \alpha \Delta_2, \\
(\beta_0)_l(\alpha) &= \beta_0 - \Delta_3 + \alpha \Delta_3, & (\beta_0)_r(\alpha) &= \beta_0 - \Delta_4 + \alpha \Delta_4, \\
(\beta_1)_l(\alpha) &= \beta_1 - \Delta_5 + \alpha \Delta_5, & (\beta_1)_r(\alpha) &= \beta_1 - \Delta_6 + \alpha \Delta_6, \\
(\beta_2)_l(\alpha) &= \beta_2 - \Delta_7 + \alpha \Delta_7, & (\beta_2)_r(\alpha) &= \beta_2 - \Delta_8 + \alpha \Delta_8, \\
\eta_l(\alpha) &= \eta - \Delta_9 + \alpha \Delta_9, & \eta_r(\alpha) &= \eta - \Delta_{10} + \alpha \Delta_{10}, \\
(\nu_1)_l(\alpha) &= \nu_1 - \Delta_{11} + \alpha \Delta_{11}, & (\nu_1)_r(\alpha) &= \nu_1 - \Delta_{12} + \alpha \Delta_{12}, \\
(\nu_2)_l(\alpha) &= \nu_2 - \Delta_{13} + \alpha \Delta_{13}, & (\nu_2)_r(\alpha) &= \nu_2 - \Delta_{14} + \alpha \Delta_{14}, \\
\theta_l(\alpha) &= \theta - \Delta_{15} + \alpha \Delta_{15}, & \theta_r(\alpha) &= \theta - \Delta_{16} + \alpha \Delta_{16}, \\
\lambda_l(\alpha) &= \lambda - \Delta_{17} + \alpha \Delta_{17}, & \lambda_r(\alpha) &= \lambda - \Delta_{18} + \alpha \Delta_{18}, \\
\sigma_l(\alpha) &= \sigma - \Delta_{19} + \alpha \Delta_{19}, & \sigma_r(\alpha) &= \sigma - \Delta_{20} + \alpha \Delta_{20}, \\
\gamma_l(\alpha) &= \gamma - \Delta_{21} + \alpha \Delta_{21}, & \gamma_r(\alpha) &= \gamma - \Delta_{22} + \alpha \Delta_{22}, \\
(\gamma_1)_l(\alpha) &= \gamma_1 - \Delta_{23} + \alpha \Delta_{23}, & (\gamma_1)_r(\alpha) &= \gamma_1 - \Delta_{24} + \alpha \Delta_{24}.
\end{aligned}$$

To further simplify the analysis and gain deeper insights into the system's dynamics, the concept of signed distance for the fuzzy numbers as defined in Definition 3.1 was employed to defuzzify the system. This approach allowed us to obtain a system of defuzzified differential equations (72), which captured the essential characteristics of the system's behaviour under the influence of fuzzy parameters.

In so doing, the following system of equations is obtained:

$$\begin{aligned}
d\left(\frac{\widetilde{dB}}{dt}, \widetilde{0}\right) &= d\left(\widetilde{s}, \widetilde{0}\right) B \left(1 - \frac{B}{L}\right) - d\left(\widetilde{\beta}_0, \widetilde{0}\right) BW - d\left(\widetilde{\beta}_1, \widetilde{0}\right) BN - d\left(\widetilde{\beta}_2, \widetilde{0}\right) B^2 H, \\
d\left(\frac{\widetilde{dW}}{dt}, \widetilde{0}\right) &= d\left(\widetilde{\beta}_0 \otimes \widetilde{\eta}, \widetilde{0}\right) BW \left(1 - \frac{W}{K(B)}\right) - d\left(\widetilde{\nu}_1, \widetilde{0}\right) WN - d\left(\widetilde{\nu}_2, \widetilde{0}\right) WH, \\
d\left(\frac{\widetilde{dN}}{dt}, \widetilde{0}\right) &= d\left(\widetilde{\theta}, \widetilde{0}\right) N \left(1 - \frac{N}{M}\right) + d\left(\widetilde{\lambda} \otimes \widetilde{\beta}_1, \widetilde{0}\right) BN - d\left(\widetilde{\sigma}, \widetilde{0}\right) NW, \\
d\left(\frac{\widetilde{dH}}{dt}, \widetilde{0}\right) &= d\left(\widetilde{\gamma}, \widetilde{0}\right) N - d\left(\widetilde{\gamma}_1, \widetilde{0}\right) H.
\end{aligned}$$

Upon evaluating and substituting the values of the respective signed distance gives the defuzzified system (72) in the form of crisp model:

$$\begin{cases} \frac{dB}{dt} = S_1 B \left(1 - \frac{B}{L}\right) - S_2 BW - S_3 BN - S_4 B^2 H, \\ \frac{dW}{dt} = S_5 BW \left(1 - \frac{W}{K(B)}\right) - S_6 WN - S_7 WH, \\ \frac{dN}{dt} = S_8 N \left(1 - \frac{N}{M}\right) + S_9 BN - S_{10} NW, \\ \frac{dH}{dt} = S_{11} N - S_{12} H, \end{cases} \quad (72)$$

with initial conditions:

$$X(t) = (B(t), W(t), N(t), H(t)) : X(0) \geq 0,$$

where,

$$\begin{aligned}
S_1 &= s + \frac{1}{4} (\Delta_2 - \Delta_1), \quad S_2 = \beta_0 + \frac{1}{4} (\Delta_4 - \Delta_3), \quad S_3 = \beta_1 + \frac{1}{4} (\Delta_6 - \Delta_5), \\
S_4 &= \beta_2 + \frac{1}{4} (\Delta_8 - \Delta_7), \quad S_5 = \frac{\beta_0 \eta}{2} + \frac{\eta}{4} (\Delta_4 - \Delta_3) + \frac{\beta_0}{4} (\Delta_{10} - \Delta_9) + \frac{1}{4} (\Delta_3 \Delta_9 + \Delta_4 \Delta_{10}), \\
S_6 &= \nu_1 + \frac{1}{4} (\Delta_{12} - \Delta_{11}), \quad S_7 = \nu_2 + \frac{1}{4} (\Delta_{14} - \Delta_{13}), \quad S_8 = \theta + \frac{1}{4} (\Delta_{16} - \Delta_{15}), \\
S_9 &= \frac{\lambda \beta_1}{2} + \frac{\lambda}{4} (\Delta_6 - \Delta_5) + \frac{\beta_1}{4} (\Delta_{18} - \Delta_{17}) + \frac{1}{4} (\Delta_5 \Delta_{17} + \Delta_6 \Delta_{18}),
\end{aligned}$$

$$\begin{aligned} S_{10} &= \sigma + \frac{1}{4} (\Delta_{20} - \Delta_{19}), \quad S_{11} = \gamma + \frac{1}{4} (\Delta_{22} - \Delta_{21}), \\ S_{12} &= \gamma_1 + \frac{1}{4} (\Delta_{24} - \Delta_{23}). \end{aligned}$$

#### 4.3.1 Boundedness and persistence of the solutions

The validity of the model system (72) was confirmed by the boundedness and persistence of the solutions. Following Chen [15], the region of attraction that describes the boundedness of the system and is discussed as stated in Lemma 4.7. Furthermore, the conditions for the system to remain permanent are stated in Theorem 4.9.

**Lemma 4.7.** *The set*

$$\Omega_f = \{(B, W, N, H) \in \mathbb{R}_+^4 : 0 \leq B \leq L, 0 \leq W \leq K(L), 0 \leq N \leq N_m, 0 \leq H \leq H_m\}$$

*is region of attraction for model system (72) and attract all solutions initiating in the interior of positive orthant, where,*

$$N_m = \frac{M}{S_8} (S_8 + S_9 L),$$

*and*

$$H_m = \frac{S_{11} N_m}{S_{12}}.$$

Similarly, suppose the initial condition of the system is given by:

$$y_0 = (B_0, W_0, N_0, H_0),$$

then, following Chaudhary [14], the system (72) with initial condition  $(y_0)$  is uniformly persistence if there exists positive constants  $\tau_1$  and  $\tau_2$  such that each non-negative solution

$$B(t, y_0), W(t, y_0), N(t, y_0), \text{ and } H(t, y_0)$$

of the system with initial conditions  $y_0 \in \mathbb{R}_+^4$  satisfies:

$$\begin{aligned} \min \left\{ \liminf_{t \rightarrow \infty} B(t, y_0), \liminf_{t \rightarrow \infty} W(t, y_0), \liminf_{t \rightarrow \infty} N(t, y_0), \liminf_{t \rightarrow \infty} H(t, y_0) \right\} &\geq \tau_1, \\ \max \left\{ \limsup_{t \rightarrow \infty} B(t, y_0), \limsup_{t \rightarrow \infty} W(t, y_0), \limsup_{t \rightarrow \infty} N(t, y_0), \limsup_{t \rightarrow \infty} H(t, y_0) \right\} &\leq \tau_2. \end{aligned}$$

**Theorem 4.9.** *The fuzzy model system (72) with initial condition  $y_0$  is uniformly persistent provided the following inequalities hold:*

$$S_1 > S_2 K(L) + S_3 N_m, S_5 > S_6 N_m + S_7 H_m, \text{ and } S_8 > S_{10} F_2.$$

*Proof.* From Lemma 4.7 it is important to note that  $0 \leq B(t) \leq L$ ,  $0 \leq W(t) \leq K(L)$ ,  $0 \leq N(t) \leq N_m$  and  $0 \leq H(t) \leq H_m$  for sufficient large  $t$ . Similarly from the equations of the model system (72) the following is obtained

$$\liminf_{t \rightarrow \infty} B(t) \geq \frac{L (S_1 - S_2 K(L) - S_3 N_m)}{S_1 + L S_4 H_m} = F_1 \text{ (say),}$$

such that  $S_1 > S_2 K(L) + S_3 N_m$ ,

$$\liminf_{t \rightarrow \infty} W(t) \geq \frac{K(F_1) (S_5 - S_6 N_m - S_7 H_m)}{S_5} = F_2 \text{ (say),}$$

such that  $S_5 > S_6 N_m + S_7 H_m$ ,

$$\liminf_{t \rightarrow \infty} N(t) \geq \frac{M(S_8 - S_{10} F_2)}{S_8} = F_3 \text{ (say),}$$

such that  $S_8 > S_{10} F_2$ , and,

$$\liminf_{t \rightarrow \infty} H(t) \geq \frac{S_{11} F_3}{S_{12}} = F_4 \text{ (say),}$$

respectively. Thus, choosing

$$\tau_1 = \min (F_1, F_2, F_3, F_4) \text{ and } \tau_2 = \max (L, K(L), N_m, H_m)$$

the system becomes permanent. □

Based on Lemma 4.7 and Theorem 4.9, the solutions of fuzzy model (70) are bounded and permanent. This allows for the use of the model to investigate the effect of human-forest interaction on forest biomass. The next subsections explore the presence of equilibrium points of the fuzzy model and their stabilities.

### 4.3.2 Equilibrium analysis

The equilibrium points of the fuzzy model (70) were obtained by solving the equations after equating them to zero. It was observed that the model has six equilibrium points (five boundary equilibria and one interior equilibrium point), namely  $E^{(0)}$ ,  $E^{(1)}$ ,  $E^{(2)}$ ,  $E^{(3)}$ ,  $E^{(4)}$ , and  $E_*$ .

$$E^{(0)}(0, 0, 0, 0), \quad E^{(1)}(L, 0, 0, 0), \quad E^{(2)}(B^{(2)}, W^{(2)}, 0, 0),$$

$$E^{(3)} \left( 0, 0, M, \frac{M \mathbf{S}_{11}}{\mathbf{S}_{13}} \right), \quad E^{(4)}(B^{(4)}, 0, N^{(4)}, H^{(4)}), \quad E_*(B_*, W_*, N_*, H_*).$$

The existence of  $E^{(0)}$ ,  $E^{(1)}$  and  $E^{(3)}$  is trivial hence omitted. Theorem 4.10 provides sufficient conditions for the existence of  $E^{(2)}$ ,  $E^{(4)}$ , and  $E_*$ .

**Theorem 4.10.** *Existence of equilibrium points  $E^{(2)}$ ,  $E^{(4)}$ , and  $E_*$ :*

(i)  $E^{(2)}$  and  $E^{(4)}$  exist provided

$$\mathbf{S}_1 > \max \{ \mathbf{S}_2 K_0, \mathbf{S}_3 M \}.$$

(ii)  $E_*$  exists provided the following inequalities hold

$$\begin{aligned} \mathbf{S}_1 &> \mathbf{S}_3 M, \\ \mathbf{S}_5 L &> \mathbf{S}_6 \left( \frac{M (\mathbf{S}_8 + \mathbf{S}_9 L)}{\mathbf{S}_8} \right) + \mathbf{S}_7 \left( \frac{\mathbf{S}_{11} M (\mathbf{S}_8 + \mathbf{S}_9 L)}{\mathbf{S}_8 \mathbf{S}_{12}} \right), \\ \mathbf{S}_5 &> \frac{\mathbf{S}_6 \mathbf{S}_9 M}{\mathbf{S}_8} + \frac{\mathbf{S}_7 \mathbf{S}_9 \mathbf{S}_{11} M}{\mathbf{S}_8 \mathbf{S}_{12}}. \end{aligned}$$

The proof of Theorem 4.10 is similar to the proof of the existence of equilibrium points in the crisp model (15).

### 4.3.3 Stability analysis

The linear stability of the boundary equilibria were analysed by using the inferences of the eigenvalues (Table 6) of the matrix of the linearised system ( $\mathcal{J}$ ) evaluated at the equilibrium point.

$$\mathcal{J} = \begin{pmatrix} \mathbf{a}_1 & -\mathbf{S}_2 B & -\mathbf{S}_3 B & -\mathbf{S}_4 B^2 \\ \mathbf{a}_2 & \mathbf{a}_3 & -\mathbf{S}_6 W & -\mathbf{S}_7 W \\ \mathbf{S}_9 N & -\mathbf{S}_{10} N & \mathbf{a}_4 & 0 \\ 0 & 0 & \mathbf{S}_{11} & -\mathbf{S}_{12} \end{pmatrix}$$

where,

$$\begin{aligned} \mathbf{a}_1 &= \mathbf{S}_1 \left( 1 - \frac{B}{L} \right) - \frac{\mathbf{S}_1 B}{L} - \mathbf{S}_2 W - \mathbf{S}_3 N - 2\mathbf{S}_4 B H, \\ \mathbf{a}_2 &= \mathbf{S}_5 W \left( 1 - \frac{W}{K(B)} \right) + \frac{K_1 \mathbf{S}_5 W^2}{(K(B))^2}, \end{aligned}$$

$$\begin{aligned} \alpha_3 &= \mathbf{S}_5 B \left( 1 - \frac{W}{K(B)} \right) - \frac{\mathbf{S}_5 B W}{K(B)} - \mathbf{S}_6 N - \mathbf{S}_7 H, \\ \alpha_4 &= \mathbf{S}_8 \left( 1 - \frac{N}{M} \right) - \frac{\mathbf{S}_8 N}{M} + \mathbf{S}_9 B - \mathbf{S}_{10} W. \end{aligned}$$

Accordingly, Theorem 4.11 provides the summary of the stability analysis of the boundary equilibria. The proof of the theorem is similar to the stability analysis of the crisp model (15).

**Theorem 4.11.** *Stability analysis of the equilibria  $E^{(0)}$ ,  $E^{(1)}$ ,  $E^{(2)}$ ,  $E^{(3)}$ , and  $E^{(4)}$ :*

- (i) *The equilibrium point  $E^{(0)}$  is always unstable;*
- (ii) *The equilibrium point  $E^{(1)}$  is always unstable;*
- (iii) *The equilibrium point  $E^{(2)}$  is unstable provided  $\mathbf{S}_8 + \mathbf{S}_9 B^{(2)} > \mathbf{S}_{10} W^{(2)}$ ;*
- (iv) *The equilibrium point  $E^{(3)}$  is unstable provided  $\mathbf{S}_1 > \mathbf{S}_3 M$ ;*
- (v) *The equilibrium point  $E^{(4)}$  is unstable if  $\mathbf{S}_5 B^{(4)} > \mathbf{S}_6 N^{(4)} + \mathbf{S}_7 H^{(4)}$ .*

On the other hand, assessing the stability of an interior equilibrium point  $E_*$  using eigenvalue inferences is complex. To address this, the Lyapunov direct method was employed after linearising the system and utilising a positive definite function (57) to establish the conditions for the local stability. On utilising Theorem (3.1), the conditions for  $E_*$  to be locally asymptotically stable are stated in the Theorem 4.12.

**Theorem 4.12.** *The model system (72) is locally asymptotically stable around the equilibrium point  $E_*$  provided the following conditions are satisfied:*

$$\begin{aligned} \left( \frac{W_*}{K(B_*)} + \frac{K'(B_*) B_* W_*}{(K(B_*))^2} \right)^2 &< \left( \frac{2B_*(\mathbf{S}_1 + \mathbf{S}_4 H_* L)}{3\mathbf{S}_2 L K(B_*)} \right), \\ \left( \frac{\mathbf{S}_2 \mathbf{S}_6}{\mathbf{S}_5} + \frac{\mathbf{S}_3 \mathbf{S}_{10}}{\mathbf{S}_9} \right)^2 &< \left( \frac{2\mathbf{S}_2 B_*}{3K(B_*)} \right) \left( \frac{\mathbf{S}_3 \mathbf{S}_8}{\mathbf{S}_9 M} \right), \\ (\mathbf{S}_{11})^2 &< \frac{2\mathbf{S}_3 \mathbf{S}_8 \mathbf{S}_{12}}{3\mathbf{S}_9 M}, \\ \left( \frac{\mathbf{S}_7}{\mathbf{S}_5} \right)^2 &< \frac{4\mathbf{S}_{12} B_*}{9\mathbf{S}_2 K(B_*)}, \\ \mathbf{S}_4^2 &< 2\mathbf{S}_{12} \left( \frac{\mathbf{S}_1 + \mathbf{S}_3 H_* L}{3B_*^2 L} \right). \end{aligned}$$

Furthermore, the Lyapunov direct method with the positive define function (63) was used to determine the conditions for global stability of the system at  $E_*$  in the region of attraction  $\Omega_f$ . Theorem 4.13 specifies the conditions for the global stability of the interior equilibrium point.

**Theorem 4.13.** *The interior equilibrium  $E_*$  if exists is globally stable inside the region of attraction  $\Omega_f$  provided the following conditions hold:*

$$\begin{aligned} \left( \frac{LK(L)\lambda_1}{K_0^2} + \frac{W_*}{K(B_*)} \right) &< \left( \frac{\mathbf{S}_1 + \mathbf{S}_4 H_* L}{3\mathbf{S}_2 K(B_*)} \right), \\ \left( \frac{\mathbf{S}_2 \mathbf{S}_6}{\mathbf{S}_5} + \frac{\mathbf{S}_3 \mathbf{S}_{10}}{\mathbf{S}_9} \right)^2 &< \left( \frac{2\mathbf{S}_2 L}{3K(B_*)} \right) \left( \frac{\mathbf{S}_3 \mathbf{S}_8}{\mathbf{S}_9 M} \right), \\ (\mathbf{S}_{11})^2 &< \frac{2\mathbf{S}_3 \mathbf{S}_8 \mathbf{S}_{12}}{3\mathbf{S}_9 M}, \\ \left( \frac{\mathbf{S}_7}{\mathbf{S}_5} \right)^2 &< \frac{4\mathbf{S}_{12} L}{9\mathbf{S}_2 K(B_*)}, \\ \mathbf{S}_4^2 &< 2\mathbf{S}_{12} \left( \frac{\mathbf{S}_1 + \mathbf{S}_3 H_* L}{3L^3} \right). \end{aligned}$$

The proof of both Theorems 4.12 and 4.13 is similar to the proof of Theorems 4.4 and 4.5, respectively.

#### 4.3.4 Numerical simulation

Numerical simulation is carried out to study the effects of fuzzy parameters on the dynamics of the system. Since the problem was not a case study for a specific species, the simulation used the estimated parameter values in Table 12 and some hypothetical parametric values of  $\Delta_i$  which were selected based on the conditions stated in model (70).

---

|                          |                          |                         |                         |                         |                         |
|--------------------------|--------------------------|-------------------------|-------------------------|-------------------------|-------------------------|
| $\Delta_1 = 0.79,$       | $\Delta_2 = 0.05,$       | $\Delta_3 = 0.05,$      | $\Delta_4 = 0.01,$      | $\Delta_5 = 0.0025,$    | $\Delta_6 = 0.001,$     |
| $\Delta_7 = 0.00039,$    | $\Delta_8 = 0.0001,$     | $\Delta_9 = 0.89,$      | $\Delta_{10} = 0.005,$  | $\Delta_{11} = 0.0019,$ | $\Delta_{12} = 0.0004,$ |
| $\Delta_{13} = 0.00009,$ | $\Delta_{14} = 0.00005,$ | $\Delta_{15} = 0.09,$   | $\Delta_{16} = 0.005,$  | $\Delta_{17} = 0.045,$  | $\Delta_{18} = 0.006,$  |
| $\Delta_{19} = 0.0008,$  | $\Delta_{20} = 0.0009,$  | $\Delta_{21} = 0.0078,$ | $\Delta_{22} = 0.0002,$ | $\Delta_{23} = 0.0038,$ | $\Delta_{24} = 0.0005,$ |
| $\Delta_{25} = 0.045,$   | $\Delta_{26} = 0.006.$   |                         |                         |                         |                         |

---

Upon using these parametric values, numerical solution were calculated by taking advantage of MATLAB2022a built in ODE solver `ode45` (Algorithm 3). Arbitrary, starting with initial condition  $B(0) = 2, W(0) = 2, N(0) = 10, H(0) = 1$ , the dynamics of forest biomass, wildlife population, human population and human activities converge to the equilibrium levels (Fig. 34). From the figure, it can be noted that equilibrium levels of four compartments are

different in two different environments (crisp and fuzzy) suggesting that fuzzy parameters have impact on the dynamics of the model, thus adjusting the values of delta to control fuzziness of parameter will leads to decisions based on uncertain or imprecise information.

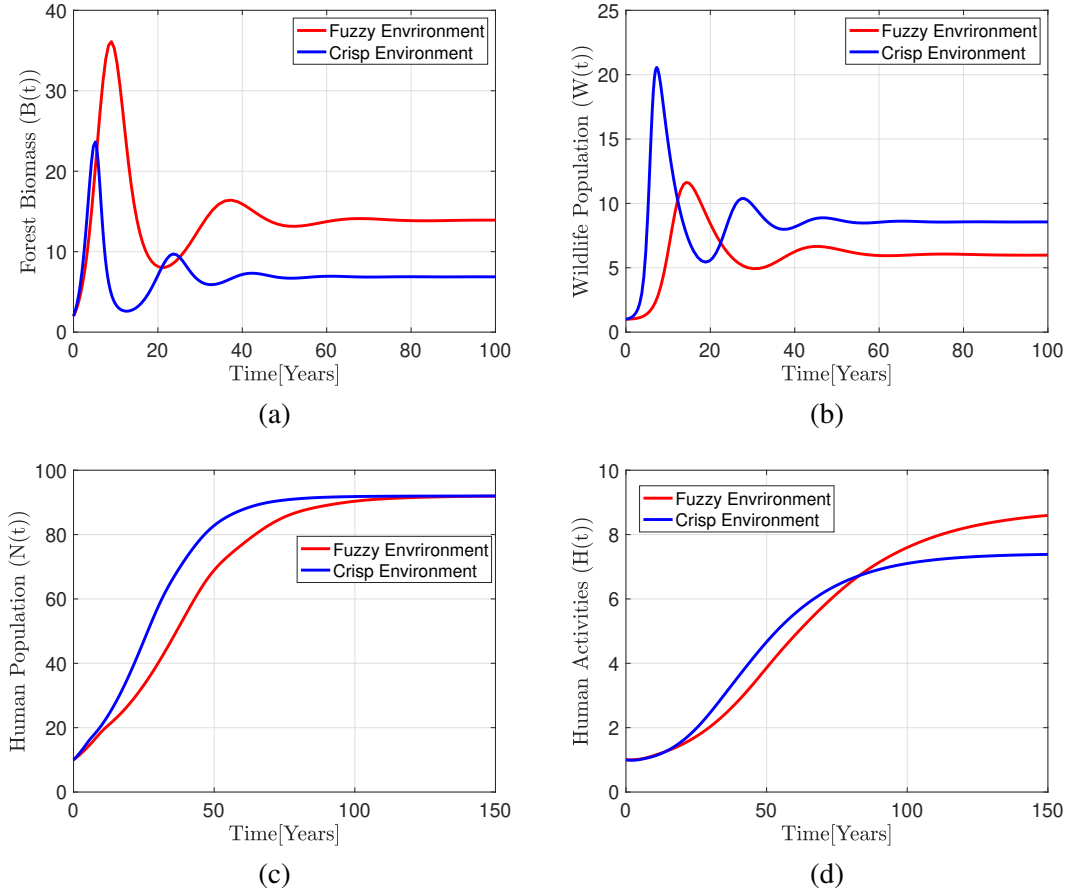


**Figure 34:** Evolution of the dynamics of (a) forest biomass, (b) wildlife population, (c) human population and (d) human activities with time in crisp and fuzzy environments with the same initial condition

Furthermore, taking different initial conditions, the phase diagrams in  $B - W - N$  and  $B - W - H$  were plotted and are shown in Fig. 35. It can be observed that regardless of the starting points, the solutions converge to the interior equilibrium point suggesting that the interior equilibrium point is globally asymptotically stable for the fuzzy model. In other planes, non-linear stability can be demonstrated in similar way. Based on the simulation results presented, it can be concluded that the fuzzy model is more trustworthy than the corresponding crisp model. This is because crisp models are a particular case of fuzzy models, and the use of fuzzy parameters in the model may leads to a more accurate representation of the dynamics of forest depletion.

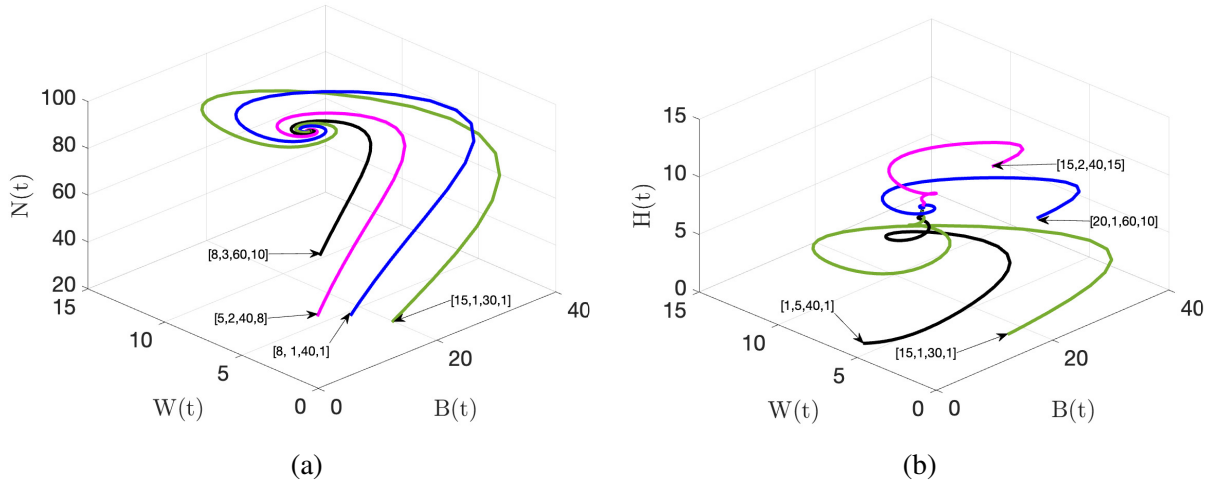


**Figure 35:** Non-linear stability of the interior equilibrium point  $E_*$  in (a)  $B - W - N$  and (b)  $B - W - H$  planes

Furthermore, the findings highlights the need for a multi-disciplinary approach in studying and managing ecological systems. This includes collaboration among ecologists, mathematicians, economists, and policy-makers to develop models that reflect the complexities of the system and identify sustainable management practices.

#### Remark 4.5

The impact of anthropogenic activities on forest biomass and forest-dependent wildlife was studied using a non-linear mathematical model both in crisp and fuzzy environments. To establish the existence of equilibrium points and their stabilities, the stability theory of differential equations was utilised. Moreover, numerical simulation was performed to validate the analytical results. The findings revealed that anthropogenic activities have a significant influence on both forest biomass and forest-dependent wildlife population. As human population growth increases, it puts pressure on available forest biomass, resulting in depletion of both forest biomass and the wildlife population dependent on it. The model suggests that to achieve ecological balance, the intrinsic growth rate of forest biomass should be greater than its utilisation rate. Furthermore, the intrinsic growth rate of forest-dependent wildlife population due to forest biomass should be higher than the rates of its depletion.

#### **4.4 Forest Biomass and Forest-Dependent Wildlife Conservation Model**

Based on the analysis of models (15) and (70), it is observed that uncontrolled human population growth and related activities pose a threat to the sustainability of forest biomass and wildlife species. As a result, it is crucial to implement resource conservation measures through various management and control strategies to ensure long-term coexistence. In this section, a non-linear model for forest biomass and wildlife conservation is presented and analysed. Technological and economic measures are proposed as conservation strategies [24, 34].

##### **4.4.1 Economic and technological efforts for resources conservation**

Economic measures for forest management are specific incentives designed and implemented to influence all stakeholders to conserve the forest or use forest resources sustainably [24]. These efforts can take the form of property rights [69, 122], markets and charge systems [23, 71], fiscal instruments [45], and livelihood support [113].

Property rights are often granted to local communities living around forest areas or depending on forest resources [122]. Strengthening property rights and transferring them to forest users and beneficiaries can lead to the emergence of forest markets and scarcity prices [69]. This scarcity may allow users and owners of the forest to benefit from conservation efforts or be required to pay on-site costs. Market and charge systems aim to address inefficiencies and weaknesses in pricing and market structure, which may give forest users and beneficiaries incorrect signals and encourage unsustainable use of forests due to affordability or profitability [71]. An example of a market and charge systems measure is the royalty rates and rationalization of timber prices in Kenya [24]. Besides, fiscal measures are budgetary policies that impose taxes and subsidies on products and services dependent on or impacting forest biomass [57]. Livelihood support measures aim to improve and diversify the means of subsistence for those whose activities have an impact on forest biomass [36]. Examples of fiscal and livelihood support measures include the manipulation of market prices through the imposition of certain taxes and subsidies and the distribution of benefits from protected areas to the local community [45, 19].

In contrast, the use of technology has the potential to enhance forest management by improving efficiency, reducing costs, and aiding in the fight against deforestation and forest conservation,

as noted in a recent study [104]. For example, the use of drones, sensors, and GPS systems can assist in mapping and monitoring forests, allowing farmers to optimize tree growth and increase the success rate of reforestation initiatives [34]. The use of drone technology in seedling planting is believed to be more effective than traditional methods, which are slow, expensive, and have a greater likelihood of failed germination [64]. Additionally, advancements in biotechnology can aid in the production of high-quality seedlings with superior genetic characteristics, making reforestation vast areas simpler and more efficient [64]. Moreover, remote sensing and big data analysis technologies can provide more precise and comprehensive forest health and growth information, aiding in progress tracking and identifying areas requiring improvement [83]. The advancement of technology can also help involve local communities in forest management by utilising mobile apps and social media to promote reforestation efforts and raise awareness about the importance of forests [85].

#### 4.4.2 Model formulation

The non-linear mathematical model was formulated by extending model system (15) by adding three compartments: population pressure  $P$ , economic measures  $E_f$  and technological efforts  $T_e$ . Based on the assumptions and descriptions provided in Chapter 3, the compartmental model diagram for the conservation model is presented in Fig. 36.

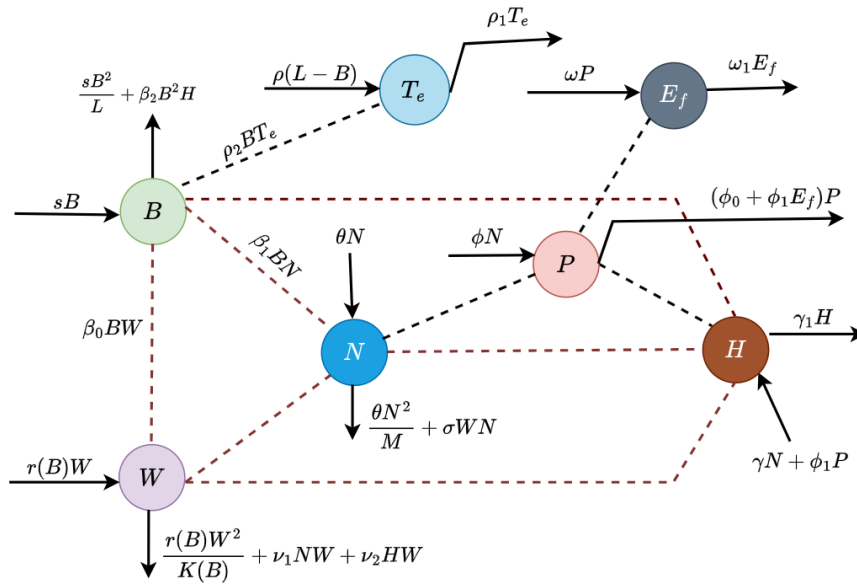


**Figure 36:** Forest biomass and forest-dependent wildlife conservation model flow diagram

Utilising Fig. 36, the mathematical model (73) is formulated:

$$\left\{ \begin{array}{l} \frac{dB}{dt} = sB \left( 1 - \frac{B}{L} \right) - \beta_0 BW - \beta_1 BN - \beta_2 B^2 H + \rho_2 B T_e, \\ \frac{dW}{dt} = r(B)W \left( 1 - \frac{W}{K(B)} \right) - \nu_1 WN - \nu_2 WH, \\ \frac{dN}{dt} = \theta N \left( 1 - \frac{N}{M} \right) + \lambda \beta_1 BN - \sigma NW, \\ \frac{dP}{dt} = \phi N - \phi_0 P - \phi_1 E_f P, \\ \frac{dH}{dt} = \gamma N + \phi_2 P - \gamma_1 H, \\ \frac{dT_e}{dt} = \rho(L - B) - \rho_1 T_e, \\ \frac{dE_f}{dt} = \omega P - \omega_1 E_f, \end{array} \right. \quad (73)$$

such that,

$$B(0) \geq 0, W(0) \geq 0, N(0) \geq 0, P(0) \geq 0, H(0) \geq 0, T_e(0) \geq 0, E_f(0) \geq 0,$$

$$0 \leq \lambda \leq 1.$$

The first equation of the proposed model captures the logistic growth of forest biomass, with an intrinsic growth rate of  $s$  and a carrying capacity of  $L$ . The second term represents the predation rate of forest-dependent wildlife species through natural processes, which is modelled using bilinear interaction ( $\beta_0 BW$ ) [92]. However, due to the use of forest resources by the human population, forest biomass experiences a reduction in its growth rate, as represented by the term  $\beta_1 BN$  [56]. The utilisation of forest resources in the region primarily encompasses activities such as timber harvesting, extraction of medicinal substances, and grazing, among others. Furthermore, forest land is often cleared for agricultural purposes, the construction of housing complexes, and the establishment of industrial infrastructure. As a result, this clearance of forested land diminishes the overall extent of forests within the region, consequently exerting negative impacts on the capacity of forestry resources to sustainably support ecological and human needs. This aspect is explicitly modelled using the fourth term,  $\beta_2 B^2 H$  [56], which accounts for the loss of forest land that cannot be used for forest regeneration. The fifth term  $\rho_2 B T_e$  expresses the growth of forest biomass after application of technological efforts to restore the cleared forest.

The second equation of the model captures the dynamics of the wildlife population that rely on forest. It follows a logistic growth pattern with a growth rate  $r(B)$  and a carrying capacity

$K(B)$ , both are functions of forest biomass to reflect the species' dependence on the forest ecosystem. The direct depletion of wildlife population by human activities, such as poaching, is modelled using the second term  $(\nu_1 WN)$  [56]. Meanwhile, the indirect depletion of wildlife caused by human activities, such as habitat destruction or fragmentation, is modelled using the third term  $(\nu_2 WH)$ . The third equation of the model describes the dynamics of the human population, assuming a logistic growth pattern. This growth pattern is characterised by an intrinsic growth rate  $\theta$ , and a carrying capacity  $M$ . The term  $\lambda\beta_1 BN$  in the model represents the additional growth of the human population due to the use of forest resources for their livelihood [38]. Furthermore, human population is impacted by interactions with wildlife population. These interactions may lead to resources competition and human safety concerns as human populations expand and encroach upon natural habitats [70]. This particular aspect is captured in the model through the inclusion of the third term  $(\sigma NW)$ .

The fourth equation describes the dynamics of population pressure, which increases proportionally to the human population with a growth rate  $\phi$ . Additionally, population pressure is subject to natural depletion  $(\phi_0 P)$  and depletion through economic measures  $(\phi_1 E_f P)$  designed to support livelihoods and regulate resource exploitation [38]. The dynamics of human activities in the forest ecosystem are modelled in equations five. It is assumed that human activities grow proportionally to the density of human population  $(\gamma N)$  and population pressure  $(\phi_2 P)$  [92]. Furthermore, the growth of human activities is driven by the benefits obtained from the forest and deplete naturally with depletion rate  $\gamma_1$ .

The implementation of technological efforts to accelerate the rate of reforestation is described in equation six. The rate of technological efforts is postulated to be proportional to the difference between the carrying capacity of the forest and the current level of forest biomass  $(\rho(L - B))$ , reflecting the need to balance economic development and ecological sustainability [56]. Technological efforts are subject to natural depletion  $(\rho_1 T_e)$ . Finally, equation seven models the implementation of economic measures to support livelihoods and control the exploitation of resources. The rate at which economic measures are instituted is assumed to be directly proportional to the magnitude of population pressure  $(\omega P)$ , reflecting the need to address social and economic dimensions of forest management [38]. Economic measures are subject to natural depletion as well  $(\omega_1 E_f)$ .

#### 4.4.3 Basic properties of the model

Before carrying out model analysis, the basic properties such as boundedness and posedness of the model were established. Well-posedness of a mathematical model refers to the property of a model that ensures the existence, uniqueness, and stability of its solutions. Thus, a model is said to be well-posed if non-negative initial conditions lead to non-negative solutions [58]. This property is important to ensure that the solutions obtained from the model are meaningful and can be interpreted in a scientifically rigorous manner. Therefore, let

$$\mathbb{R}_+^7 = \{B(t), W(t), N(t), P(t), H(t), T_e(t), E_f(t)\}$$

be the set containing positive coordinate of  $\mathbb{R}^7$ . Now, consider the model system (73) has initial values

$$\mathbb{R}_+^7(0) = \{B(0), W(0), N(0), P(0), H(0), T_e(0), E_f(0)\},$$

if

$$B(0) \geq 0, W(0) \geq 0, N(0) \geq 0, P(0) \geq 0, H(0) \geq 0, T_e(0) \geq 0, \text{ and } E_f(0) \geq 0,$$

then:

$$B(t) \geq 0, W(t) \geq 0, N(t) \geq 0, P(t) \geq 0, H(t) \geq 0, T_e(t) \geq 0, \text{ and } E_f(t) \geq 0.$$

This has been verified by showing that  $\mathbb{R}_+^7$  is positively invariant with respect to system (73) by examining the direction of the vector fields on each coordinate plane. For instance, consider the  $B - W - N$  space, on this plane  $P = H = T_e = E_f = 0$  and

$$\dot{P} \big|_{P=H=T_e=E_f=0} = \phi N > 0, \quad \dot{H} \big|_{P=H=T_e=E_f=0} = \gamma N > 0,$$

$$\dot{T}_e \big|_{P=H=T_e=E_f=0} = \rho(L - B) > 0, \quad \dot{E}_f \big|_{P=H=T_e=E_f=0} = 0.$$

Further, taking  $W - N - P$  plane, on this plane  $B = H = T_e = E_f = 0$  and

$$\dot{B} \big|_{B=H=T_e=E_f=0} = 0, \quad \dot{H} \big|_{B=H=T_e=E_f=0} = \gamma N > 0,$$

$$\dot{T}_e \big|_{B=H=T_e=E_f=0} = \rho L > 0, \quad \dot{E}_f \big|_{B=H=T_e=E_f=0} = 0.$$

With these counter examples, it is observed that some of the vector fields are tangents to the coordinate and others are pointing to the interior of  $\mathbb{R}_+^7$  suggesting that all solutions starting in  $\mathbb{R}_+^7$  remain in  $\mathbb{R}_+^7$  for all  $t > 0$ . Therefore, the system is well posed and solutions obtained from the model are meaningful ecologically.

The bounds of the variables in conservation model are stated in Lemma 4.8. For the proof, refer Lemma 4.4.

**Lemma 4.8.** *The set*

$$\Omega_c = \left\{ (B, W, N, P, H, T_e, E_f) \in \mathbb{R}_+^7 : 0 \leq B \leq L, 0 \leq W \leq K(L), 0 \leq N \leq N_m, \right. \\ \left. 0 \leq P \leq P_m, 0 \leq H \leq H_m, 0 \leq T_e \leq \frac{\rho L}{\rho_1}, 0 \leq E_f \leq \frac{\omega}{\omega_1} P_m \right\},$$

*is the region of attraction for model system (73) and attract all solutions initiating in the interior of positive orthant, where,*

$$N_m = \frac{M}{\theta} (\theta + \lambda \beta_1 L), \quad P_m = \frac{\phi}{\phi_0} N_m, \quad \text{and} \quad H_m = \frac{\gamma N_m + \phi_2 P_m}{\gamma_1}.$$

With this lemma, it is true that in the region  $\mathbb{R}_+^7$  the model is ecologically and mathematically well posed, hence can be used to study the conservation of forest biomass and wildlife protection.

#### 4.4.4 Equilibrium analysis

The model system (73) has five boundary equilibria and one interior equilibrium point.

$$\mathbf{E}_1 \left( 0, 0, 0, 0, 0, \frac{\rho L}{\rho_1}, 0 \right), \quad \mathbf{E}_2 (L, 0, 0, 0, 0, 0, 0), \quad \mathbf{E}_3 \left( B_3, W_3, 0, 0, 0, \frac{\rho(L - B_3)}{\rho_1}, 0 \right), \\ \mathbf{E}_4 \left( 0, 0, M, P_4, H_4, \frac{\rho L}{\rho_1}, E_{f_4} \right), \quad \mathbf{E}_5 \left( B_5, 0, N_5, P_5, H_5, \frac{\rho(L - B_5)}{\rho_1}, E_{f_5} \right), \\ \overline{\mathbf{E}} (\overline{B}, \overline{W}, \overline{N}, \overline{P}, \overline{H}, \overline{T_e}, \overline{E_f}),$$

where,

$$P_4 = \frac{\sqrt{(\omega_1 \phi_0)^2 + 4M\omega_1 \phi_1 \omega \phi} - \omega_1 \phi_0}{2\omega \phi_0}, \\ E_{f_4} = \frac{\omega P_4}{\omega_1}, \quad H_4 = \frac{\gamma M + \phi_2 P_4}{\gamma_1}.$$

#### Remark 4.6

It can be noted that existence of  $\mathbf{E}_1$ ,  $\mathbf{E}_2$ , and  $\mathbf{E}_4$  is obvious. Further, Theorem 4.14 summarises the existence of  $\mathbf{E}_3$  and  $\mathbf{E}_5$ . The proof can be shown in a similar way as the proof for the existence of  $E_2$  and  $E_4$ , respectively, in model (15).

**Theorem 4.14.** *The Equilibrium points  $\mathbf{E}_3$  and  $\mathbf{E}_5$  exist provided condition (74) holds*

$$s > \max \left\{ \beta_0 K_0, \beta_1 M - \frac{\rho \rho_2}{\rho_1} \right\}. \quad (74)$$

The existence of interior equilibrium point  $\overline{E}$  is shown as follows: Notably, at  $\overline{E}$ , the values  $\overline{B}$ ,  $\overline{W}$ ,  $\overline{N}$ ,  $\overline{P}$ ,  $\overline{H}$ ,  $\overline{T_e}$ , and  $\overline{E_f}$  correspond to the positive solutions of the algebraic equations (75a)-(75g).

$$s \left( 1 - \frac{\overline{B}}{L} \right) - \beta_0 \overline{W} - \beta_1 \overline{N} - \beta_2 \overline{B} \overline{H} + \rho_2 \overline{T_e} = 0, \quad (75a)$$

$$r(\overline{B}) \left( 1 - \frac{\overline{W}}{K(\overline{B})} \right) - \nu_1 \overline{N} - \nu_2 \overline{H} = 0, \quad (75b)$$

$$\theta \left( 1 - \frac{\overline{N}}{M} \right) + \lambda \beta_1 \overline{B} - \sigma \overline{W} = 0, \quad (75c)$$

$$\phi \overline{N} - \phi_0 \overline{P} - \phi_1 \overline{E_f} \overline{P} = 0, \quad (75d)$$

$$\gamma \overline{N} + \phi_2 \overline{P} - \gamma_1 \overline{H} = 0, \quad (75e)$$

$$\rho \left( L - \overline{B} \right) - \rho_1 \overline{T_e} = 0, \quad (75f)$$

$$\omega \overline{P} - \omega_1 \overline{E_f} = 0. \quad (75g)$$

With proper substitutions and some algebraic manipulations on the equations (75a-75g) yields the isoclines (76) and (77).

$$s \left( 1 - \frac{\overline{B}}{L} \right) - \beta_0 \overline{W} - \beta_1 \mathbf{h}_1(\overline{B}, \overline{W}) - \beta_2 \overline{B} \mathbf{h}_3(\overline{B}, \overline{W}) + \frac{\rho \rho_2}{\rho_1} (L - \overline{B}) = 0 = \mathcal{H}_1(\overline{B}, \overline{W}), \quad (\text{say}) \quad (76)$$

$$r(\overline{B}) \left( 1 - \frac{\overline{W}}{K(\overline{B})} \right) - \nu_1 \mathbf{h}_1(\overline{B}, \overline{W}) - \nu_2 \mathbf{h}_3(\overline{B}, \overline{W}) = 0 = \mathcal{H}_2(\overline{B}, \overline{W}) \quad (\text{say}), \quad (77)$$

where,

$$\overline{N} = \frac{M}{\theta} \left( \theta + \lambda \beta_1 \overline{B} - \sigma \overline{W} \right) = \mathbf{h}_1(\overline{B}, \overline{W}), \quad (78)$$

$$\overline{P} = \frac{\sqrt{\omega_1^2 \phi_0^2 + 4 \omega_1 \omega \phi_1 \phi \mathbf{h}_1(\overline{B}, \overline{W})} - \omega_1 \phi_0}{2 \omega \phi_1} = \mathbf{h}_3(\overline{B}, \overline{W}), \quad (79)$$

$$\overline{H} = \frac{\gamma \mathbf{h}_1(\overline{B}, \overline{W}) + \phi_2 \mathbf{h}_2(\overline{B}, \overline{W})}{\gamma_1} = \mathbf{h}_2(\overline{B}, \overline{W}), \quad (80)$$

$$\overline{T_e} = \frac{\rho \rho_2}{\rho_1} (L - \overline{B}), \quad (81)$$

$$\overline{E_f} = \frac{\omega \mathbf{h}_2(\overline{B}, \overline{W})}{\omega_1}, \quad (82)$$

$$\theta + \lambda \beta_1 \overline{B} > \sigma \overline{W}. \quad (83)$$

From the isocline (76) the following inferences are analysed:

(a) When  $\overline{W} = 0$ , gives  $\mathcal{H}_1(\overline{B}, 0) = \mathbf{R}_1(\overline{B})$  (say).

$$s \left( 1 - \frac{\overline{B}}{L} \right) - \beta_1 \mathbf{h}_1(\overline{B}) - \beta_2 \overline{B} \mathbf{h}_3(\overline{B}) + \frac{\rho \rho_2}{\rho_1} (L - \overline{B}) = 0 = \mathbf{R}_1(\overline{B}). \quad (84)$$

Further equation (84) leads to the following inferences:

(i) When  $\overline{B} = 0$ , gives equation (85):

$$s + \frac{\rho \rho_2}{\rho_1} L - \beta_1 M, \quad (85)$$

which is positive provided inequality (86) holds,

$$s + \frac{\rho \rho_2}{\rho_1} L > \beta_1 M. \quad (86)$$

(ii) When  $\overline{B} = L$ , gives equation (87):

$$-\beta_1 \mathbf{h}_1(L) - \beta_2 L \mathbf{h}_3(L) < 0. \quad (87)$$

(iii) The derivative of  $\mathbf{R}_1(\overline{B})$  with respect to  $\overline{B}$  gives:

$$\mathbf{R}'_1(\overline{B}) = -\frac{s}{L} - \beta_1 \mathbf{h}'_1(\overline{B}) - \beta_2 \overline{B} \mathbf{h}'_3(\overline{B}) - \beta_2 \mathbf{h}_3(\overline{B}) < 0.$$

These considerations (i-iii) suggest that equation (84) has a positive unique solution in  $(0, L)$ .

(b)  $\overline{B} < 0$  as  $\overline{W} \rightarrow \infty$ .

(c) If the derivative of  $\overline{W}$  with respect to  $\overline{B}$  from isocline (76) is defined as  $\left( \frac{d\overline{W}}{d\overline{B}} \right)_1$ , then

$$\left( \frac{d\overline{W}}{d\overline{B}} \right)_1 < 0. \quad (88)$$

Furthermore, the isocline (77) gives the following inferences:

(a) When  $\overline{W} = 0$ , gives  $\mathcal{H}_2(\overline{B}, 0) = \mathbf{R}_2(\overline{B})$  (say).

$$r(\overline{B}) - \nu_1 \mathbf{h}_1(\overline{B}) - \nu_2 \mathbf{h}_3(\overline{B}) = 0 = \mathbf{R}_2(\overline{B}). \quad (89)$$

Equation (89) yields the following conclusions:

(i) When  $\bar{B} = 0$ , gives equation:

$$\mathbf{R}_2(0) = -\nu_1 M - \nu_2 \beta_3,$$

such that,

$$\beta_3 = \frac{\gamma M - \omega_1 \phi_0 + \sqrt{(\omega_1 \phi_0)^2 + 4M\omega_1\omega\phi_1\phi}}{2\omega\phi_1}.$$

Since  $\beta_3$  is positive, then  $\mathbf{R}_2(0)$  is negative.

(ii) When  $\bar{B} = L$ , gives  $\mathbf{R}_2(L)$ ,

$$\mathbf{R}_2(L) = r(L) - \nu_1 \mathbf{h}_1(L) - \nu_2 \mathbf{h}_3(L), \quad (90)$$

which is positive provided:

$$r(L) > \nu_1 \mathbf{h}_1(L) + \nu_2 \mathbf{h}_3(L). \quad (91)$$

(iii) The derivative  $\mathbf{R}_2(\bar{B})$  with respect to  $\bar{B}$  is given as:

$$\mathbf{R}_2'(\bar{B}) = r'(\bar{B}) - \nu_1 \mathbf{h}_1'(\bar{B}) - \nu_2 \mathbf{h}_3'(\bar{B}) \quad (92)$$

which is positive provided:

$$r'(\bar{B}) > \nu_1 \mathbf{h}_1'(\bar{B}) + \nu_2 \mathbf{h}_3'(\bar{B}). \quad (93)$$

The inferences (i)-(iii) suggest that  $\mathbf{R}_2(\bar{B})$  has a unique non negative solution in the interval  $(0, L)$ .

(b)  $\bar{W} > 0$  as  $\bar{B} \rightarrow \infty$

(c) If the derivative of  $\bar{W}$  with respect to  $\bar{B}$  from isocline (77) is denoted by  $\left(\frac{d\bar{W}}{d\bar{B}}\right)_2$ , then

$$\left(\frac{d\bar{W}}{d\bar{B}}\right)_2 > 0. \quad (94)$$

Therefore, with these considerations, it is true that for uniqueness of  $(\bar{B}, \bar{W})$  in the regions  $(0, L)$  and  $(0, K(L))$ , conditions (88) and (94) must hold. This completes the proof for the existence of unique and positive roots  $(\bar{B}, \bar{W})$  in the interval  $(0, L)$  and  $(0, K(L))$ . Therefore, once the values of  $(\bar{B}, \bar{W})$  are known, the values of  $\bar{N}$ ,  $\bar{P}$ ,  $\bar{H}$ ,  $\bar{T}_e$  and  $\bar{E}_f$  can be evaluated from equations (78), (79), (80), (81), and (82), respectively.

#### 4.4.5 Physical interpretation of the equilibrium points

- (i)  $E_1$ : The boundary equilibrium point represents a scenario in which there is a complete absence of forest biomass and wildlife population, as well as a complete absence of human population, its associated pressure and activities, and economic measures. However, technological efforts are at their maximum level, indicating a significant investment in reforestation and forest restoration. This scenario can occur in areas that have experienced extensive deforestation or forest degradation due to human activities, such as unsustainable logging or conversion of forestland to agricultural land. It is important to note that this scenario may not be feasible in reality, as it is unlikely that all human activities can be completely eliminated. Nevertheless, this equilibrium point serves as a theoretical benchmark for the maximum potential of technological efforts to restore forest ecosystems.
- (ii)  $E_2$ : The equilibrium point when forest biomass is at its maximum carrying capacity ( $L$ ) and all other variables are not present implies that the forest ecosystem has reached its optimal state, while other variables, such as wildlife population, human population, and human activities are not exerting any significant pressure on the ecosystem. Nevertheless, while the state is theoretically possible, not likely to occur in natural systems, however, it represents a state of ecological balance where the forest is able to sustain its biomass and support its associated wildlife population, while also accommodating human activities. It is an ideal state that many conservation and management efforts aim to achieve, as it promotes sustainable use and conservation of forest ecosystems.
- (iii)  $E_3$ : The point represents a hypothetical state where the forest and wildlife populations are thriving due to the availability of sufficient forest biomass, and technological efforts are being used to maintain and enhance the forest ecosystem. The scenario is difficult to achieve in real-world situations. However, it signifies a balance between growth and predation, which is essential for the sustainability of the ecosystem.
- (iv)  $E_4$ : In this scenario, the absence of forest biomass and forest-dependent wildlife population means that the ecosystem is severely degraded and in need of urgent restoration efforts. However, the existence of human population, population pressure, human activities, economic measures, and technological efforts implies that the region has the potential to invest in conservation and restoration efforts. This equilibrium point highlights the need for sustainable and integrated approaches that consider the ecological, social, and

economic aspects of forest conservation and restoration. Thus, in long run forest biomass and forest-dependent wildlife will exist.

- (v)  $E_5$ : The absence of forest-dependent wildlife population in the presence of other variables such as forest biomass, human population, population pressure, human activities, economic measures, and technological efforts signify a potentially unsustainable use of forest resources. This scenario suggests that the forest ecosystem is being exploited for its resources without regard for the preservation of its biodiversity. It highlights the importance of considering not just the economic benefits but also the ecological and social values of forests in decision-making processes. Thus the point highlights the need for sustainable forest management practices that take into account the needs of both human and wildlife populations should be prioritised to ensure the long-term health and resilience of forest ecosystems.
- (vi)  $\bar{E}$ : The point represents the coexistence of the variables. The point implies that there is a balance between the components of the ecosystem and they are able to coexist and interact with each other in a sustainable manner. This is the state that many conservation and management efforts aim to achieve.

#### 4.4.6 Stability analysis

The local stability analysis of the equilibrium points of the system was carried out by inferring the sign of the eigenvalues of the appropriate Jacobian matrix [49]. Thus, the overall Jacobian matrix ( $J_c$ ) of the system (73) is given as:

$$J_c = \begin{pmatrix} j_{11} & -\beta_0 B & -\beta_1 B & 0 & -\beta_2 B^2 & \rho_2 B & 0 \\ j_{21} & j_{22} & -\nu_1 W & 0 & -\nu_2 W & 0 & 0 \\ \lambda\beta_1 N & -\sigma N & j_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi & -E\phi_1 - \phi_0 & 0 & 0 & -\phi_1 P \\ 0 & 0 & \gamma & \phi_2 & -\gamma_1 & 0 & 0 \\ -\rho & 0 & 0 & 0 & 0 & -\rho_1 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 & -\omega_1 \end{pmatrix}$$

where,

$$\begin{aligned} j_{11} &= s \left( 1 - \frac{B}{L} \right) - \frac{sB}{L} - \beta_0 W - \beta_1 N - 2\beta_2 BH + \rho_2 T_e, \\ j_{21} &= r'(B)W \left( 1 - \frac{W}{K(B)} \right) + \frac{r(B)W^2 K'(B)}{K(B)^2}, \\ j_{22} &= r(B) \left( 1 - \frac{W}{K(B)} \right) - \frac{r(B)W}{K(B)} - \nu_1 N - \nu_2 H, \\ j_{33} &= \theta \left( 1 - \frac{N}{M} \right) - \frac{\theta N}{M} + \lambda \beta_1 B - \sigma W. \end{aligned}$$

Using the inferences from Table 6, the linear stability analysis of equilibrium points  $E_1$ ,  $E_2$ ,  $E_3$ ,  $E_4$ , and  $E_5$  is provided in Theorem 4.15. The proof of this theorem is similar to the proof of Theorem 4.3.

**Theorem 4.15.** *Local stability analysis of the equilibrium points  $E_1$ ,  $E_2$ ,  $E_3$ ,  $E_4$ , and  $E_5$ :*

- (i) *At  $E_1$ , the system always exhibits unstable behaviour;*
- (ii) *At  $E_2$ , the system always exhibits unstable behaviour;*
- (iii) *At  $E_3$ , the system always exhibits unstable behaviour provided  $\theta + \lambda \beta_1 B_3 > \sigma W_3$ ;*
- (iv) *At  $E_4$ , the system exhibits unstable behaviour provided equation (86) holds;*
- (v) *At  $E_5$ , the system exhibits unstable behaviour provided  $r(B_5) > \nu_1 N_5 + \nu_2 H_5$ .*

Following Theorem 3.1, the local stability behaviour of the system (73) at  $\bar{E}$  is established using Lyapunov's stability theory by linearising the system by utilising the following transformations:  $B = \bar{B} + b$ ,  $W = \bar{W} + w$ ,  $N = \bar{N} + n$ ,  $P = \bar{P} + \wp$ ,  $H = \bar{H} + h$ ,  $T_e = \bar{T}_e + t_e$ ,  $E_f = \bar{E}_f + e_f$ , where,  $b$ ,  $w$ ,  $n$ ,  $\wp$ ,  $h$ ,  $t_e$  and  $e_f$  are small perturbations around the equilibrium point  $\bar{E}$ .

After several algebraic manipulations yield a linearised system (95):

$$\left\{ \begin{array}{l} \frac{db}{dt} = - \left( \frac{s\bar{B}}{L} + \beta_2 \bar{B}\bar{H} \right) b - \beta_0 \bar{B}w - \beta_1 \bar{B}n - \beta_2 \bar{B}^2 \bar{h} + \rho_2 \bar{B}t_e, \\ \frac{dw}{dt} = \left( r'(\bar{B})\bar{W} \left( 1 - \frac{\bar{W}}{K(\bar{B})} \right) + \frac{r(\bar{B})\bar{W}^2 K'(\bar{B})}{K(\bar{B})^2} \right) b \\ \quad - \left( \frac{r(\bar{B})\bar{W}}{K(\bar{B})} \right) w - \nu_1 \bar{W}n - \nu_2 \bar{W}\bar{h}, \\ \frac{dn}{dt} = \lambda\beta_1 \bar{N}b - \sigma \bar{N}w - \frac{\theta \bar{N}}{M}n, \\ \frac{d\wp}{dt} = \phi n - \left( \phi_0 + \phi_1 \bar{E}_f \right) \wp - \phi_1 \bar{P}e_f \\ \frac{d\bar{h}}{dt} = \gamma n + \phi_2 \wp + -\gamma_1 \bar{h}, \\ \frac{d(t_f)}{dt} = -\rho b - \rho_1 t_e, \\ \frac{d(e_f)}{dt} = \omega \wp - \omega_1 e_f. \end{array} \right. \quad (95)$$

On utilising system (95), Theorem 4.16 specifies the conditions under which a system (73) is locally asymptotically stable around the interior equilibrium point  $\bar{E}$ .

**Theorem 4.16.** *The model system (73) is locally asymptotically stable around the interior equilibrium point  $\bar{E}$ , provided conditions (96)-(102) hold:*

$$\left( \frac{\bar{W}}{K(\bar{B})} - \frac{K'(\bar{B})\bar{B}\bar{W}}{(K(\bar{B}))^2} \right)^2 < \left( \frac{s + \beta_2 L \bar{H}}{\beta_0 L} \right) \left( \frac{2\bar{B}}{3K(\bar{B})} \right), \quad (96)$$

$$\beta_2^2 < \frac{\gamma_1 (s + \beta_2 L \bar{H})}{2L\bar{B}^2}, \quad (97)$$

$$\left( \frac{\nu_1}{\eta} + \frac{\sigma}{\lambda} \right)^2 < \frac{4\theta\beta_0\bar{B}}{9\lambda MK(\bar{B})}, \quad (98)$$

$$\nu_2^2 < \frac{\eta\gamma_1 r(\bar{B})}{3K(\bar{B})}, \quad (99)$$

$$\phi^2 < \frac{2\theta\phi_1 \bar{P} (\phi_1 \bar{E}_f + \phi_0)}{3\lambda M}, \quad (100)$$

$$\gamma^2 < \frac{\theta\gamma_1}{3\lambda M}, \quad (101)$$

$$\phi_2^2 < \frac{\gamma_1 \omega \left( \phi_1 \overline{E_f} + \phi_0 \right)}{2\phi_1 \overline{P}}. \quad (102)$$

*Proof.* Taking the positive definite function (103):

$$\mathbf{V} = \frac{1}{2} \left( \frac{1}{\overline{B}} b^2 + \frac{\kappa_1}{\overline{W}} w^2 + \frac{\kappa_2}{\overline{N}} n^2 + \kappa_3 \wp^2 + \kappa_4 \hbar^2 + \kappa_5 t_e^2 + \kappa_6 e_f^2 \right), \quad (103)$$

where  $\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5$  and  $\kappa_6$  are positive constants. The time derivative of the function along the solutions of linearised system (95) gives:

$$\frac{d\mathbf{V}}{dt} = \frac{b}{\overline{B}} \frac{db}{dt} + \kappa_1 \frac{w}{\overline{W}} \frac{dw}{dt} + \kappa_2 \frac{n}{\overline{N}} \frac{dn}{dt} + \kappa_3 \wp \frac{d\wp}{dt} + \kappa_4 \hbar \frac{d\hbar}{dt} + \kappa_5 t_e \frac{dt_e}{dt} + \kappa_6 e_f \frac{de_f}{dt},$$

$$\begin{aligned} \frac{d\mathbf{V}}{dt} = & - \left( \frac{s}{L} + \beta_2 \overline{H} \right) b^2 - \beta_0 bw - \beta_1 bn - \beta_2 \overline{B} b \hbar + \rho_2 b t_e - \kappa_1 \left( \frac{r(\overline{B})}{K(\overline{B})} \right) w^2 \\ & - \kappa_1 \nu_1 wn - \kappa_1 \nu_2 w \hbar + \kappa_1 \left( r'(\overline{B}) \left( 1 - \frac{\overline{W}}{K(\overline{B})} \right) + \frac{r(\overline{B}) \overline{W} K'(\overline{B})}{K(\overline{B})^2} \right) bw \\ & + \kappa_2 \lambda \beta_1 bn - \kappa_2 \sigma nw - \frac{\kappa_2 \theta}{M} n^2 + \kappa_3 \phi n \wp - \kappa_3 \left( \overline{E_f} \phi_1 + \phi_0 \right) \wp^2 - \kappa_3 \phi_1 \overline{P} e_f \wp \\ & \kappa_4 \gamma n \hbar + \kappa_4 \phi_2 \wp \hbar - \kappa_4 \gamma_1 \hbar^2 - \kappa_5 \rho b t_e - \kappa_5 \rho_1 t_e^2 + \kappa_6 \omega \wp e_f - \kappa_6 \omega_1 e_f^2. \end{aligned}$$

Arbitrary, choosing:

$$\kappa_1 = \frac{1}{\eta}, \quad \kappa_2 = \frac{1}{\lambda}, \quad \kappa_3 = \frac{\omega}{\phi_1 \overline{P}}, \quad \kappa_4 = 1, \quad \kappa_5 = \frac{\rho_2}{\rho} \text{ and } \kappa_6 = 1,$$

reduce  $\frac{d\mathbf{V}}{dt}$  to:

$$\begin{aligned} \frac{d\mathbf{V}}{dt} = & - \left( \frac{s}{L} + \beta_2 \overline{H} \right) b^2 - \kappa_1 \left( \frac{r(\overline{B})}{K(\overline{B})} \right) w^2 - \frac{\kappa_2 \theta}{M} n^2 - \kappa_3 \left( \overline{E_f} \phi_1 + \phi_0 \right) \wp^2 \\ & - \kappa_4 \gamma_1 \hbar^2 - \kappa_5 \rho_1 t_e^2 - \kappa_6 \omega_1 e_f^2 - \kappa_1 \left( \frac{r'(\overline{B}) \overline{W}}{K(\overline{B})} - \frac{r(\overline{B}) \overline{W} K'(\overline{B})}{K(\overline{B})^2} \right) bw \\ & - \beta_2 \overline{B} b \hbar - (\kappa_1 \nu_1 + \kappa_2 \sigma) wn - \kappa_1 \nu_2 w \hbar + \kappa_3 \phi n \wp + \kappa_4 \gamma n \hbar + \kappa_4 \phi_2 \wp \hbar. \end{aligned}$$

The interior equilibrium point  $\overline{E}$  is locally asymptotically stable provided  $\frac{d\mathbf{V}}{dt} < 0$ . Thus,  $\frac{d\mathbf{V}}{dt}$  is negative definite whenever conditions (96)-(102) hold.  $\square$

Provided Lemma 4.8 exists, Theorem 4.17 specifies the conditions under which a system (73) is globally asymptotically stable around the interior equilibrium point  $\overline{E}$ .

**Theorem 4.17.** *The interior equilibrium point  $\bar{E}$  is globally stable in the region of attraction  $\Omega_c$  whenever the conditions (104)-(110) hold:*

$$\left( \frac{\lambda_0 L K(L)}{K_0^2} + \frac{\bar{W}}{K(\bar{B})} \right)^2 < \frac{2(s + \beta_2 L \bar{H})}{3\beta_0 K(\bar{B})}, \quad (104)$$

$$\beta_2^2 < \frac{\gamma_1 (s + \beta_2 L \bar{H})}{2L^3}, \quad (105)$$

$$\left( \frac{\nu_1}{\eta} + \frac{\sigma}{\lambda} \right)^2 < \frac{4\beta_0 \theta L}{9\lambda M K(\bar{B})}, \quad (106)$$

$$\nu_2^2 < \frac{\eta \gamma_1 r(L)}{3K(\bar{B})}, \quad (107)$$

$$\phi^2 < \left( \frac{2\phi_1 \theta \bar{P}}{3\omega \lambda M} \right) \left( \frac{\omega_1 \phi_0 + \omega \phi_1 P_m}{\omega_1} \right), \quad (108)$$

$$\gamma^2 < \frac{\theta \gamma_1}{3\lambda M}, \quad (109)$$

$$\phi_2^2 < \left( \frac{\omega \gamma_1}{2\phi_1 \bar{P}} \right) \left( \frac{\phi_0 \omega_1 + \phi_1 \omega P_m}{\omega_1} \right). \quad (110)$$

*Proof.* Consider the positive definite function  $\mathcal{U}$  about  $\bar{E}$  :

$$\begin{aligned} \mathcal{U} = & \left( B - \bar{B} - \bar{B} \ln \frac{B}{\bar{B}} \right) + m_1 \left( W - \bar{W} - \bar{W} \ln \frac{W}{\bar{W}} \right) + m_2 \left( N - \bar{N} - \bar{N} \ln \frac{N}{\bar{N}} \right) \\ & + \frac{m_3}{2} (P - \bar{P})^2 + \frac{m_4}{2} (H - \bar{H})^2 + \frac{m_5}{2} (T_f - \bar{T}_f)^2 + \frac{m_6}{2} (E_f - \bar{E}_f)^2. \end{aligned}$$

where  $m_1, m_2, m_3, m_4, m_5$  and  $m_6$  are positive constants. It can be easily checked that the function  $\mathcal{U}$  is positive definite as follows: Assume  $y^* > 0$ , let  $g(y) = (y - y^*) - y^* \ln \frac{y}{y^*}$ , then  $g(y^*) = 0$  and  $g'(y) = 1 - \frac{y^*}{y}$ . It is worth noting that  $g'(y) > 0$  when  $y < y^*$  and  $g'(y) < 0$  when  $y > y^*$ . This suggests that the function  $g(y)$  has an absolute minimum 0 at  $y = y^*$  in the interval  $(0, \infty)$ . This property suggests that the function  $\mathcal{U}$  is positive definite with respect to  $\bar{E}$ .

Thus, the time derivative of function  $\mathcal{U}$  gives:

$$\begin{aligned} \frac{d\mathcal{U}}{dt} = & \left( \frac{B - \bar{B}}{B} \right) \frac{dB}{dt} + m_1 \left( \frac{W - \bar{W}}{W} \right) \frac{dW}{dt} + m_2 \left( \frac{N - \bar{N}}{N} \right) \frac{dN}{dt} + m_3 (P - \bar{P}) \frac{dP}{dt} \\ & + m_4 (H - \bar{H}) \frac{dH}{dt} + m_5 (T_e - \bar{T}_e) \frac{dT_e}{dt} + m_6 (E_f - \bar{E}_f) \frac{dE_f}{dt}. \end{aligned}$$

Substituting the values of  $dB/dt$ ,  $dW/dt$ ,  $dN/dt$ ,  $dP/dt$ ,  $dH/dt$ ,  $dT_e/dt$  and  $dE_f/dt$  from the system (73) gives:

$$\begin{aligned}
\frac{d\mathcal{U}}{d}t = & (B - \bar{B}) \left[ - \left( \frac{s}{L} + \beta_2 \bar{H} \right) (B - \bar{B}) - \beta_0(W - \bar{W}) - \beta_1(N - \bar{N}) - \beta_2 B(H - \bar{H}) \right] \\
& + m_1 (W - \bar{W}) \left[ \beta_0 \eta (B - \bar{B}) - \frac{\beta_0 \eta B}{K(\bar{B})} (W - \bar{W}) - \nu_1(N - \bar{N}) - \nu_2(H - \bar{H}) \right] \\
& + m_1 (W - \bar{W}) \left[ \left( \frac{\beta_0 \eta \lambda_0 B W}{K_0^2} + \frac{\alpha \eta \bar{W}}{K(\bar{B})} \right) (B - \bar{B}) \right] \\
& + m_2 (N - \bar{N}) \left[ - \frac{\theta}{M} (N - \bar{N}) + \lambda \beta_1 (B - B^*) - \sigma(W - \bar{W}) \right] \\
& + m_3 (P - \bar{P}) \left[ \phi(N - \bar{N}) - (\phi_0 + \phi_1 E_f) (P - \bar{P}) - \phi_1 \bar{P} (E_f - \bar{E}_f) \right] \\
& + m_4 (H - \bar{H}) \left[ -\gamma_1 + \gamma(N - \bar{N}) + \phi_2 (P - \bar{P}) \right] + \rho_2 (B - \bar{B}) (T_e - \bar{T}_e) \\
& + m_5 (T_e - \bar{T}_e) \left[ -\rho(B - \bar{B}) - \rho_1 (T_e - \bar{T}_e) \right] + m_6 (E_f - \bar{E}_f) \left[ \omega(P - \bar{P}) - \omega_1 (E_f - \bar{E}_f) \right].
\end{aligned}$$

Arbitrary, choosing  $m_1 = \frac{1}{\eta}$ ,  $m_2 = \frac{1}{\lambda}$ ,  $m_3 = \frac{\omega}{\phi_1 \bar{P}}$ ,  $m_4 = 1$ ,  $m_5 = \frac{\rho_2}{\rho}$ , and  $m_6 = 1$ ,  $\frac{d\mathcal{U}}{dt}$  is reduced to:

$$\begin{aligned}
\frac{d\mathcal{U}}{d}t = & (B - \bar{B}) \left[ - \left( \frac{s}{L} + \beta_2 \bar{H} \right) (B - \bar{B}) - \beta_2 B (H - \bar{H}) \right] \\
& + m_1 (W - \bar{W}) \left[ \left( \frac{\beta_0 \eta \lambda_0 B W}{K_0^2} + \frac{\beta_0 \eta \bar{W}}{K(\bar{B})} \right) (B - \bar{B}) \right] \\
& + m_1 (W - \bar{W}) \left[ - \frac{\beta_0 \eta B}{K(\bar{B})} (W - \bar{W}) - \nu_1 (N - \bar{N}) - \nu_2 (H - \bar{H}) \right] \\
& + m_2 (N - \bar{N}) \left[ - \frac{\theta}{M} (N - \bar{N}) - \sigma (W - \bar{W}) \right] \\
& + m_3 (P - \bar{P}) \left[ \phi (N - \bar{N}) - (\phi_0 + \phi_1 E_f) (P - \bar{P}) \right] \\
& + m_4 (H - \bar{H}) \left[ -\gamma_1 (H - \bar{H}) + \gamma (N - \bar{N}) + \phi_2 (P - \bar{P}) \right] \\
& - m_5 \rho_1 (T_e - \bar{T}_e)^2 - m_6 \omega_1 (E_f - \bar{E}_f)^2.
\end{aligned}$$

It is worth noting that  $\frac{d\mathcal{U}}{d}t$  is negative definite provided conditions (104)-(110) hold.  $\square$

#### 4.4.7 Parameter estimation and model fitting

The estimated parameter values presented in Table 13 were obtained by minimizing the sum of the squares of residuals between simulated solution of the model ( $Y_l$ ) by using the literature values and the synthetic data ( $Y_g$ ) generated randomly by adding iid Gaussian noise to  $Y_l$  [58]. The MATLAB built-in function `fminsearch` which uses Nelder-Mead simplex algorithm as described in Algorithm 1 was used obtain the local minimizers of the sum of the squares of residuals.

**Table 13:** Estimated parameter values

| Parameter | Initial value | Source  | Estimated | Parameter  | Initial value | Source   | Estimated |
|-----------|---------------|---------|-----------|------------|---------------|----------|-----------|
| $s$       | 0.8           | [56]    | 0.69      | $\sigma$   | 0.001         | [92]     | 0.0012    |
| $L$       | 100           | [56]    | 91        | $\omega_1$ | 0.1           | [76]     | 0.1067    |
| $\alpha$  | 0.05          | [49]    | 0.0487    | $\gamma$   | 0.004         | Assumed  | 0.0039    |
| $\eta$    | 0.9           | [49]    | 0.8045    | $\gamma_1$ | 0.01          | [92]     | 0.0429    |
| $\beta_1$ | 0.003         | [92]    | 0.0029    | $\phi_0$   | 0.5           | [76]     | 0.4323    |
| $\beta_2$ | 0.0004        | Assumed | 0.0004    | $\phi$     | 0.1           | [76]     | 0.09      |
| $\nu_1$   | 0.002         | [92]    | 0.002     | $\phi_1$   | 0.01          | [75]     | 0.011     |
| $\nu_2$   | 0.0001        | [92]    | 0.0001    | $\phi_2$   | 0.2           | Assumed  | 0.18      |
| $\theta$  | 0.5           | [92]    | 0.75      | $\rho$     | 0.01          | [74, 76] | 0.028     |
| $M$       | 100           | [56]    | 93        | $\rho_1$   | 0.03          | [74, 76] | 0.0738    |
| $K_0$     | 10            | Assumed | 9.0227    | $\rho_2$   | 0.02          | [76]     | 0.0215    |
| $K_1$     | 3             | [49]    | 2.966     | $\omega$   | 0.05          | [77]     | 0.0524    |
| $\lambda$ | 0.05          | [92]    | 0.045     |            |               |          |           |

The best fits of the generated synthetic data by using the estimated parameter values is shown in Figs. 37a-37g. The evolution of the model (73) solution by using literature values (red line) and estimated parameter values (dotted black line) are shown in Figs. 38a-38g. It can be seen that all solutions are superimposed suggesting that the model prediction is the best fit of the real data.

The sample autocorrelations of the residuals are shown in Fig. 39, it can be observed that almost all the autocorrelation of residuals of the corresponding estimates are inside the region which represents the observed correlation within the residuals are insignificant at 5% level and are almost evenly distributed around zero. Thus, the residuals are independent and uncorrelated, suggesting that the model has the ability to fit the data.

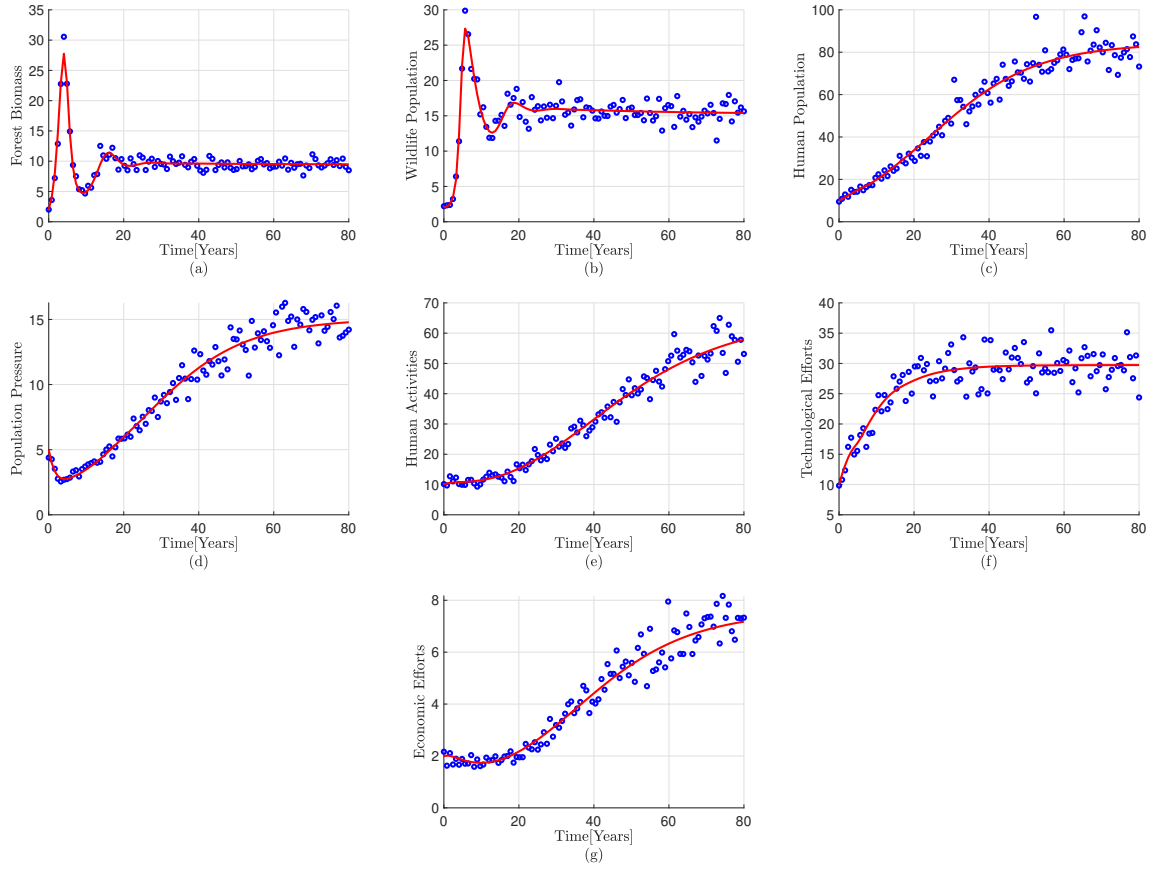


**Figure 37:** Model fitting (lines) corresponding to estimated parameters for (a) Forest biomass (b) Wildlife population (c) Human population (d) population pressure (e) Human activities (f) Technological efforts and (g) Economic efforts

#### 4.4.8 Uncertainty and Sensitivity (US) analysis of the input parameters

The LHS/PRCC approach, as described in Section 3.5, was employed to conduct uncertainty and sensitivity analysis of the model's input parameters. In this method, each parameter was treated as a random variable, and probability distribution functions (pdfs) for each parameter of interest were generated, as illustrated in Fig. 40. Subsequently, parameter values were randomly selected from these samples, ensuring that each value was used only once during the analysis.

For the LHS matrix, a sample size of 200 was utilised, and the parameters were assumed to follow uniform probability distribution functions due to limited information available about their distributions [63]. The parameter ranges were randomly selected, with the mean values taken from estimated and literature values (Table 13 and Table 14, respectively).

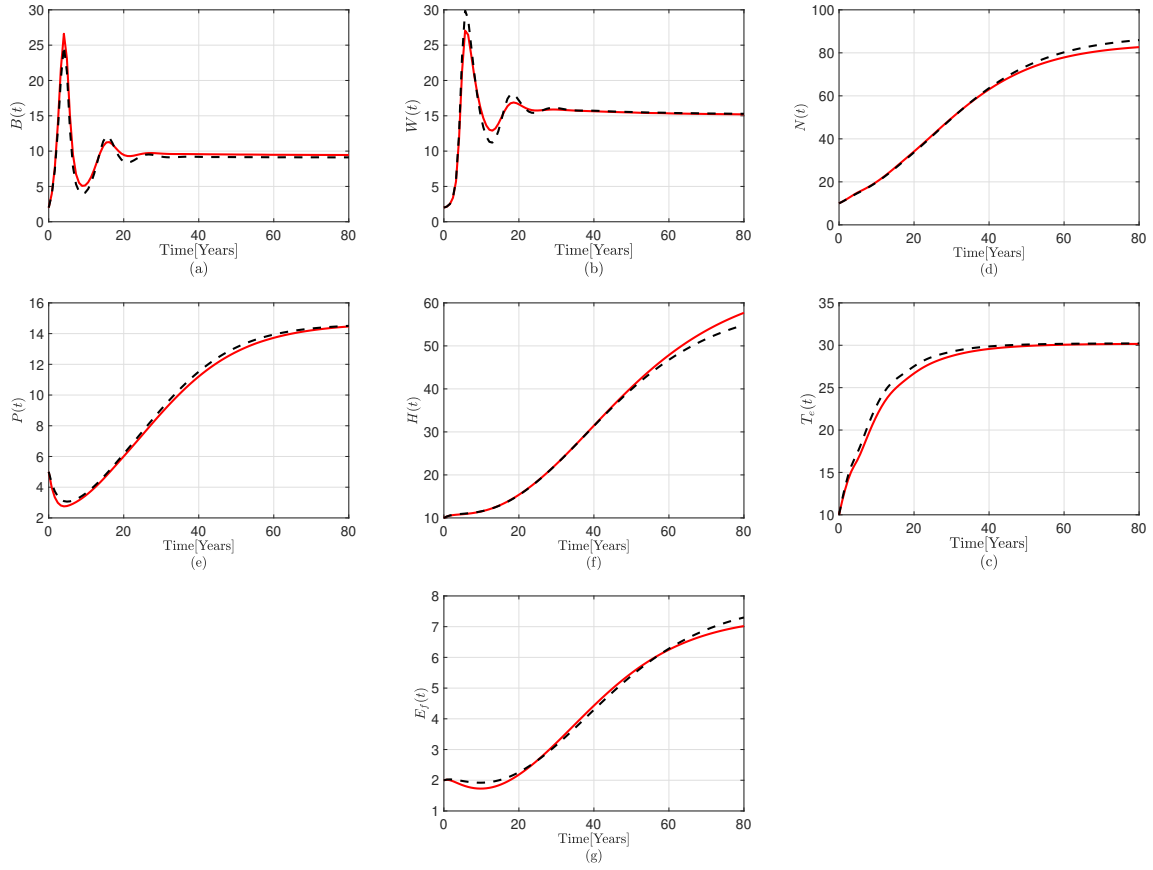


**Figure 38:** Evolution of model solutions using literature values (solid red lines) versus estimated values (dash black lines) for (a) Forest biomass (b) Wildlife population (c) Technological efforts (d) Human population (e) Population pressure, (f) Human activities and (g) Economic efforts

It was found that a sample size of 200 produced similar results to tests with larger sample sizes and was deemed efficient in identifying the most significant parameters. Increasing the sample size did not yield significant improvements, as the approach consistently identified the same set of most significant parameters in various tests.

The PRCC values were computed for each model output variable and LHS-sampled parameter to determine their impact on the model output. The primary objective was to identify the significant parameters that affect the densities of forest biomass and forest-dependent wildlife population, as well as population pressure and human activities. A positive PRCC value indicates a positive correlation between the input parameter and the model output. This means that as the value of the input parameter increases, the model output tends to increase, and as the value of the input parameter decreases, the model output tends to decrease.

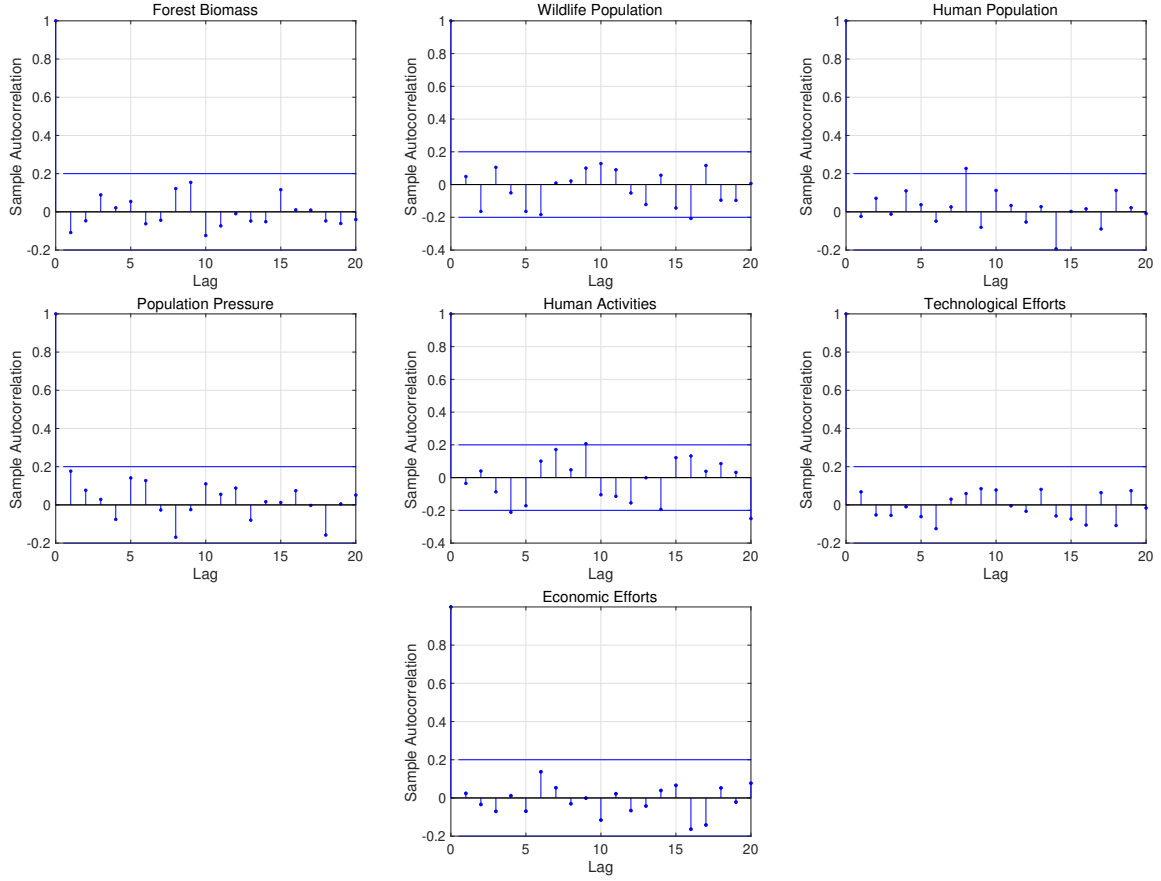


**Figure 39:** The sample autocorrelation of the residuals for the model outputs arranged from the top left corner as follows: forest biomass, wildlife population, human population, population pressure, human activities, technological efforts, and economic efforts

On the other hand, a negative PRCC value suggests an inverse correlation between the input parameter and the model output. In this case, as the value of the input parameter increases, the model output tends to decrease, and as the value of the input parameter decreases, the model output tends to increase.

To investigate the significance of parameters over the entire time period of model (73) dynamics, PRCCs were computed at various time intervals and plotted against time. As there were no known critical time periods, the analysis was exploratory in nature, aiming to identify any time-dependent correlations over the entire duration. The PRCCs were computed for a period of 100 years with respect to forest biomass  $B(t)$ , forest-dependent wildlife  $W(t)$ , population pressure  $P(t)$ , and human activities  $H(t)$ , as shown in Fig. 41. The shaded region represents PRCCs that are insignificantly different from zero [63].

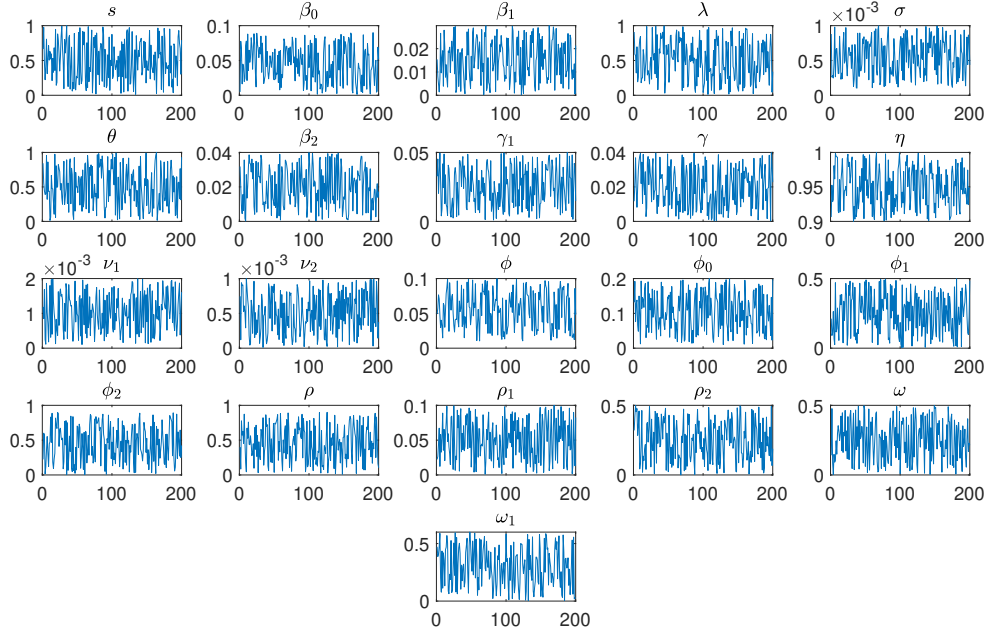


**Figure 40:** Probability distribution functions (pdfs) of the twenty one sampled input parameter values obtained from LHS using a sample size of 200

For  $B(t)$  and  $W(t)$ , the most significant parameters over the entire time span were  $\rho$ ,  $\rho_2$ , and  $\beta_2$ , as their PRCC values were significantly different from zero. Parameters like  $\gamma_1$ ,  $\rho_1$ , and  $\phi_2$  were insignificant at the early time points but became more impactful as time progressed, while the remaining parameters remained insignificant throughout the entire duration.

Regarding  $H(t)$ , the significant parameters for the entire time span were  $\gamma$ ,  $\gamma_1$ ,  $\phi_1$ , and  $\phi_2$ . Initially, the effects of  $\omega$ ,  $\theta$ , and  $\omega_1$  were insignificant, but they became significant as time passed. Parameter  $\phi$  showed significance in the early time points but lost its impact as time progressed. For the model output  $P(t)$ , the most significant parameters over the entire time intervals were  $\phi$ ,  $\phi_1$ ,  $\omega$ , and  $\omega_1$ , while  $\phi_0$  was insignificant only at the early time points. However, parameter  $\phi_0$  was significant in the early time points but became insignificant as time progressed. These findings shed light on the varying importance of parameters over time, which can be crucial for understanding the dynamics of the system and developing effective conservation and management strategies.

**Table 14:** Mean values and ranges of parameters values used in LHS

| Parameter | Range                      | Mean value | Source | Parameter  | Range              | Mean value | Source |
|-----------|----------------------------|------------|--------|------------|--------------------|------------|--------|
| $s$       | [0.01 - 1]                 | 0.69       | Fitted | $\sigma$   | $[10^{-3} - 0.1]$  | 0.0012     | Fitted |
| $L$       | -                          | 100        | [56]   | $\omega_1$ | [0.003 - 0.8]      | 0.1067     | Fitted |
| $\beta_0$ | $[10^{-4} - 0.09]$         | 0.0487     | Fitted | $\gamma$   | $[10^{-4} - 0.4]$  | 0.0039     | Fitted |
| $\eta$    | [0.8- 1]                   | 0.8045     | Fitted | $\gamma_1$ | $[10^{-3} - 0.05]$ | 0.0429     | Fitted |
| $\beta_1$ | $[10^{-4} - 0.3]$          | 0.0029     | Fitted | $\phi_0$   | [0.01 - 0.9]       | 0.4323     | Fitted |
| $\beta_2$ | $[10^{-5} - 0.04]$         | 0.0004     | Fitted | $\phi$     | [0.01 - 0.9]       | 0.09       | Fitted |
| $\nu_1$   | $[2 \times 10^{-4} - 0.2]$ | 0.002      | Fitted | $\phi_1$   | $[10^{-4} - 0.5]$  | 0.011      | Fitted |
| $\nu_2$   | $[2 \times 10^{-5} - 0.1]$ | 0.0001     | Fitted | $\phi_2$   | [0.02 - 0.9]       | 0.18       | Fitted |
| $\theta$  | [0.01 - 1]                 | 0.75       | Fitted | $\rho$     | [0.01 - 0.8]       | 0.028      | Fitted |
| $M$       | -                          | 100        | [56]   | $\rho_1$   | $[10^{-3} - 0.5]$  | 0.0738     | Fitted |
| $K_0$     | -                          | 20         | [49]   | $\rho_2$   | $[10^{-4} - 0.6]$  | 0.0215     | Fitted |
| $K_1$     | -                          | 3          | [49]   | $\omega$   | [0.01 - 0.5]       | 0.0524     | Fitted |
| $\lambda$ | [0 - 1]                    | 0.045      | Fitted |            |                    |            |        |

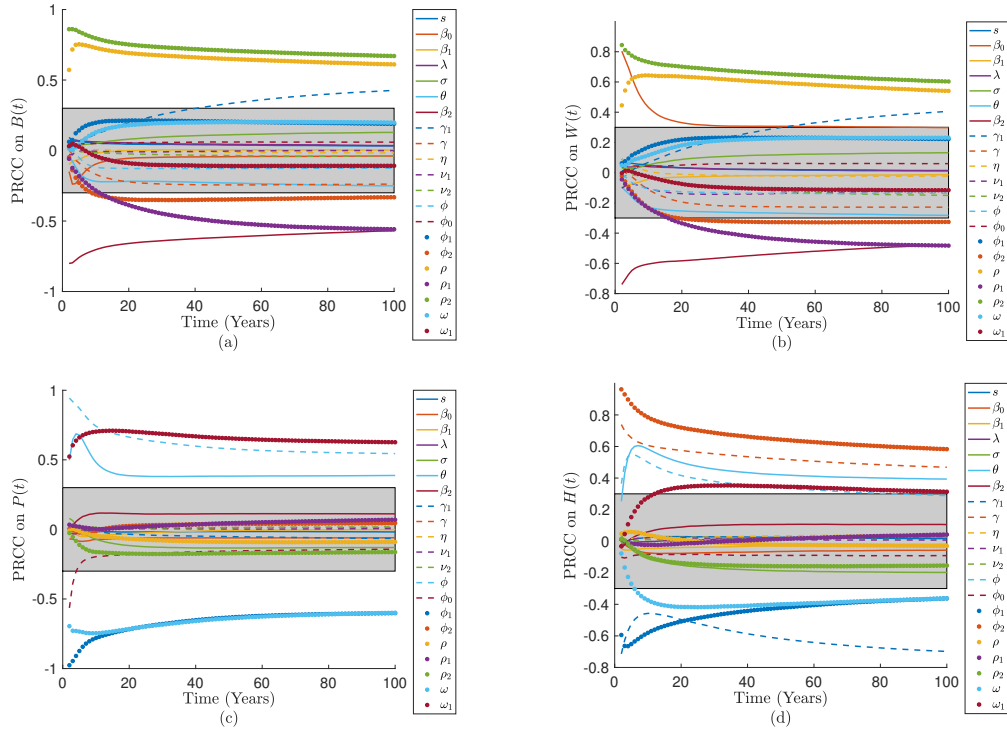


**Figure 41:** PRCC of the model parameters with respect to (a) Forest biomass ( $B(t)$ ), (b) Forest-dependent wildlife population ( $W(t)$ ), (c) Population pressure ( $P(t)$ ) and (d) Human activities ( $H(t)$ ) plotted over a time course (sample size =200)

The sensitivity analysis was further extended to explore the time invariance of the system dynamics. Specifically, consideration was given to two time points:  $t = 8$ , representing an early stage in the system dynamics, and  $t = 80$ , representing a later stage. These specific time points were arbitrarily chosen to assess the impact of parameters at different stages of the system's evolution. The results are presented in Figs. 42 and 43 for forest biomass and forest-dependent wildlife population, and Figs. 44 and 45 for population pressure and human activities, respectively.

From these figures, it can be observed that the results are consistent with the time-varying sensitivity analysis. The sensitivity of the system to different parameters remains consistent at both time points, implying that certain parameters consistently play crucial roles in influencing the dynamics of the system, regardless of the specific time. This insight provides valuable information for understanding the long-term behaviour of the system and can be beneficial in devising sustainable conservation and management strategies.

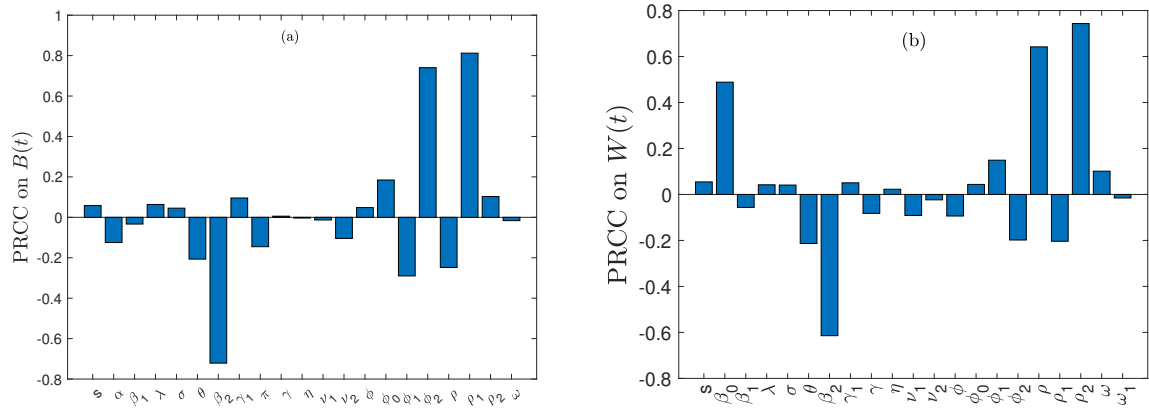


**Figure 42:** PRCC of the model parameters with respect to (a) Forest biomass ( $B(t)$ ) and (b) Forest-dependent wildlife population ( $W(t)$ ) at time point  $t = 8$

A significance test was performed to determine whether the PRCC values for the two parameters in relation to forest biomass  $B(t)$  varied significantly. Since the PRCC values can range from  $-1$  to  $1$ , the Fisher transformation was applied to remove the skewness in the distribution of PRCC values before computing the p-values [6]. Pairwise comparisons were carried out for parameters whose PRCCs significantly differed from zero ( $p < 0.05$ ), with the null hypothesis stating that the compared sensitive parameters are not significantly different.

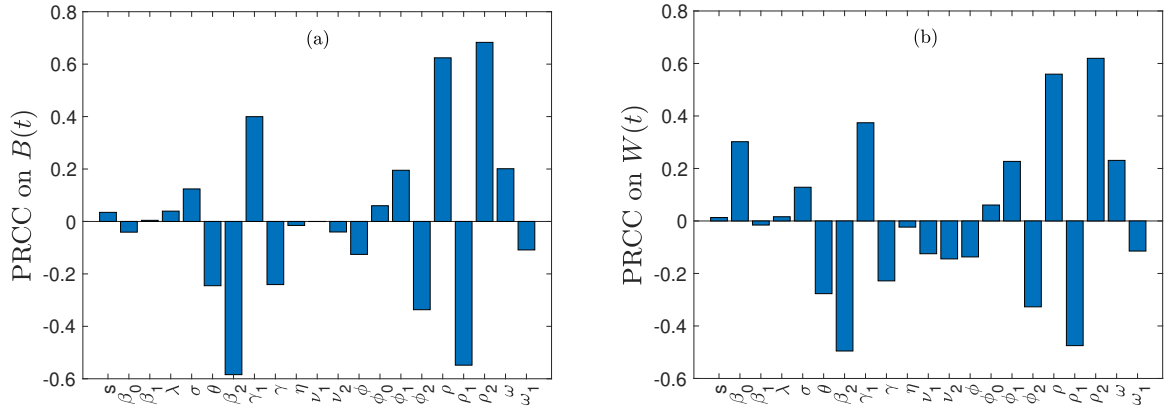


**Figure 43:** PRCC of the model parameter with respect to (a) Forest biomass ( $B(t)$ ) and (b) Forest-dependent wildlife population ( $W(t)$ ) at time point  $t = 80$

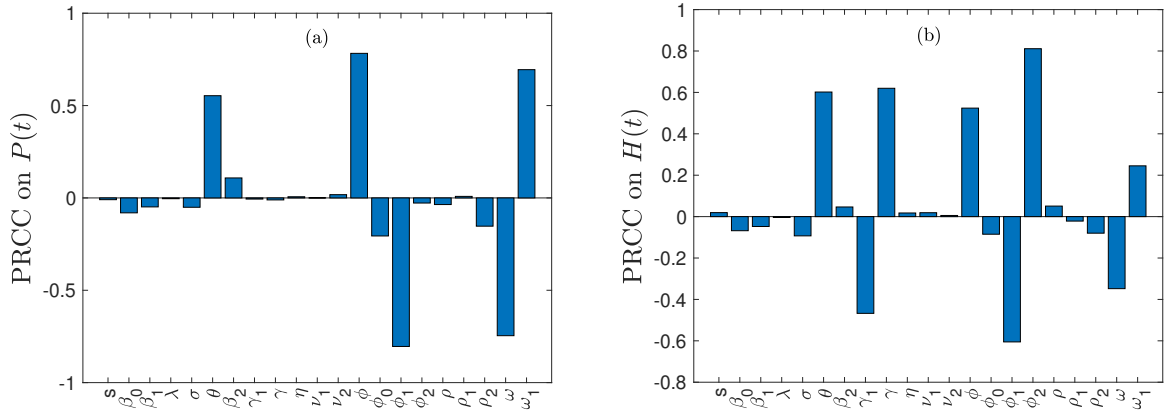


**Figure 44:** PRCC of the model parameters with respect to (a) Population pressure ( $P(t)$ ) and (b) Human activities ( $H(t)$ ) at time point  $t = 8$

The computed p-values between the compared parameters are presented in Table 15. Additionally, False Discovery Rate (FDR) adjusted p-values (Table 16) were computed to reduce the chance of making a Type 1 statistical error [18]. In this analysis, p-values less than 0.05 are considered significantly different, leading to the rejection of the null hypothesis. Using the results from Table 16, the outcomes of the pairwise comparison are presented in Table 17. The table indicates "TRUE" when the compared PRCC values are significantly different ( $p < 0.05$ ) and "FALSE" when they are not significantly different ( $p > 0.05$ ), thus confirming the rejection of the null hypothesis. The results show that most of the sensitive parameters indeed differ significantly.

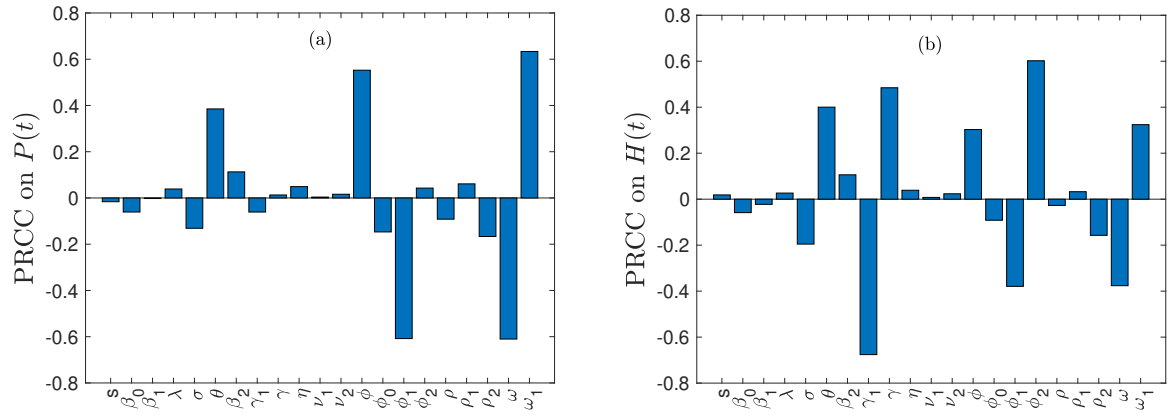


**Figure 45:** PRCC of the model parameters with respect to (a) Population pressure ( $P(t)$ ) and (b) Human activities ( $H(t)$ ) at time point  $t = 80$

**Table 15:** Pairwise PRCC comparisons (Unadjusted p-values)

|            | $\beta_2$ | $\theta$ | $\gamma_1$ | $\gamma$ | $\phi$ | $\phi_1$ | $\phi_2$ | $\rho$   | $\rho_1$ | $\rho_2$ | $\omega$  | $\omega_1$ |
|------------|-----------|----------|------------|----------|--------|----------|----------|----------|----------|----------|-----------|------------|
| $\beta_2$  |           | 6.80e-11 | 0          | 1.36e-11 | 0      | 0        | 1.50e-05 | 0        | 0.12     | 0        | 0         | 0          |
| $\theta$   |           |          | 0          | 0.81     | 0.006  | 0        | 0.028    | 0        | 6.61e-16 | 0        | 0         | 0.002      |
| $\gamma_1$ |           |          |            | 0        | 0      | 9.3e-07  | 0        | 2.21e-11 | 0        | 2.21e-11 | 9.336e-07 | 0          |
| $\gamma$   |           |          |            |          | 0.011  | 0        | 0.015    | 0        | 0        | 0        | 0         | 0.005      |
| $\phi$     |           |          |            |          |        | 1.31e-13 | 7.077e-7 | 0        | 0        | 0        | 1.31e-13  | 0.8215     |
| $\phi_1$   |           |          |            |          |        |          | 0        | 0        | 0        | 0        | 1         | 7.04e-13   |
| $\phi_2$   |           |          |            |          |        |          |          | 0        | 4.40e-9  | 0        | 0         | 2.16e-7    |
| $\rho$     |           |          |            |          |        |          |          |          | 0        | 1        | 0         | 0          |
| $\rho_1$   |           |          |            |          |        |          |          |          |          | 0        | 0         | 0          |
| $\rho_2$   |           |          |            |          |        |          |          |          |          |          | 0         | 0          |
| $\omega$   |           |          |            |          |        |          |          |          |          |          |           | 7.04e-13   |
| $\omega_1$ |           |          |            |          |        |          |          |          |          |          |           |            |

**Table 16:** Pairwise PRCC comparisons (FDR Adjusted p-values)

|            | $\beta_2$ | $\theta$ | $\gamma_1$ | $\gamma$ | $\phi$  | $\phi_1$  | $\phi_2$  | $\rho$   | $\rho_1$  | $\rho_2$ | $\omega$ | $\omega_1$ |
|------------|-----------|----------|------------|----------|---------|-----------|-----------|----------|-----------|----------|----------|------------|
| $\beta_2$  |           | 9.13e-11 | 0          | 1.95e-11 | 0       | 0         | 1.753e-05 | 0        | 0.1332    | 0        | 0        | 0          |
| $\theta$   |           |          | 0          | 0.84     | 0.006   | 0         | 0.030     | 0        | 1.072e-15 | 0        | 0        | 0.003      |
| $\gamma_1$ |           |          |            | 0        | 0       | 1.141e-06 | 0         | 3.04e-11 | 0         | 3.04e-11 | 1.1e-06  | 0          |
| $\gamma$   |           |          |            |          | 0.01269 | 0         | 0.016     | 0        | 0         | 0        | 0        | 0.006      |
| $\phi$     |           |          |            |          |         | 2.02e-13  | 8.9e-7    | 0        | 0         | 0        | 2.0e-13  | 0.8472     |
| $\phi_1$   |           |          |            |          |         |           | 0         | 0        | 0         | 0        | 1        | 1.0e-12    |
| $\phi_2$   |           |          |            |          |         |           |           | 0        | 5.8e-9    | 0        | 0        | 2.7e-7     |
| $\rho$     |           |          |            |          |         |           |           |          | 0         | 1        | 0        | 0          |
| $\rho_1$   |           |          |            |          |         |           |           |          |           | 0        | 0        | 0          |
| $\rho_2$   |           |          |            |          |         |           |           |          |           |          | 0        | 0          |
| $\omega$   |           |          |            |          |         |           |           |          |           |          |          | 1.0e-12    |
| $\omega_1$ |           |          |            |          |         |           |           |          |           |          |          |            |

**Table 17:** Parameters different after FDR adjustment?

|            | $\beta_2$ | $\theta$ | $\gamma_1$ | $\gamma$ | $\phi$ | $\phi_1$ | $\phi_2$ | $\rho$ | $\rho_1$ | $\rho_2$ | $\omega$ | $\omega_1$ |
|------------|-----------|----------|------------|----------|--------|----------|----------|--------|----------|----------|----------|------------|
| $\beta_2$  |           | TRUE     | TRUE       | TRUE     | TRUE   | TRUE     | TRUE     | TRUE   | FALSE    | TRUE     | TRUE     | TRUE       |
| $\theta$   |           |          | TRUE       | FALSE    | TRUE   | TRUE     | TRUE     | TRUE   | TRUE     | TRUE     | TRUE     | TRUE       |
| $\gamma_1$ |           |          |            | TRUE     | TRUE   | TRUE     | TRUE     | TRUE   | TRUE     | TRUE     | TRUE     | TRUE       |
| $\gamma$   |           |          |            |          | TRUE   | TRUE     | TRUE     | TRUE   | TRUE     | TRUE     | TRUE     | TRUE       |
| $\phi$     |           |          |            |          |        | TRUE     | TRUE     | TRUE   | TRUE     | TRUE     | TRUE     | FALSE      |
| $\phi_1$   |           |          |            |          |        |          | TRUE     | TRUE   | TRUE     | TRUE     | FALSE    | TRUE       |
| $\phi_2$   |           |          |            |          |        |          |          | TRUE   | TRUE     | TRUE     | TRUE     | TRUE       |
| $\rho$     |           |          |            |          |        |          |          |        | TRUE     | FALSE    | TRUE     | TRUE       |
| $\rho_1$   |           |          |            |          |        |          |          |        |          | TRUE     | TRUE     | TRUE       |
| $\rho_2$   |           |          |            |          |        |          |          |        |          |          | TRUE     | TRUE       |
| $\omega$   |           |          |            |          |        |          |          |        |          |          |          | TRUE       |
| $\omega_1$ |           |          |            |          |        |          |          |        |          |          |          |            |

#### 4.4.9 Numerical simulation

To check the feasibility of the model analysis regarding stability and permanence conditions of the model system (73), numerical simulation was carried out using MATLAB 2022b software. The parameter values in Table 13 and the initial conditions  $B(0) = 5$ ,  $W(0) = 5$ ,  $N(0) = 10$ ,  $P(0) = 5$ ,  $H(0) = 25$ ,  $T_e(0) = 25$ , and  $E_f(0) = 2$ , arbitrary selected were used in the simulation. The obtained values of  $\overline{E}$  were  $\overline{B} = 9.40$ ,  $\overline{W} = 15.10$ ,  $\overline{N} = 84.74$ ,  $\overline{P} = 14.78$ ,  $\overline{H} = 66.54$ ,  $\overline{T}_e = 30.3$ , and  $\overline{E}_f = 7.35$ .

The conditions for local stability (Theorem 4.16) and global stability (Theorem 4.17) were all satisfied, suggesting that the interior equilibrium  $\overline{E}$  is both locally and globally asymptotically stable for the selected set of parameter values and initial conditions. The eigenvalues of the Jacobian Matrix corresponding to  $\overline{E}$  for the model system (73) are:

$$-0.2928 + 0.5085i, -0.2928 - 0.5085i, -0.5625, -0.4857, -0.0490, -0.0924, -0.1177.$$

It can be observed that out of the seven eigenvalues, five have negative values, while the other two have negative real parts. This indicates that the interior equilibrium point  $\overline{E}$  is locally asymptotically stable.

The evolution of the model's solution over time is shown in Figs. 46a-g. A comparison of these figures reveals that the forest biomass ( $\overline{B} = 9.40$ ) and forest-dependent wildlife population ( $\overline{W} = 15.10$ ) converge to their equilibrium levels in less than 60 years. In contrast, the model system (15) takes more than 60 years to reach steady states with lower values (Fig. 24). This

implies that the inclusion of technological efforts ( $T_e$ ) and economic measures ( $E_f$ ) in the model significantly contributes to the conservation of resources.

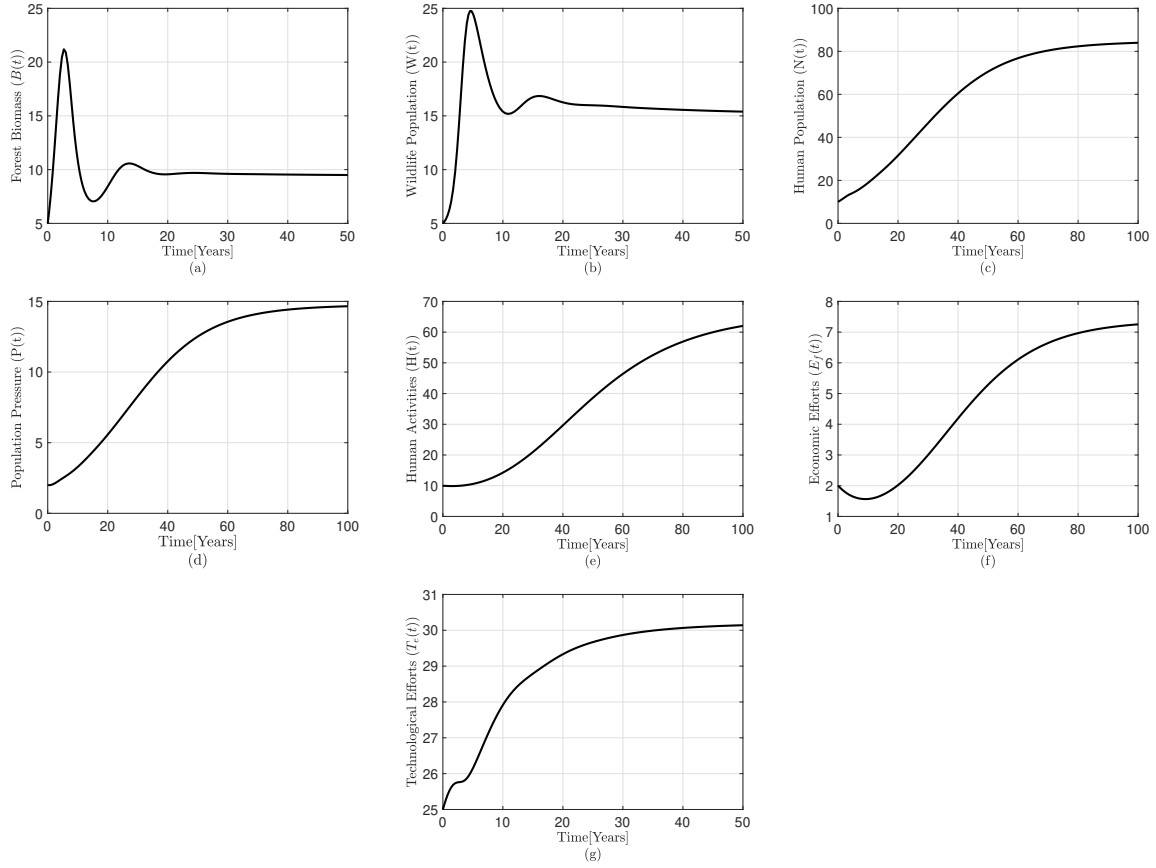


**Figure 46:** Temporal evolution of: (a) Forest biomass, (b) Wildlife population in forest habitats, (c) Human population, (d) Population pressure, (e) Human activities, (f) Economic efforts, and (g) Technological efforts. Forest biomass, wildlife population and Technological efforts were simulated for the period of 50 years, whereas remaining variables were simulated for the period of 100 years

The non-linear stability of the equilibrium point  $\overline{E}$  is demonstrated using solution trajectories with different initial conditions, as shown in Figs. 47 and 48a, b. It can be observed that all the solution trajectories initiating inside the region of attraction ( $\Omega_c$ ) approached the equilibrium values  $(\overline{B}, \overline{W})$ ,  $(\overline{P}, \overline{H}, \overline{E}_f)$  and  $(\overline{N}, \overline{P}, \overline{E}_f)$ , respectively. This shows that the equilibrium  $\overline{E}$  is globally asymptotically stable in the  $B - W$ ,  $P - H - E_f$ , and  $N - P - E_f$  planes, respectively. In other planes, non-linear stability can be demonstrated in a similar way.

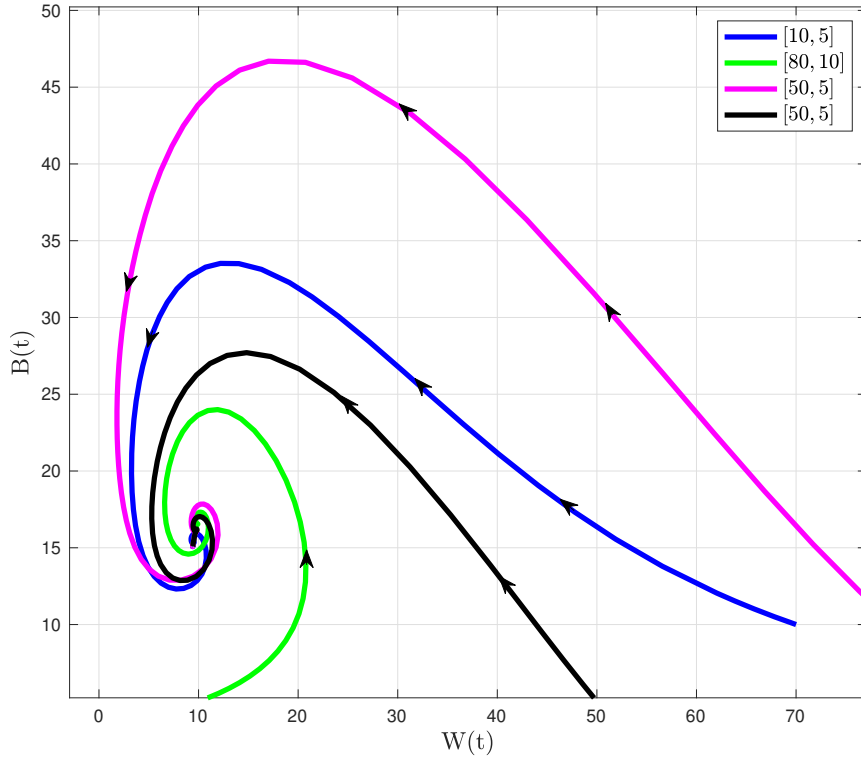


**Figure 47:** Non-linear stability of the equilibrium point  $\bar{E}$  in  $B - W$  plane

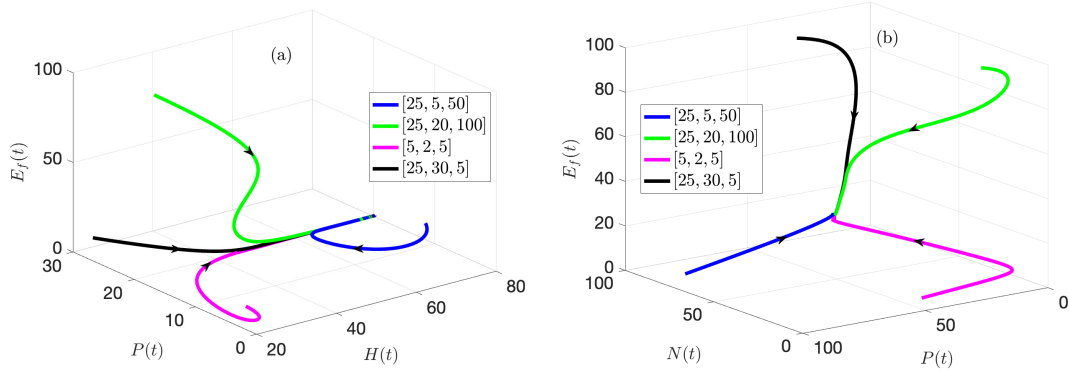


**Figure 48:** Non-linear stability of the equilibrium point  $\bar{E}$  in (a)  $P - H - E$  and (b)  $N - P - E$  planes

The behaviour of the system was studied systematically varying the values of crucial parameters, such as  $\beta_1$ ,  $\beta_2$ ,  $\phi$ ,  $\phi_2$ ,  $\rho$ ,  $\omega$ , and  $\gamma$ , both individually and in combinations. In Fig. 49 temporal variations of  $B$ ,  $W$ ,  $N$ , and  $H$  for different values of  $\beta_1$  and  $\beta_2$  are shown. The insights gained from these simulations are striking. When  $\beta_1$  and  $\beta_2$  are increased, either independently or together, the densities of forest biomass and forest-dependent wildlife

population undergo a decline. However, the decline in the density of forest-dependent wildlife population is more pronounced compared to that of forest biomass. This suggests that the density of forest-dependent wildlife population is significantly more susceptible to extinction risks, both directly and indirectly, as the human population and its associated activities increase.

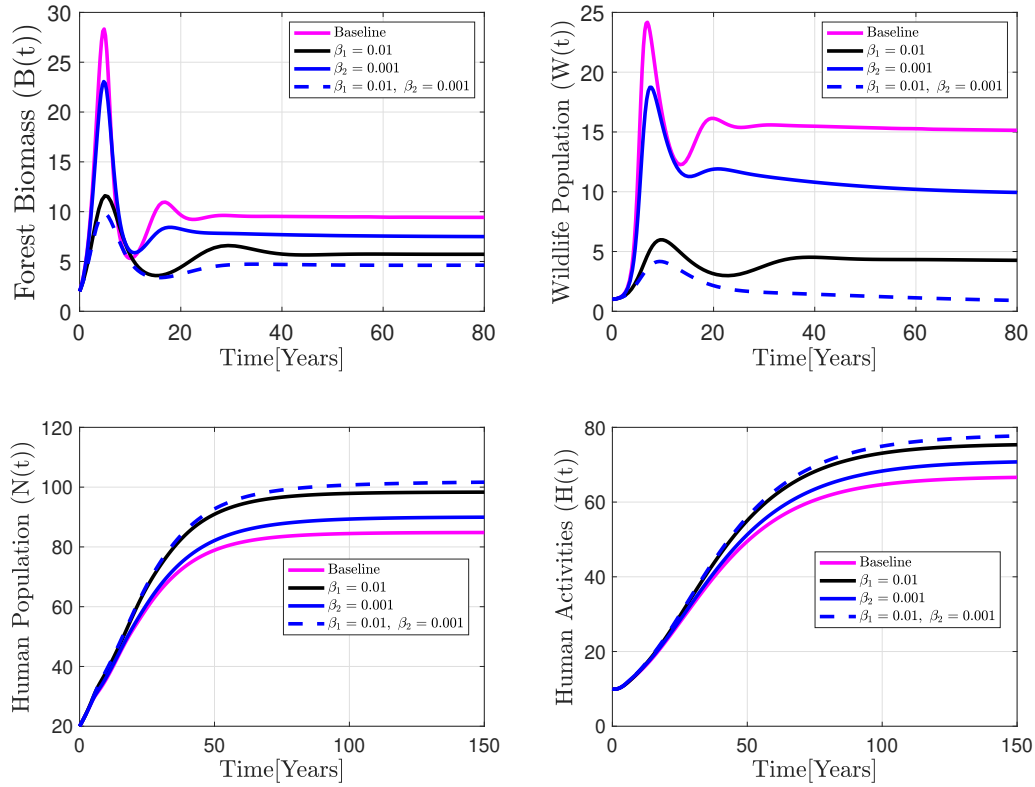


**Figure 49:** Variation of forest biomass ( $B$ ), forest-dependent wildlife population ( $W$ ), human population ( $N$ ) and human activities ( $H$ ) with time ( $t$ ) for different values of  $\beta_1$  and  $\beta_2$

In Fig. 50, the variations of model variables with respect to time for different values of  $\phi$  and  $\phi_2$ , both individually and in combination are presented. The findings suggest that an increase in population pressure due to higher values of  $\phi$  and  $\phi_2$  leads to a rise in human activities, resulting in a greater exploitation of forest resources. Consequently, the density of forest biomass and forest-dependent wildlife populations experience a significant decrease. Notably, the decline in the forest-dependent wildlife population is more pronounced compared to that of forest biomass. These results indicate that the forest-dependent wildlife population is at a heightened risk of extinction, either directly or indirectly, as human activities continue to increase.

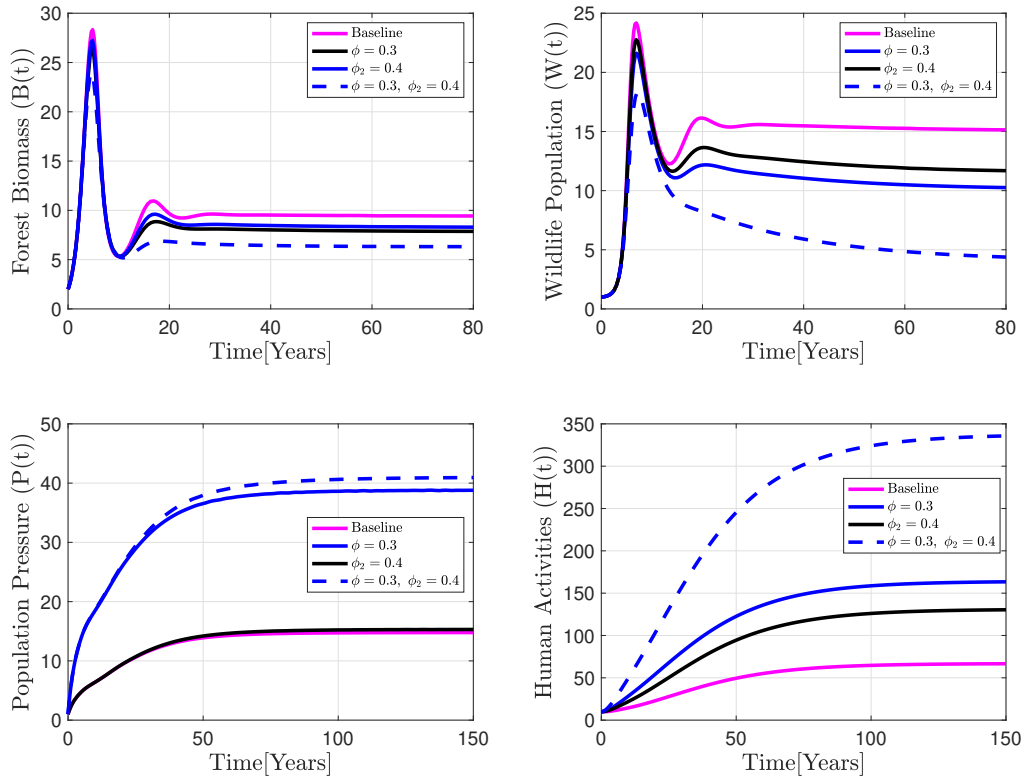


**Figure 50:** Variation of forest biomass ( $B$ ), wildlife population ( $W$ ), population pressure ( $P$ ) and human activities ( $H$ ) with time ( $t$ ) for different values of  $\phi$  and  $\phi_2$

The variations of  $B$ ,  $W$ ,  $P$ , and  $H$  with time for different values of  $\beta_2$  and  $\gamma$ , and the results are presented in Fig. 51. It can be observed that with increasing human activities and resource exploitation, the densities of forest biomass and the population of forest-dependent wildlife decrease. Furthermore, the graph illustrates that this corresponding change is more prominent when  $\beta_2$  changes from 0.0004 to 0.004. This finding suggests that parameter  $\beta_2$  must be controlled and kept below a certain threshold to ensure the survival of the forest-dependent wildlife population.

Furthermore, the variations of  $B$ ,  $W$ ,  $P$  and  $H$  with  $t$  for different values of  $\rho$  and  $\omega$  is also studied and illustrated in Fig. 52. It can be observed that implementing technological efforts significantly increases the densities of both forest biomass and forest-dependent wildlife populations. On the other hand, using economic measures to reduce population pressure leads to an increase in the density of forest biomass.

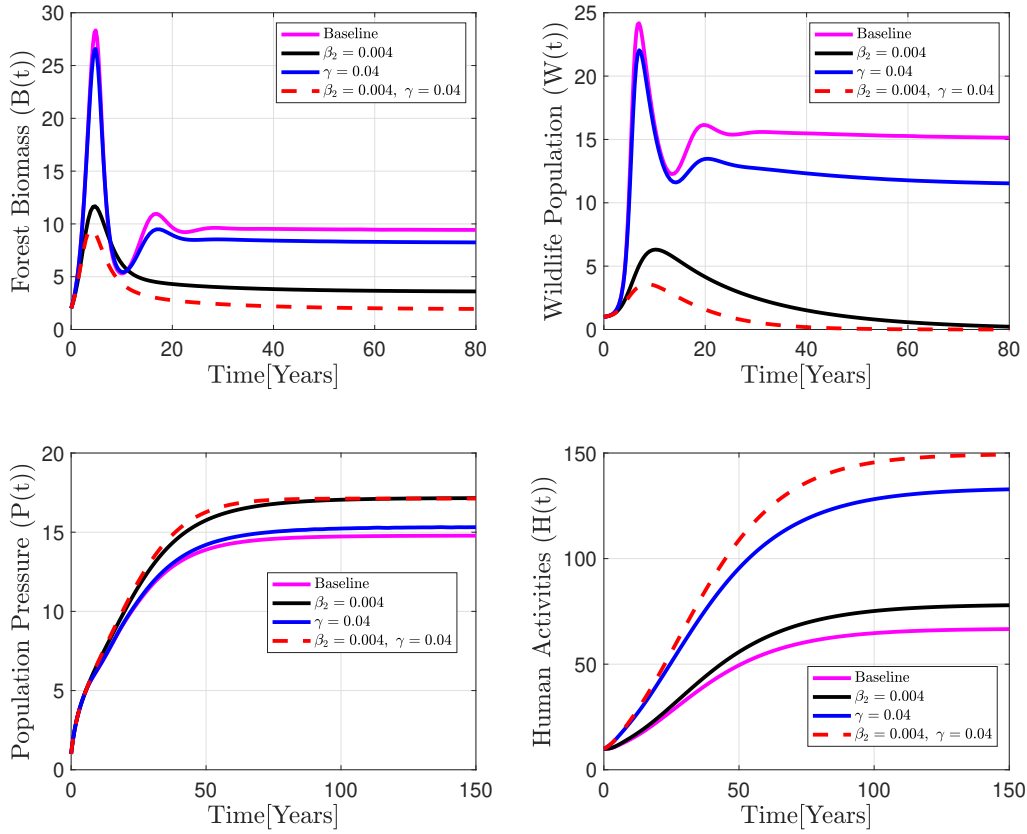


**Figure 51:** Variation of forest biomass ( $B$ ), wildlife population ( $W$ ), population pressure ( $P$ ) and human activities ( $H$ ) with time ( $t$ ) for different values of  $\beta_2$  and  $\gamma$

This suggests that a combined approach, utilising both technological efforts and economic measures, is crucial for effective resource conservation. By implementing these measures concurrently over time, we can better preserve the forest biomass and maintain the well-being of the forest-dependent wildlife populations while meeting the demand of human population.

Similarly, as revealed in Fig. 52, for better resources management it is suggested that the strategies, that is, technological efforts and economic measures should be implemented simultaneously. Thus, the study also investigated the impact of parametric variations of the implementation rate of technological efforts ( $\rho$ ) and the implementation rate of economic measures ( $\omega$ ) on the system's behaviour. These parameters were varied simultaneously while other parameters were kept fixed. Valuable insights were gained as shown in Fig. 53. The results reveal a compelling pattern, as these parameters exceed certain threshold, the variables representing human population, population pressure, and human activities experience extinction while forest biomass and forest-dependent wildlife keep on increasing.

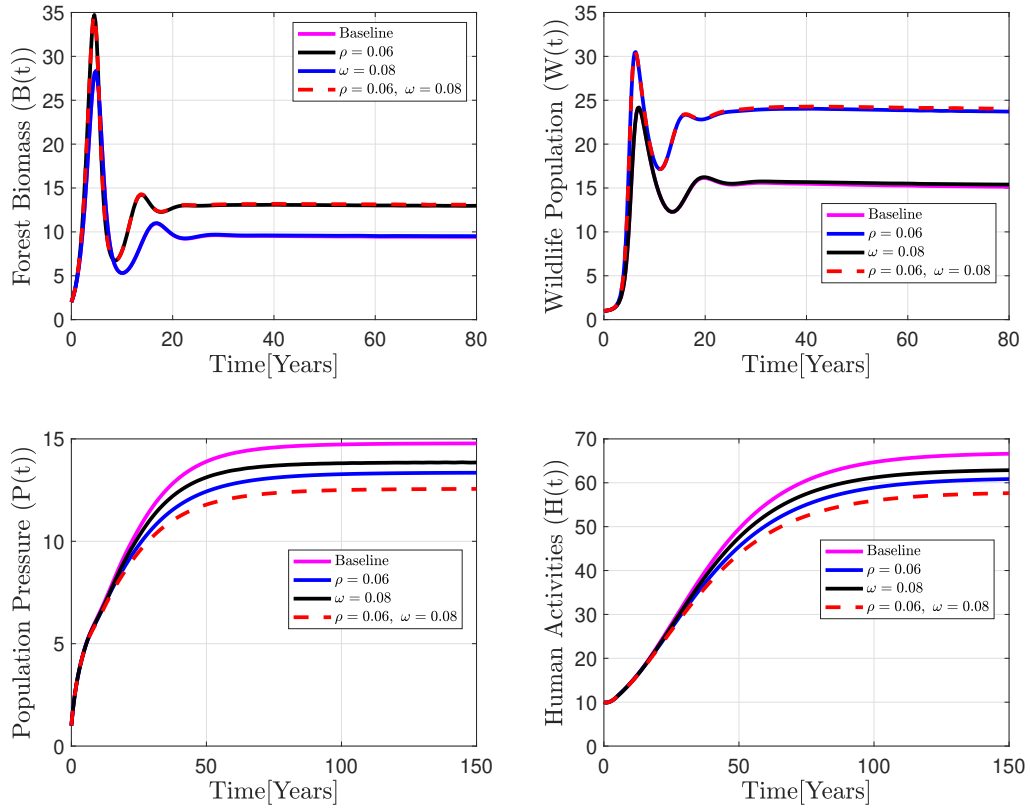


**Figure 52:** Variation of forest biomass ( $B$ ), wildlife population ( $W$ ), population pressure ( $P$ ) and human activities ( $H$ ) with time ( $t$ ) for different values of  $\rho$  and  $\omega$

This phenomenon serves as a sobering reminder that, despite the noble intentions behind these measures their excessive utilisation can lead to system destabilisation. Hence, it is crucial to exercise caution and employ these measures at a limited level to maintain the system's stability and avoid adverse consequences.

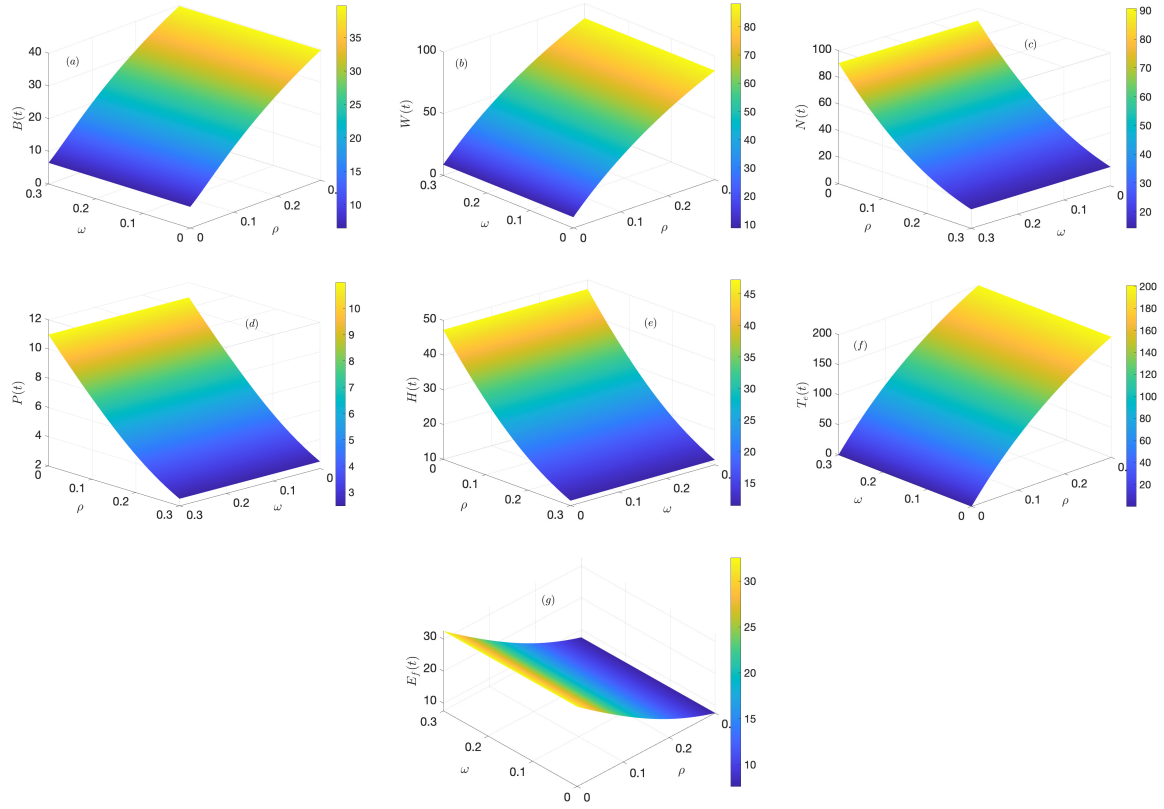


**Figure 53:** Evolution of (a) Forest biomass, (b) Wildlife population in forest habitats, (c) Human population, (d) Population pressure, (e) Human activities, (f) Technological efforts, and (g) Economic efforts for different values of  $\rho$  and  $\omega$ . The remaining parameter values were kept fixed as in Table 13

## CHAPTER FIVE

### CONCLUSION AND RECOMMENDATIONS

#### 5.1 Conclusion

The depletion of forest biomass and the decline of forest-dependent wildlife population are pressing issues that have significant ecological and societal implications. Human activities such as agriculture, livestock keeping, logging that lead to deforestation, land use changes and fragmentation have resulted in the loss of forest biomass and declining populations of wildlife species that rely on forests. In this study, depletion and conservation of forest biomass and forest-dependent wildlife population due to human population and associated activities is studied. The general objective of the study was to develop and analyse models in crisp and fuzzy environment to study the impact of anthropogenic activities on forest biomass and forest-dependent wildlife population with management measures. The study was guided by the following specific objectives: (a) Formulate univariate time series model for forest land loss forecast in Tanzania; (b) Develop and analyse a mathematical model to describe the impact of anthropogenic activities on forest biomass and forest-dependent wildlife population; (c) Examine how the uncertain parameters represented by fuzzy numbers affect the behaviour of the model that simulates forest biomass and forest-dependent wildlife population depletion due to anthropogenic activities; (d) Evaluate the effects of economic measures and technological efforts on conservation of forest biomass and forest-dependent wildlife population.

The study considered Tanzania where the local population heavily depends on forest resources for their livelihoods, including timber, non-timber forest products, and fuelwood. This region also supports a rich biodiversity of wildlife species, where the forest provides essential habitats and resources for their survival. In particular, the study emphasized the critical role of forest biomass in sustaining the populations of forest-dependent wildlife species, which are entirely reliant on the forest for their survival. Furthermore, the study also acknowledged the existence of various human activities carried out in the area, such as agricultural expansion, and hunting, which may have a significant impact on the forest ecosystem and the wildlife populations that depend on it.

The study presents four different models with the aim of understanding the impact of human activities on forest biomass and wildlife populations. Each model is designed to achieve specific

objectives, such as predicting the forest loss land in Tanzania using univariate time series data, assessing the effect of forest biomass depletion on the survival of forest-dependent wildlife populations, evaluating the role of economic and technological factors in resource conservation, and examining the impact of fuzzy parameters on model dynamics. The following conclusion are drawn from the findings:

- (i) Using Tanzania as case study, the formulated univariate time series model for forecasting forest land loss in Tanzania using the Box–Jenkins methodology predicts an increase in the loss of forest land in Tanzania. The model accurately predicts the next six observations with a 96.2% accuracy rate ( $MAPE = 0.0377$ ), which indicates the reliability and effectiveness of the model. This forecast suggests that if no appropriate measures are taken to curb deforestation, the loss of forest land in Tanzania will continue to increase. It highlights the urgent need for sustainable forest management practices and effective conservation policies to prevent further loss of valuable forest resources in Tanzania.
- (ii) The depletion of forest biomass due to anthropogenic activities has severe implications for the survival of wildlife species and the ecological balance of forest ecosystems. The findings of this study suggests that achieving ecological balance requires the intrinsic growth rate of forest biomass to be higher than the rate of its utilisation. Similarly, the intrinsic growth rate of wildlife populations that depend on forests should be higher than the rate of depletion, and the intrinsic growth rate of the human population should be significantly lower than the rate of decrease. Achieving these goals will require a combination of measures, including effective conservation strategies, sustainable forest management practices, and policies to address population growth and associated activities. Such measures should involve interdisciplinary collaborations and stakeholder engagement to ensure the sustainable use and conservation of forest ecosystems for future generations.
- (iii) The use of fuzzy parameters in modelling the depletion of forest biomass and forest-dependent wildlife population provides a more trustworthy representation of their dynamics. Fuzzy parameters enables the model to incorporate uncertainties and variations in factors such as climate, geography, and human activities, which may influence the dynamics of the system. This provides a more reliable representation of the system's behaviour, thus enhancing decision-making processes.

- (iv) The findings reveal that economic measures and technological efforts are potential approaches to conserve forest biomass and forest-dependent wildlife population. However, it is important to note that these efforts should be approached with caution and must be guided by a comprehensive understanding of the complex dynamics of forest ecosystems otherwise the system will be destabilised. Such understanding requires interdisciplinary research and collaboration between various stakeholders, including local communities, policymakers, economists, technologists, and conservation practitioners. The involvement of these different groups can promote the sustainable use and conservation of forest ecosystems, while also addressing the socio-economic needs of the local communities who rely on these resources for their livelihoods. Therefore, the need for a holistic approach to forest conservation is emphasised, which considers the uncertainties and complexities of natural systems, and involves the active participation of different stakeholders.

## **5.2 Recommendations**

Based on the conclusions drawn from the study, the following recommendations are made:

### **(i) For stakeholders**

To ensure the sustainable use of forest resources, stakeholders may prioritize immediate actions alongside advocating for policy changes and improvements. In the short term, promoting sustainable forest management practices is vital. It is recommended that industries and communities dependent on the forest to adopt measures like selective logging, reforestation, and biodiversity conservation. This can be achieved through awareness campaigns and capacity-building programs. On the other hand, financial incentives play a crucial role in driving sustainable practices. It is recommended that tax breaks or subsidies be provided to companies and individuals engaged in responsible forest management and conservation efforts. Moreover, investing in ecotourism initiatives can diversify income sources for local communities, thereby reducing their reliance on resource-intensive activities that harm the forest. Similarly, collaboration with local communities and indigenous groups is essential in protecting forest ecosystems. Acknowledging and respecting their traditional knowledge and practices will foster a sense of ownership and stewardship over the forests they inhabit.

However, to achieve lasting and widespread impact, stakeholders must advocate for pol-

icy changes and improvements. First and foremost, reforming royalty rates can provide a powerful incentive for sustainable practices. Adjusting royalty rates to discourage non-sustainable resource extraction while encouraging responsible management will drive positive change. Tax policies should also be leveraged to reward environmentally responsible companies with lower tax rates while imposing penalties on those contributing to deforestation or environmental degradation. Such measures will send a clear signal that sustainable practices are economically advantageous.

Moreover, the integration of remote sensing and geospatial analysis technologies in monitoring forest biomass, wildlife populations, and deforestation is critical. Advocating for government support and funding in advancing these technologies will enhance monitoring capabilities and decision-making processes. Establishing collaborative governance platforms involving government agencies, NGOs, local communities, and private sector representatives is another crucial step. This collective approach will ensure the co-development and implementation of conservation policies and initiatives, leading to more effective outcomes.

Finally, strengthening the enforcement of existing environmental laws and regulations is paramount. Combating illegal logging, poaching, and other harmful activities through stringent law enforcement will deter destructive practices and uphold the integrity of conservation efforts. By combining immediate actions and advocating for policy changes, stakeholders can make substantial progress towards sustainable forest management and conservation. These efforts are fundamental to preserving biodiversity, ecosystem services, and the well-being of present and future generations.

## **(ii) For future studies**

The dynamical system model outcomes are bounded from the qualitative view points into the role of impreciseness depending on the hypothetical set of parameter values. Thus, experimental investigations are essential to enrich in-depth understanding and to develop a more realistic model.

Similarly, the uncertainty and sensitivity analysis using the LHS/PRCC method provided valuable insights into the most significant parameters of the model. However, the approach assumes input model parameters and model outputs exhibit monotonicity and non-linear relationship which may not hold true always. Thus, to obtain a more compre-

hensive evaluation of model uncertainties, future research could combine the approach with other sensitivity indices such as variance-based methods (e-FAST and Sobol) that do not require the assumption of monotonicity. By combining these approaches, a better understanding of how specific parameter changes impact the output variability can be studied. Nevertheless, the issue of computational costs should be taken into account as most of the variance-based methods are prone to computational resources. Similarly, exploring optimal control strategies and assessing the cost-effectiveness of conservation measures could provide valuable insights for forest conservation efforts.

## REFERENCES

- [1] M. A. Ali, S. Means, H. Ho, and J. Heffernan, “Global sensitivity analysis of a single–cell HBV model for viral dynamics in the liver,” *Infectious Disease Modelling*, vol. 6, no. 2021, pp. 1220–1235, 2021. [Online]. Available: <https://doi.org/10.1016/j.idm.2021.10.003>
- [2] B. F. Alves de Oliveira, M. J. Bottino, P. Nobre, and C. A. Nobre, “Deforestation and climate change are projected to increase heat stress risk in the Brazilian Amazon,” *Communications Earth & Environment*, vol. 2, no. 1, pp. 1–15, 2021.
- [3] C. Anyigor and J. Afiukwa, “Application of MATLAB ordinary differential equation function solver (ode45) in modelling and simulation of batch reaction kinetics,” *American Journal of Scientific and Industrial Research*, vol. 4, no. 3, pp. 285–287, 2013.
- [4] R. Bellman, *Stability theory of differential equations*. Massachusetts, USA: Courier Corporation, 2008, vol. 3.
- [5] F. Berkes, C. Folke, and J. Colding, *Linking social and ecological systems: Management practices and social mechanisms for building resilience*. Ednburgh, UK: Cambridge University Press, 2000.
- [6] S. M. Blower and H. Dowlatabadi, “Sensitivity and uncertainty analysis of complex models of disease transmission: an HIV model, as an example,” *International Statistical Review/Revue Internationale de Statistique*, vol. 62, no. 2, pp. 229–243, 1994.
- [7] M. Bologna and G. Aquino, “Deforestation and world population sustainability: A quantitative analysis,” *Scientific Reports*, vol. 10, no. 1, pp. 1–9, 2020.
- [8] J. Börner, D. Schulz, S. Wunder, and A. Pfaff, “The effectiveness of forest conservation policies and programs,” *Annual Review of Resource Economics*, vol. 12, no. 1, pp. 45–64, 2020.
- [9] S. P. Brooks and G. O. Roberts, “Assessing convergence of Markov chain Monte Carlo algorithms,” *Statistics and Computing*, vol. 8, no. 4, pp. 319–335, 1998.
- [10] J. Buongiorno and J. K. Gilles, *Decision methods for forest resource management*. California, USA: Academic Press, 2003.

- [11] J. Carr, *Applications of Centre Manifold Theory*. New York, NY, USA: Springer Science & Business Media, 1981, vol. 35.
- [12] C. Castillo-Chavez and B. Song, “Dynamical models of tuberculosis and their applications,” *Mathematical Biosciences & Engineering*, vol. 1, no. 2, p. 361–404, 2004.
- [13] J. M. Chase, S. A. Blowes, T. M. Knight, K. Gerstner, and F. May, “Ecosystem decay exacerbates biodiversity loss with habitat loss,” *Nature*, vol. 584, no. 7820, pp. 238–243, 2020.
- [14] M. Chaudhary, J. Dhar, and O. P. Misra, “A mathematical model for the conservation of forestry biomass with an alternative resource for industrialization: A modified Leslie Gower interaction,” *Modeling Earth Systems and Environment*, vol. 1, no. 4, pp. 1–10, 2015.
- [15] F. Chen, “On a nonlinear nonautonomous predator–prey model with diffusion and distributed delay,” *Journal of Computational and Applied Mathematics*, vol. 180, no. 1, pp. 33–49, 2005.
- [16] F. Chien and S. Shateyi, “Volterra–Lyapunov stability analysis of the solutions of babesiosis disease model,” *Symmetry*, vol. 13, no. 7, pp. 1–14, 2021.
- [17] J. E. Cinner, J. Zamborain-Mason, E. Maire, A. S. Hoey, N. A. Graham, D. Mouillot, S. Villéger, S. Ferse, and S. Lockie, “Linking key human-environment theories to inform the sustainability of coral reefs,” *Current Biology*, vol. 32, no. 12, pp. 2610–2620, 2022.
- [18] J. Y. Dai, J. L. Stanford, and M. LeBlanc, “A multiple-testing procedure for high-dimensional mediation hypotheses,” *Journal of the American Statistical Association*, vol. 117, no. 537, pp. 198–213, 2022.
- [19] L. L. de Bruyn, T. M. P. Duong, P. Kristiansen, G. R. Marshall, and J. Wilkes, “The role of livelihood initiatives in reducing non-wood forest product reliance in protected areas of Southern Vietnam: Opportunities and challenges,” in *Non-Wood Forest Products of Asia: Knowledge, Conservation and Livelihood*, R. Manzoor, N. A. Khan, and M. Hossain, Eds. Cham, Switzerland: Springer Nature, 2022, vol. 25, pp. 221–251.

- [20] S. Devi and N. Gupta, “Comparative study of the effects of different growths of vegetation biomass on CO<sub>2</sub> in crisp and fuzzy environments,” *Natural Resource Modeling*, vol. 33, no. 2, p. e12263, 2020.
- [21] A. Dirk, “Asymptotic stability of nonautonomous systems by Liapunov’s direct method,” *Systems & Control Letters*, vol. 25, no. 4, pp. 273–280, 1995.
- [22] B. Dubey and A. S. Narayanan, “Modelling effects of industrialization, population and pollution on a renewable resource,” *Nonlinear Analysis: Real World Applications*, vol. 11, no. 4, pp. 2833–2848, 2010.
- [23] L. Emerton, *Mount Kenya: The economics of community conservation*. London, United Kingdom: International Institute for Environment and Development, 1999.
- [24] —, *Using economic incentives for biodiversity conservation*. Gland, Switzerland: IUCN–The World Conservation Union, 2000.
- [25] J. Fan, “A century of integrated research on the human-environment system in Chinese human geography,” *Progress in Human Geography*, vol. 46, no. 4, pp. 988–1008, 2022.
- [26] FAO, “Global Forest Resources Assessment 2020 Report: United Republic of Tanzania,” FAO, Rome, Italy, Tech. Rep., 2020.
- [27] FAO and UNEP, “The state of the world’s forests 2020: Forests, biodiversity and people,” FAO and UNEP, Rome, Italy, Tech. Rep., 2020.
- [28] I. Farahbakhsh, C. Bauch, and M. Anand, “Modelling coupled human–environment complexity for the future of the biosphere: strengths, gaps and promising directions,” *Philosophical Transactions of the Royal Society B*, vol. 377, no. 1857, pp. 1–13, 2022.
- [29] R. L. Farsiu, S. R. Boyd, and S. L. Taylor, “Challenges and opportunities in mathematical modeling of biological systems,” *Frontiers in Bioengineering and Biotechnology*, vol. 8, no. 1, pp. 1–18, 2021.
- [30] H. I. Freedman, *Deterministic mathematical models in population ecology*. New York, NY, USA: Marcel Dekker Incorporated, 1980, vol. 57.
- [31] H. I. Freedman and J. W. So, “Global stability and persistence of simple food chains,” *Mathematical Biosciences*, vol. 76, no. 1, pp. 69–86, 1985.

- [32] H. Freedman and S. G. Ruan, “Uniform persistence in functional differential equations,” *Journal of Differential Equations*, vol. 115, no. 1, pp. 173–192, 1995.
- [33] S. R. Freitas, T. J. Hawbaker, and J. P. Metzger, “Effects of roads, topography, and land use on forest cover dynamics in the Brazilian atlantic forest,” *Forest Ecology and Management*, vol. 259, no. 3, pp. 410–417, 2010.
- [34] C. Gallemore, I. Delabre, K. Jespersen, and T. Liu, “To see and be seen: Technological change and power in deforestation driving global value chains,” *Global Networks*, vol. 22, no. 4, pp. 615–630, 2022.
- [35] C. Garcia-Perez and S. Albeverio, “Challenges in mathematical modeling: Perspectives from applied mathematicians,” *Mathematical Biosciences and Engineering*, vol. 16, no. 1, pp. 1–25, 2019.
- [36] K. Gashu and O. Aminu, “Participatory forest management and smallholder farmers’ livelihoods improvement nexus in Northwest Ethiopia,” *Journal of Sustainable Forestry*, vol. 38, no. 5, pp. 413–426, 2019.
- [37] H. R. Gimblett, *Integrating geographic information systems and agent-based modeling techniques for simulating social and ecological processes*. New York, NY USA: Oxford University Press, 2002.
- [38] M. D. Goshu and M. F. Endalew, “Mathematical modeling on conservation of depleted forestry resources,” *Natural Resource Modeling*, vol. 35, no. 2, pp. 1–21, 2022.
- [39] A. Götz and G. Mayer, “Interval analysis: Theory and applications,” *Journal of Computational and Applied Mathematics*, vol. 121, no. 1-2, pp. 421–464, 2000.
- [40] A. Grzesiek, M. Mrozińska, P. Giri, S. Sundar, and A. Wyłomańska, “The covariation-based yule–walker method for multidimensional autoregressive time series with  $\alpha$ -stable distributed noise,” *International Journal of Advances in Engineering Sciences and Applied Mathematics*, vol. 13, no. 4, pp. 394–414, 2021.
- [41] Ö. A. Gümüş, “Global and local stability analysis in a nonlinear discrete–time population model,” *Advances in Difference Equations*, vol. 2014, no. 1, pp. 1–9, 2014.

- [42] H. Haario, E. Saksman, and J. Tamminen, “An Adaptive Metropolis Algorithm,” *Bernoulli*, vol. 40, no. 8, pp. 54–58, 2001.
- [43] S. Haddadan, Y. Zhuang, C. Cousins, and E. Upfal, “Fast doubly-adaptive MCMC to estimate the Gibbs partition function with weak mixing time bounds,” *Advances in Neural Information Processing Systems*, vol. 34, no. 2, pp. 25 760–25 772, 2021.
- [44] J. K. Hale, *Ordinary Differential Equations*. New York, NY, USA: Wiley–Intscience, 1969.
- [45] D. Heine, E. Hayde, and M. G. Faure, “Letting commodity tax rates vary with the sustainability of production,” in *Designing fiscal instruments for sustainable forests*. Washington DC, USA: The World Bank, 2021, pp. 145–171.
- [46] J. C. Helton and F. J. Davis, “Latin Hypercube Sampling and the propagation of uncertainty in analyses of complex systems,” *Reliability Engineering & System Safety*, vol. 81, no. 1, pp. 23–69, 2003.
- [47] Z. C. Hlaing, C. Kamiyama, and O. Saito, “Interaction between rural people’s basic needs and forest products: A case study of the Katha District of Myanmar,” *International Journal of Forestry Research*, vol. 2017, no. 2105012, pp. 1–18, 2017.
- [48] S. Jere, B. Kasense, and B. B. Bwalya, “Univariate time–series analysis of second–hand car importation in Zambia,” *Open Journal of Statistics*, vol. 7, no. 4, pp. 718–730, 2017.
- [49] K. Jyotsna and A. Tandon, “A mathematical model studying the survival of forest resource–dependent wildlife population in the presence of population pressure–induced mining activities,” *Natural Resource Modeling*, vol. 30, no. 4, pp. 1–25, 2017.
- [50] —, “A mathematical model to study the impact of mining activities and pollution on forest resources and wildlife population,” *Journal of Biological Systems*, vol. 25, no. 02, pp. 207–230, 2017.
- [51] J. R. Kideghesho, “Realities on deforestation in Tanzania: Trends, Drivers and the way Forward,” in *Precious Forests*, 1<sup>st</sup> ed., M. Zlatic, Ed. IntechOpen, 2015, ch. 2.
- [52] H. K. Kindsvater, N. K. Dulvy, C. Horswill, M. J. Juan-Jordá, M. Mangel, and J. Matthiopoulos, “Overcoming the data crisis in biodiversity conservation,” *Trends in Ecology & Evolution*, vol. 33, no. 9, pp. 676–688, 2018.

- [53] J. La–Salle and S. Lefschetz, *Stability by Liapunov’s direct method with applications*. New York, NY, USA: Academic Press, 1961.
- [54] J. C. Lagarias, J. A. Reeds, M. H. Wright, and P. E. Wright, “Convergence properties of the Nelder–Mead simplex method in low dimensions,” *SIAM Journal on optimization*, vol. 9, no. 1, pp. 112–147, 1998.
- [55] K. Lata and A. K. Misra, “Modeling the effect of economic efforts to control population pressure and conserve forestry resources,” *Nonlinear Analysis: Modelling and Control*, vol. 22, no. 4, pp. 473–488, 2017.
- [56] K. Lata, A. K. Misra, and J. B. Shukla, “Modeling the effect of deforestation caused by human population pressure on wildlife species,” *Nonlinear Analysis: Modelling and Control*, vol. 23, no. 3, pp. 303–320, 2018.
- [57] V. N. Lazhentsev, S. I. Chuzhmarova, and A. I. Chuzhmarov, “Tax administration in the forest management system and its influence on the budgets of northern territories,” *Ekonomicheskie i Sotsialnye Peremeny*, vol. 13, no. 1, pp. 125–143, 2020.
- [58] M. Y. Li, *An introduction to mathematical modeling of infectious diseases*. Springer Nature, 2018, vol. 2.
- [59] S. Liao and J. Wang, “Global stability analysis of epidemiological models based on Volterra–Lyapunov stable matrices,” *Chaos, Solitons & Fractals*, vol. 45, no. 7, pp. 966–977, 2012.
- [60] D. V. Lingner, A. V. Rodrigues, L. Z. Oliveira, A. L. De Gasper, and A. C. Vibrans, “Modelling changes in forest attributes driven by human activities at different spatial scales in the subtropical Atlantic forest,” *Biodiversity and Conservation*, vol. 29, no. 4, pp. 1283–1299, 2020.
- [61] J. Liu, T. Dietz, S. R. Carpenter, M. Alberti, C. Folke, E. Moran, A. N. Pell, P. Deadman, T. Kratz, J. Lubchenco *et al.*, “Complexity of coupled human and natural systems,” *Science*, vol. 317, no. 5844, pp. 1513–1516, 2007.
- [62] T. B. Luzyanina and G. A. Bocharov, “Markov chain Monte Carlo parameter estimation of the ODE compartmental cell growth model,” *Mathematical Biology and Bioinformatics*, vol. 13, no. 2, pp. 376–391, 2018.

- [63] S. Marino, I. B. Hogue, C. J. Ray, and D. E. Kirschner, “A methodology for performing global uncertainty and sensitivity analysis in systems biology,” *Journal of Theoretical Biology*, vol. 254, no. 1, pp. 178–196, 2008.
- [64] D. C. Marvin, L. P. Koh, A. J. Lynam, S. Wich, A. B. Davies, R. Krishnamurthy, E. Stokes, R. Starkey, and G. P. Asner, “Integrating technologies for scalable ecology and conservation,” *Global Ecology and Conservation*, vol. 7, pp. 262–275, 2016.
- [65] K. Matandiko, “In Tanzania, wild animals kill one person every 52 hours,” *The Citizens*, 17 Jan 2023, available at: <https://shorturl.at/jzABV> (Accessed: May 16th, 2023).
- [66] M. M. Mayengo, “The Volterra-Lyapunov matrix theory for global stability analysis of alcohol-related health risks model,” *Results in Physics*, pp. 1–9, 2022.
- [67] I. S. Mbalawata, S. Särkkä, M. Vihola, and H. Haario, “Adaptive metropolis algorithm using variational bayesian adaptive Kalman filter,” *Computational Statistics & Data Analysis*, vol. 83, pp. 101–115, 2015.
- [68] T. E. McKone and A. W. Deshpande, “Can fuzzy logic bring complex environmental problems into focus?” *Environmental Science & Technology*, vol. 39, no. 2, pp. 42A–47A, 2005.
- [69] J. A. McNeely, “Economic incentives for conserving biodiversity: Lessons for Africa,” *Ambio*, pp. 144–150, 1993.
- [70] S. Mekonen, “Coexistence between human and wildlife: the nature, causes and mitigations of human wildlife conflict around Ble mountains national park, Southeast Ethiopia,” *BMC Ecology*, vol. 20, no. 51, pp. 1–9, 2020.
- [71] C. Mersmann, “Links between trade and sustainable forest management: An overview,” *Unasylva*, vol. 55, no. 219, pp. 3–9, 2004.
- [72] F. L. M. Ming’ate and Z. Lonzi, “The role of forest resource and resource users’ boundaries in improving the livelihoods of communities adjacent to Arabuko–Sokoke forest reserve, Kenya,” *Management*, vol. 7, no. 1, pp. 54–58, 2022.
- [73] A. K. Misra and A. Jha, “Modeling the effect of population pressure on the dynamics of carbon dioxide gas,” *Journal of Applied Mathematics and Computing*, vol. 67, pp. 623–640, 10 2021.

- [74] A. K. Misra and K. Lata, “Depletion and conservation of forestry resources: A mathematical model,” *Differential Equations and Dynamical Systems*, vol. 23, no. 1, pp. 25–41, 2015.
- [75] A. K. Misra, K. Lata, and J. B. Shukla, “Effects of population and population pressure on forest resources and their conservation: a modeling study,” *Environment, Development and Sustainability*, vol. 16, no. 2, pp. 361–374, 2014.
- [76] A. Misra and K. Lata, “A mathematical model to achieve sustainable forest management,” *International Journal of Modeling, Simulation, and Scientific Computing*, vol. 6, no. 4, pp. 1–18, 2015.
- [77] A. Misra, K. Lata, and J. Shukla, “A mathematical model for the depletion of forestry resources due to population and population pressure augmented industrialization,” *International Journal of Modeling, Simulation, and Scientific Computing*, vol. 5, no. 1, pp. 1–16, 2014.
- [78] J. Mohamed, “Time series modeling and forecasting of somaliland consumer price index: A comparison of ARIMA and regression with ARIMA errors,” *American Journal of Theoretical and Applied Statistics*, vol. 9, no. 4, pp. 143–53, 2020.
- [79] M. Mohieldin and P. Caballero, “Protect, restore and promote sustainable use of terrestrial ecosystems, sustainably manage forests, combat desertification, and halt and reverse land degradation and halt biodiversity loss,” *UN Chronicle*, vol. 51, no. 4, pp. 34–35, 2015.
- [80] R. E. Moore, *Interval analysis*. New York, NY USA: Prentice-Hall Englewood Cliffs, 1966, vol. 4.
- [81] R. Morgan, “Linearization and stability analysis of nonlinear problems,” *Rose–Hulman Undergraduate Mathematics Journal*, vol. 16, no. 2, pp. 1–20, 2015.
- [82] D. Mukherjee and A. B. Roy, “Uniform persistence and global stability of two prey–predator pairs linked by competition,” *Mathematical Biosciences*, vol. 99, no. 1, pp. 31–45, 1990.
- [83] S. Mukhopadhyay and S. Sinha, “Applications of geospatial technology on the forest management in three districts of North Bengal, India,” in *Conservation, Management*

*and Monitoring of Forest Resources in India*, M. Sahana, G. Areendran, and K. Raj, Eds. Cham, Switzerland: Springer, 2022, ch. 12, pp. 305–324.

- [84] N. Myers, R. A. Mittermeier, C. G. Mittermeier, G. A. B. Da Fonseca, and J. Kent, “Biodiversity hotspots for conservation priorities,” *Nature*, vol. 403, no. 6772, pp. 853–858, 2000.
- [85] M. C. Oliveira and L. Siqueira, “Digitalization between environmental activism and counter-activism: The case of satellite data on deforestation in the Brazilian Amazon,” *Earth system governance*, vol. 12, p. 100135, 2022.
- [86] R. Owusu, J. N. Kimengsi, and F. Moyo, “Community-based forest landscape restoration (FLR): determinants and policy implications in Tanzania,” *Land Use Policy*, vol. 109, pp. 1–11, 2021.
- [87] D. Pal, G. S. Mahapatra, and G. P. Samanta, “Stability and bionomic analysis of fuzzy parameter based prey–predator harvesting model using UFM,” *Nonlinear Dynamics*, vol. 79, pp. 1939–1955, 2015.
- [88] P. Panja, “Fuzzy parameter based mathematical model on forest biomass,” *Biophysical Reviews and Letters*, vol. 13, no. 04, pp. 179–193, 2018.
- [89] —, “Is the forest biomass a key regulator of global warming? a mathematical modelling study,” *Geology, Ecology, and Landscapes*, vol. 6, no. 1, pp. 66–74, 2022.
- [90] P. Panja, S. K. Mondal, and J. Chattopadhyay, “Dynamical study in fuzzy threshold dynamics of a cholera epidemic model,” *Fuzzy Information and Engineering*, vol. 9, no. 3, pp. 381–401, 2017.
- [91] R. Pathak, M. Agarwal, Y. Takeuchi, and J. B. Shukla, “Modeling the depletion of forest resources due to population and industrialization and its effect on survival of wildlife species,” *International Journal of Mathematical Modelling and Analysis of Complex Systems (IJMMACS): Nature and Society*, vol. 3, no. 1, pp. 37–65, 2017.
- [92] R. Pathak, V. Verma, and M. Agarwal, “Impact of human activities on forest resources and wildlife population,” *Computational Ecology and Software*, vol. 11, no. 2, pp. 83–99, 2021.

- [93] K. Pho, S. Ly, S. Ly, and T. M. Lukusa, “Comparison among Akaike Information Criterion, Bayesian Information Criterion and Vuong’s test in model selection: A case study of violated speed regulation in Taiwan,” *Journal of Advanced Engineering and Computation*, vol. 3, no. 1, pp. 293–303, 2019.
- [94] M. L. Puri and D. A. Ralescu, “Differentials of fuzzy functions,” *Journal of Mathematical Analysis and Applications*, vol. 91, no. 2, pp. 552–558, 1983.
- [95] E. Rametsteiner and A. Whiteman, “State of the world’s forests: Enhancing the socio-economic benefits from forests,” FAO, Rome, Italy, Tech. Rep., 2014.
- [96] F. A. Razali and N. F. Haron, “Forecasting data using Box-Jenkins procedure: A case study for unemployed people in malaysia,” *Gading Journal of Science and Technology*, vol. 6, no. 1, pp. 31–39, 2023.
- [97] R. Redheffer, “Volterra multipliers I,” *SIAM Journal on Algebraic Discrete Methods*, vol. 6, no. 4, pp. 592–611, 1985.
- [98] —, “Volterra multipliers II,” *SIAM Journal on Algebraic Discrete Methods*, vol. 6, no. 4, pp. 612–623, 1985.
- [99] M. Renardy, C. Hult, S. Evans, J. J. Linderman, and D. E. Kirschner, “Global sensitivity analysis of biological multiscale models,” *Current Opinion in Biomedical Engineering*, vol. 11, pp. 109–116, 2019.
- [100] F. Rinaldi, “Global stability results for epidemic models with latent period,” *Mathematical Medicine and Biology: A Journal of the IMA*, vol. 7, no. 2, pp. 69–75, 1990.
- [101] H. Ritchie and M. Roser, “Forests and deforestation,” *Our World in Data*, vol. 10, no. 1, pp. 4–10, 2021, <https://ourworldindata.org/forests-and-deforestation>.
- [102] S. Seabold and J. Perktold, “Statsmodels: Econometric and statistical modeling with python,” in *Proceedings of the 9th Python in Science Conference*, vol. 57, no. 61. Austin, TX, 2010, pp. 92–96.
- [103] G. H. Shoddo, “The contribution of Gudo forest conservation culture is key to biodiversity conservation the case of Sheka Zone, southwest Ethiopia,” *Land Use Policy*, vol. 113, p. 105872, 2022.

- [104] K. Shono and Ö. Jonsson, “Global progress towards sustainable forest management: bright spots and challenges,” *International Forestry Review*, vol. 24, no. 1, pp. 85–97, 2022.
- [105] J. B. Shukla, H. I. Freedman, V. M. Pal, O. P. Misra, M. Agarwal, and A. Shukla, “Degradation and subsequent regeneration of a forestry resource: a mathematical model,” *Ecological Modelling*, vol. 44, no. 3-4, pp. 219–229, 1989.
- [106] J. Shukla, L. Kusum, and A. Misra, “Modeling the depletion of a renewable resource by population and industrialization: Effect of technology on its conservation,” *Natural Resource Modeling*, vol. 24, no. 2, pp. 242–267, 2011.
- [107] J. Shukla, R. Pathak, M. Agarwal, and Y. Takeuchi, “Modeling the depletion of forest resources by population and industrialization and their conservation by green belts plantation,” *International Journal of Mathematical Modelling and Analysis of Complex Systems (IJMMACS): Nature and Society*, vol. 3, no. 1, pp. 7–36, 2017.
- [108] S. K. Sinha, J. Pal, and K. Jyotsna, “A dynamical study on the adverse effects of industrial activities in the forest and wildlife region through modelling,” *Modeling Earth Systems and Environment*, pp. 1–13, 2022.
- [109] C. Skarbek, “A review of endemic species in the Eastern Arc Afromontane Region: Importance, inferences, and conservation,” *Macalester Reviews in Biogeography*, vol. 1, no. 1, p. 3, 2008.
- [110] A. Stuart and A. R. Humphries, *Dynamical systems and numerical analysis*. Edniburgh, United Kingdom: Cambridge University Press, 1998, vol. 2.
- [111] S. Sundar, N. Swaroop, and R. Naresh, “Modeling the effect of population and population augmented industrialization on forestry resources,” *European Journal of Engineering and Technology Research*, vol. 2, no. 1, pp. 65–70, 2017.
- [112] W. D. Sunderlin, A. Angelsen, B. Belcher, P. Burgers, R. Nasi, L. Santoso, and S. Wunder, “Livelihoods, forests, and conservation in developing countries: an overview,” *World Development*, vol. 33, no. 9, pp. 1383–1402, 2005.
- [113] B. Surya, S. Syafri, H. Sahban, and H. H. Sakti, “Natural resource conservation based on community economic empowerment: Perspectives on watershed management and slum

- settlements in Makassar City, South Sulawesi, Indonesia,” *Land*, vol. 9, no. 4, pp. 1–22, 2020.
- [114] M. K. Swamy, *Plant-derived Bioactives*. Bengaluru, India: Springer, 2020.
- [115] L. Tacconi, “Indonesia’s NDC bodes ill for the Paris Agreement,” *Nature Climate Change*, vol. 8, no. 10, pp. 842–842, 2018.
- [116] UN, *World population prospects 2022: Summary of results*. UN, 2022.
- [117] G. I. Valderrama-Bahamóndez and H. Fröhlich, “MCMC techniques for parameter estimation of ODE based models in systems Biology,” *Frontiers in Applied Mathematics and Statistics*, vol. 5, p. 55, 2019.
- [118] D. Wadhwa, “The number of extremely poor people continues to rise in Sub-Saharan Africa,” 2018. [Online]. Available: <https://blogs.worldbank.org/opendata/number-extremely-poor-people-continues-rise-sub-saharan-africa>
- [119] D. P. Warwick, “Culture and the management of family planning programs,” *Medical Anthropology*, pp. 359–376, 2023.
- [120] R. Watson, I. Baste, A. Larigauderie, P. Leadley, U. Pascual, B. Baptiste, S. Demissew, L. Dziba, G. Erpul, A. Fazel *et al.*, “Summary for policymakers of the global assessment report on biodiversity and ecosystem services of the Intergovernmental Science–Policy Platform on Biodiversity and Ecosystem Services,” *IPBES Secretariat: Bonn, Germany*, pp. 22–47, 2019.
- [121] U. A. Yakubu and M. P. A. Saputra, “Time series model analysis using Autocorrelation Function (ACF) and Partial Autocorrelation Function (PACF) for E–wallet transactions during a pandemic,” *International Journal of Global Operations Research*, vol. 3, no. 3, pp. 80–85, 2022.
- [122] Y. Yang, H. Li, L. Cheng, and Y. Ning, “Effect of land property rights on forest resources in Southern China,” *Land*, vol. 10, no. 4, pp. 1–15, 2021.
- [123] L. A. Zadeh, “Fuzzy sets,” *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

## APPENDICES

### Appendix 1: Python code for Time Series Analysis

```
1  #!/usr/bin/env python
2  # coding: utf-8
3
4  # In[2]:
5
6
7  import pandas as pd
8  import matplotlib.pyplot as plt
9  plt.rcParams.update({'figure.figsize':(5,3), 'figure.dpi':120})
10 df = pd.read_csv('Forest_Loss.csv')
11 df.info()
12 fig=df.plot()
13 plt.savefig('ForestLoss.png')
14
15
16 # In[3]:
17
18
19
20 #A Lag plot is a scatter plot of a time series against a lag of itself.
21 #It is normally used to check for autocorrelation. If there is any pattern
22 #existing in the series like the one you see below, the series is
    autocorrelated.
23 #If there is no such pattern,
24 #the series is likely to be random white noise.
25 from pandas.plotting import lag_plot
26 plt.rcParams.update({'ytick.left' : False, 'axes.titlepad':10})
27 fig, axes = plt.subplots(2, 2, figsize=(8,8), sharex=True, sharey=True, dpi
    =100)
28 for i, ax in enumerate(axes.flatten()[0:4]):
29     lag_plot(df, lag=i+1, ax=ax, c='firebrick')
30     ax.set_title('Lag ' + str(i+1))
31
32 fig.suptitle('Forest Loss in Ha', y=1)
33 plt.show()
34
35
36 # In[4]:
37
38
39 #msk = (df.index < len(df)-4)
40 #df_train = df[msk].copy()
41 #df_test = df[~msk].copy()
42 #df_test
43
44
45 # In[5]:
46
47
48 df_train = df[:25]
49 df_test = df[25:]
```

```

50 df_test
51
52
53 # In[6]:
54
55
56 from statsmodels.tsa.stattools import adfuller
57 adf_test = adfuller(df_train)
58 print('ADF Statistic: %f' % adf_test[0])
59 print(f'p-value: {adf_test[1]}')
60
61
62 # In[7]:
63
64
65 import numpy as np, pandas as pd
66 from statsmodels.graphics.tsaplots import plot_acf, plot_pacf
67 import matplotlib.pyplot as plt
68 plt.rcParams.update({'figure.figsize':(9,9), 'figure.dpi':120})
69 # Original Series
70 fig, axes = plt.subplots(3, 2, sharex=True)
71 axes[0, 0].plot(df_train); axes[0, 0].set_title('Original Series')
72 plot_acf(df_train, lags =24, ax=axes[0, 1])
73
74 # 1st Differencing
75 axes[1, 0].plot(df_train.diff()); axes[1, 0].set_title('1st Order
    Differencing')
76 plot_acf(df_train.diff().dropna(),lags =23, ax=axes[1, 1])
77
78 # 2nd Differencing
79 axes[2, 0].plot(df_train.diff().diff()); axes[2, 0].set_title('2nd Order
    Differencing')
80 plot_acf(df_train.diff().diff().dropna(),lags=22, ax=axes[2, 1])
81
82 plt.show()
83 plt.savefig('Difference.png')
84
85
86 # In[8]:
87
88
89 # PACF plot of 1st differenced series
90 plt.rcParams.update({'figure.figsize':(9,3), 'figure.dpi':120})
91 fig = plt.figure(figsize=(8, 4))
92 ax = fig.add_subplot(121)
93 ax2 = fig.add_subplot(122)
94 ax.plot(df_train.diff().diff()); ax.set_title('2st Differencing')
95 #ax.set(ylim=(0,5))
96 ax2.set(ylim = (0,5))
97 fig.tight_layout(pad=4.0)
98 plot_pacf(df_train.diff().dropna(),method = 'ywm',lags = 10, ax=ax2)
99 plt.savefig(sample.png)
100
101
102 # In[ ]:
103
104
105

```

```

106
107
108 # In[9]:
109
110
111 from statsmodels.tsa.stattools import adfuller
112 adf_test = adfuller(df_train)
113 print('ADF Statistic: %f' % adf_test[0])
114 print(f'p-value: {adf_test[1]}')
115
116
117 # In[10]:
118
119
120 plt.rcParams.update({'figure.figsize':(4,3), 'figure.dpi':120})
121 df_train_diff = df_train.diff().dropna()
122 df_train_diff.plot()
123 plt.show()
124
125
126 # In[11]:
127
128
129 plt.rcParams.update({'figure.figsize':(4,3), 'figure.dpi':120})
130 acf_diff = plot_acf(df_train_diff,lags=15)
131 apcf_diff = plot_pacf(df_train_diff,lags=10,method='ywm')
132
133
134 # In[12]:
135
136
137 adf_test = adfuller(df_train_diff)
138 print(f'p-value: {adf_test[1]}')
139
140
141 # In[13]:
142
143
144 from statsmodels.tsa.arima.model import ARIMA
145 model = ARIMA(df_train, order=(1,2,0))
146 model_fit = model.fit()
147 print(model_fit.summary())
148
149
150 # In[14]:
151
152
153 print(model_fit.summary().as_latex())
154
155
156 # In[15]:
157
158
159 import matplotlib.pyplot as plt
160 plt.rcParams.update({'figure.figsize':(7,3), 'figure.dpi':120})
161 residuals = model_fit.resid[1:]
162 fig, ax = plt.subplots(1,2)
163 residuals.plot(title='Residuals', ax=ax[0])

```

```

164 residuals.plot(title='Density', kind='kde', ax=ax[1])
165 fig.tight_layout(pad=1.0)
166 ax[1].set_ylabel('')
167 plt.savefig('residual.png')
168
169
170 # In[16]:
171
172
173 import statsmodels.api as sm
174 sm.stats.acorr_ljungbox(residuals, lags=[10], return_df=True)
175
176
177 # In[17]:
178
179
180 # Actual vs Fitted
181 import statsmodels.api as sm
182 from statsmodels.graphics.tsaplots import plot_predict
183 plt.rcParams.update({'figure.figsize':(5,3), 'figure.dpi':120})
184 import matplotlib.pyplot as plt
185 plot_predict(model_fit,dynamic=False)
186 plt.show()
187
188
189 # In[ ]:
190
191
192
193
194
195 # In[18]:
196
197
198 import pmdarima as pm
199 from statsmodels.tsa.arima_model import ARIMA
200 auto_arima = pm.auto_arima(df_train, start_p=1, start_q=1,test='adf',
201                           stepwise=True, seasonal=False,trace = True)
202
203
204 # In[19]:
205
206
207 print(auto_arima )
208
209
210 # In[20]:
211
212
213 forecast_test = model_fit.forecast(len(df_test))
214
215 df['forecast_manual'] = [None]*len(df_train) + list(forecast_test)
216
217 df.plot()
218
219
220 # In[34]:
221

```

```

222 forecast_test_auto = auto_arima.predict(len(df_test))
223 df['forecast_auto'] = [None]*len(df_train) + list(forecast_test_auto)
224 df.plot(xlim = (23,35),ylim = (1500,3010))
225 plt.savefig('test.pdf')
226 #ax.set(ylim=(0,5))
227 #plt.set(ylim = (0,5))
228
229
230
231 # In[446]:
232
233
234 from sklearn.metrics import mean_absolute_error,
    mean_absolute_percentage_error, mean_squared_error
235
236 mae = mean_absolute_error(df_test, forecast_test)
237 mape = mean_absolute_percentage_error(df_test, forecast_test)
238 rmse = np.sqrt(mean_squared_error(df_test, forecast_test))
239
240 print(f'mae - manual: {mae}')
241 print(f'mape - manual: {mape}')
242 print(f'rmse - manual: {rmse}')
243
244
245 # In[447]:
246
247
248 mae = mean_absolute_error(df_test, forecast_test_auto)
249 mape = mean_absolute_percentage_error(df_test, forecast_test_auto)
250 rmse = np.sqrt(mean_squared_error(df_test, forecast_test_auto))
251
252 print(f'mae - auto: {mae}')
253 print(f'mape - auto: {mape}')
254 print(f'rmse - auto: {rmse}')
255
256
257 # In[448]:
258
259
260 import matplotlib.pyplot as plt
261 import statsmodels.tsa.api as tsa
262 model_fit.plot_diagnostics(figsize=(8, 9))
263 plt.savefig('dig.pdf')
264
265
266 # In[449]:
267
268
269 print(auto_arima.summary())
270
271
272 # In[450]:
273
274
275 print(auto_arima.summary().as_latex())
276
277
278 # In[451]:

```

```

279
280
281 # Build Model
282 from statsmodels.tsa.arima.model import ARIMA
283 model = ARIMA(df_train, order=(1, 2, 0))
284 fitted2 = model.fit()
285 print(fitted.summary())
286
287
288 # In[453]:
289
290
291 # Forecast
292 fc= fitted2.forecast(6, alpha=0.05)
293 conf = fitted2.get_forecast(6).conf_int()
294 se = fitted2.bse
295 print(conf.to_latex())
296
297
298 # In[374]:
299
300
301 # Make as pandas series
302 fc_series = pd.Series(fc, index=df_test.index)
303 lower_series = pd.Series(conf['lower Forest Loss in Ha'], index=df_test.
    index)
304 upper_series = pd.Series(conf['upper Forest Loss in Ha'], index=df_test.
    index)
305
306
307 # In[419]:
308
309
310 # Plot
311 plt.figure(figsize=(5,4), dpi=120)
312 plt.plot(df_train, label='training series')
313 plt.plot(df_test, label='actual series')
314 plt.plot(fc_series, label='predicted series')
315 plt.fill_between(lower_series.index, lower_series, upper_series,
316                 color='k', alpha=.15)
317 plt.title('Forecast vs Actuals')
318 plt.legend(loc='upper left', fontsize=8)
319 plt.savefig('forecast')
320
321
322 # In[425]:
323
324
325 import matplotlib.pyplot as plt
326 plt.rcParams.update({'figure.figsize':(7,3), 'figure.dpi':120})
327 residuals = fitted2.resid[1:]
328 fig, ax = plt.subplots(1,2)
329 residuals.plot(title='Residuals', ax=ax[0])
330 residuals.plot(title='Density', kind='kde', ax=ax[1])
331 fig.tight_layout(pad=1.0)
332 ax[1].set_ylabel('')
333 plt.savefig('res2.png')
334

```

```

335
336 # In[15]:
337
338
339 import pandas as pd
340 import matplotlib.pyplot as plt
341 plt.rcParams.update({'figure.figsize':(5,3), 'figure.dpi':120})
342 df1 = pd.read_csv('Forest_Loss.csv')
343
344
345 # In[17]:
346
347
348 # Build Model
349 from statsmodels.tsa.arima.model import ARIMA
350 model1 = ARIMA(df1, order=(1, 2, 0))
351 fit = model.fit()
352 print(fit.summary())
353
354
355 # In[70]:
356
357
358 # Forecast
359 fc= fit.forecast(10, alpha=0.05)
360 coitn = fit.get_forecast(10).conf_int()
361 index_of_fc = np.arange(len(df1), len(df1)+10)
362 coitn
363
364
365 # In[71]:
366
367
368 # make series for plotting purpose
369 fc_series = pd.Series(fc, index=index_of_fc)
370 lower_series = pd.Series(coitn['lower Forest Loss in Ha'], index=
    index_of_fc)
371 upper_series = pd.Series(coitn['upper Forest Loss in Ha'], index=
    index_of_fc)
372
373
374 # In[72]:
375
376
377 # Plot
378 plt.rcParams.update({'figure.figsize':(4,4), 'figure.dpi':120})
379 plt.plot(df1)
380 plt.plot(fc_series, color='darkgreen')
381 plt.fill_between(lower_series.index,
382                 lower_series,
383                 upper_series,
384                 color='k', alpha=.15)
385
386 plt.title(Final Forecast of WWW Usage)
387 plt.gca().set_yscale('linear')
388 #plt.ylim(0, 7000)
389 plt.show()
390

```

```
391
392 # In[ ]:
393
394
395
396
397
398 # In[ ]:
```

## Appendix 2: MATLAB code for Bifurcation Analysis

```

1 %Ibrahim Fanuel, ibrahimf@nm-nm-aist.ac.tz
2 clear; clc;
3 %%Parameter Declaration
4     L=100; M=100; alpha=0.05; %beta1=0.003;...
5     s=0.79;
6     lambda=0.05; sigma=0.001;K0=10;K1=2;...
7     theta=0.1; beta2=0.0004; gamma1=0.05; ...
8     Pi=0.02; gamma=0.004; eta=0.9; nu1=0.002; nu2=0.0001;
9 % Define the parameter range for the plot
10 param_range = 0:0.0001:0.01;
11 %Variable
12 % Define the ODE system
13 ode_sys = @(t,y,beta1) [s*y(1)*(1-y(1)/L)-alpha*y(2)*y(1)-beta1*
14     y(3)*y(1)-beta2*(y(1))^2*y(4);...
15     alpha*eta*y(1)*y(2)*(1-(y(2)/(K0+K1*y(1)))) - nu1*y(2)*y(3)
16     - nu2*y(2)*y(4);...
17     theta*y(3)*(1-y(3)/M) + lambda*beta1*y(1)*y(3) - sigma*y(3)*
18     y(2);...
19     gamma*y(3) + Pi*beta2*(y(1))^2*y(4) - gamma1*y(4)];
20
21 % Set the options for ode45
22 options = odeset('RelTol', 1e-6, 'AbsTol', 1e-8);
23
24 % Initial conditions
25 y0 = [1; 2; 10; 1];
26 tspan = [0 200];
27 % Loop over the parameter range
28 for i = 1:length(param_range)
29     beta1 = param_range(i);
30     [t, y] = ode45(@(t,y) ode_sys(t,y,beta1), tspan, y0, options)
31     ;
32
33     % Store the state variable at a fixed time
34     state_var1(i) = y(end,1);
35     state_var2(i) = y(end,2);
36     state_var3(i) = y(end,3);
37     state_var4(i) = y(end,4);
38 end
39
40 % Plot the dynamics of the state variable with different values
41 % of the parameter
42 % Plot the first state variable with different values of the
43 % parameter
44 figure;
45 plot(param_range, state_var1, 'b-', LineWidth=3);
46 xlabel('$S_3$', 'FontSize', 20, Interpreter='latex');

```

```

41 ylabel('B(t)',FontSize=20,Interpreter='latex');
42 ax=gca; ax.FontSize = 20;
43 %title('Dynamics of the forest biomass with different Values of
    \beta_1 ',FontSize=20);
44 grid on;
45
46 % Plot the second state variable with different values of the
    parameter
47 figure;
48 plot(param_range, state_var2, 'b-',LineWidth=3);
49 xlabel('$S_3$',FontSize=20,Interpreter='latex');
50 ylabel('$W(t)$',FontSize=20,Interpreter='latex');
51 ax=gca; ax.FontSize = 20;
52 %xlim([0.0078 0.008]) %zoom
53 %title('Dynamics of the Second State Variable with Different
    Values of \mu');
54 grid on;
55
56 % Plot the first state variable with different values of the
    parameter
57 figure;
58 plot(param_range, state_var3, 'b-',LineWidth=3);
59 xlabel('$S_3$',FontSize=20,Interpreter='latex');
60 ylabel('$N(t)$',FontSize=20,Interpreter='latex');
61 ax=gca; ax.FontSize = 20;
62 %title('Dynamics of the First State Variable with Different
    Values of \mu');
63 grid on;
64
65 % Plot the second state variable with different values of the
    parameter
66 figure;
67 plot(param_range, state_var4, 'b-',LineWidth=3);
68 xlabel('$S_3$',FontSize=20,Interpreter='latex');
69 ylabel('H(t)',FontSize=20,Interpreter='latex');
70 ax=gca; ax.FontSize = 20;
71 %title('Dynamics of the Second State Variable with Different
    Values of \mu');
72 grid on;

```

## Review Article

# Modelling the Impact of Human Population and Its Associated Pressure on Forest Biomass and Forest-Dependent Wildlife Population

Ibrahim M. Fanuel<sup>1,2</sup> ,<sup>1,2</sup> Damian Kajunguri,<sup>3</sup> and Francis Moyo<sup>4</sup>

<sup>1</sup>*School of Computational and Communication Science and Engineering, Nelson Mandela African Institution of Science and Technology, Arusha, Tanzania*

<sup>2</sup>*Department of Mathematics & ICT, College of Business Education, Mwanza, Tanzania*

<sup>3</sup>*Department of Mathematics, Kabale University, Kabale, Uganda*

<sup>4</sup>*School of Life Science and Bio-Engineering, Nelson Mandela African Institution of Science and Technology, Arusha, Tanzania*

Correspondence should be addressed to Ibrahim M. Fanuel; [ibrahimf@nm-aist.ac.tz](mailto:ibrahimf@nm-aist.ac.tz)

Received 27 June 2022; Revised 19 December 2022; Accepted 26 December 2022; Published 12 January 2023

Academic Editor: Qiankun Song

Copyright © 2023 Ibrahim M. Fanuel et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Mathematical models have been widely used to explain the system originating from human-nature interaction, investigate the impacts of various components, and forecast system behaviour. This paper provides a profound reference to the current state of the art regarding the application of mathematical models to study the impact of human population and population pressure on forest biomass and forest-dependent wildlife. The review focused on two aspects, namely, model formulation and model analysis. In model formulation, the review revealed that socioeconomic status influences forest resource consumption patterns, thus, stratification of the human population based on economic status is a critical phenomenon in modelling human-nature interactions; however, this component has not been featured in the reviewed models. Regarding model analysis, in most of the reviewed work, single parameter approach was utilized to perform uncertainty quantification of the model parameter; this approach has been proven to be inadequate in measuring the uncertainty and sensitivity of the parameter. Thus, the use of correlation or variance based methods, which are multidimensional parameter space methods are of significant importance. Generally, despite the limitations of many assumptions in mathematical modelling, it is revealed that mathematical models demonstrate the ability to handle complex systems originating from interactions between humans and nature.

## 1. Introduction

Forests play a significant role in maintaining the ecological balance of the ecosphere [1]. Forest maintains microclimate and water cycle [2], prevents soil erosion, and sequesters carbon dioxide from the atmosphere [3], as well as the source of wood, energy, medicine, and fodder [4]. The World Bank estimates that about one billion people worldwide depend on the forest as a primary source of medicine [5]. Furthermore, forests provide habitat for 80%-90% of the terrestrial biodiversity [6]. The highest ratio of endemics per area of all biodiversity hotspots is found in the forests of the eastern arc of mountains in Tanzania and Kenya [7]; it occupies an

area of 2000 km<sup>2</sup> and offer habitat to about 121 endemic vertebrate species [5].

Despite the significant roles of the forest, the resources are under threat due to unrelenting pressure from the human population and its associated activities [8]. Human population growth has been shown to be significant around the world [9] putting pressure on forest resources [10]. The need for food and settlement to accommodate the rapidly growing human population is the major cause of deforestation and the destruction of wildlife habitats [11]. The challenge is exacerbated by poverty in developing countries such as Tanzania where the livelihoods of people highly depend on land and forest resources [12]. In particular,



# Mathematical model to study the impact of anthropogenic activities on forest biomass and forest-dependent wildlife population

Ibrahim M. Fanuel<sup>1,4</sup> · Silas Mirau<sup>1</sup> · Damian Kajunguri<sup>2</sup> · Francis Moyo<sup>3</sup>

Received: 15 April 2023 / Revised: 21 June 2023 / Accepted: 7 July 2023

© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

## Abstract

This paper proposes and analyses a nonlinear mathematical model to study the impact of anthropogenic activities on forest biomass and forest-dependent wildlife populations using a system of differential equations. It is assumed that the growth of forest biomass, forest-dependent wildlife populations, and the human population follow logistic equations. The effect of forest biomass depletion on the survival of forest-dependent wildlife populations is investigated by introducing a function that denotes the dependence on forest biomass. The system's behaviour near all ecologically acceptable equilibria is studied, and to confirm the analytical conclusions, a numerical simulation is performed. The model analysis shows that as forest biomass declines due to an increase in human population and its associated activities, the population of wildlife species also declines, and if no measures are taken, both forest biomass and the wildlife population may become extinct.

**Keywords** Mathematical model · Anthropogenic activities · Forest biomass · Forest-dependent wildlife

**Mathematics Subject Classification** 34A34 · 34D20 · 34D23 · 34D35

Silas Mirau, Damian Kajunguri and Francis Moyo contributed equally to this work.

Ibrahim M. Fanuel  
ibrahimf@nm-aist.ac.tz

Silas Mirau  
silas.mirau@nm-aist.ac.tz

Damian Kajunguri  
dkajung@gmail.com

Francis Moyo  
francis.moyo@nm-aist.ac.tz

<sup>1</sup> Department of Applied Mathematics and Computational Science, The Nelson Mandela African Institution of Science and Technology, Arusha, Tanzania

<sup>2</sup> Department of Mathematics, Kabale University, Kabale, Uganda

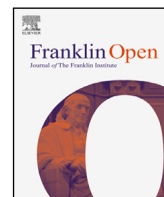
<sup>3</sup> Department of Biodiversity Conservation and Ecosystem Management, The Nelson Mandela African Institution of Science and Technology, Arusha, Tanzania

<sup>4</sup> Department of ICT and Mathematics, College of Business Education, Mwanza, Tanzania

## 1 Introduction

Forest degradation due to anthropogenic activities is the leading cause of biodiversity loss across the globe [1]. In developing countries such as Tanzania, forests are more vulnerable due to the rapid human population growth that depends hugely on forest resources for their livelihoods [2, 3]. The vast majority of terrestrial biodiversity, ranging from 80 to 90%, can be found within forests and wooded areas [3], to humans forests provide a wide range of resources, including food, wood, medicine, fresh water, and clean air. The ecosystem that supports the human population could be jeopardized if the forest goes extinct [4]. Despite the benefits that forests provide, they are threatened by a wide range of human activities [1]. These include unsustainable harvesting; human settlement encroachment; conversion of the forest land; tourism and recreational pressure; forest fires; as well as mining and fossil fuel exploitation [5].

Moreover, around 1.6 billion individuals depend entirely or partially on forest resources for their sustenance [2, 6]. It is not feasible to completely restrict people's access to and utilisation of these resources. For many years, forest conservation has emerged as a critical strategy for addressing the negative consequences of human–forest interactions



# Fuzzy modelling on the depletion of forest biomass and forest-dependent wildlife population

Ibrahim M. Fanuel<sup>a,c,\*</sup>, Silas Mirau<sup>a</sup>, Maranya Mayengo<sup>a</sup>, Francis Moyo<sup>b</sup>

<sup>a</sup> Department of Applied Mathematics and Computational Science, Nelson Mandela African Institution of Science and Technology, Arusha, Tanzania

<sup>b</sup> Department of Biodiversity Conservation and Ecosystem Management, Nelson Mandela African Institution of Science and Technology, Arusha, Tanzania

<sup>c</sup> Department of ICT and Mathematics, College of Business Education, Tanzania

## ARTICLE INFO

### MSC:

34A34

34D20

34D23

34D35

### Keywords:

Fuzzy modelling

Forest depletion

Triangular fuzzy number

Signed distance

## ABSTRACT

This paper presents a system of non-linear differential equations describing the depletion of forest biomass and forest-dependent wildlife population caused by human population and its associated activities. The model incorporates the imprecise nature of the parameters, which are treated as triangular fuzzy numbers to reflect the inherent uncertainty. We utilised  $\alpha$  cut to transform these imprecise parameters into intervals. Subsequently, employing the principles of interval mathematics, we effectively converted the related differential equation into a pair of distinct differential equations. By leveraging the signed distance of the fuzzy numbers, we further simplified the equations, resulting in a single differential equation, which led to the formulation of a defuzzified model. The existence of equilibrium points with their stability behaviour is presented. Furthermore, the existence of trans-critical bifurcation is analysed. Through numerical simulations, we observe significant differences between the solutions of system in crisp and fuzzy environments. These findings highlight the importance of using fuzzy models to accurately represent the dynamics of complex natural systems. Consequently, we conclude that fuzzy models provide a trustworthy representation of the dynamics of complex natural systems.

## 1. Introduction

Forests provide a variety of significant benefits, including ecological, economic, and social advantages [1]. Ecologically, they are crucial for preserving the world's biodiversity, providing habitat for an extensive range of plant and wildlife species [2]. To human, forests are the source of wood, energy, medicine, and fodder [3]. Furthermore, they provide vital ecosystem services such as carbon sequestration, soil conservation, and watershed protection. Forests also have cultural and spiritual significance for local communities and offer opportunities for recreation and tourism. Despite the benefits that forests provide, they are threatened by a wide range of human activities [1]. These include unsustainable harvesting; human settlement encroachment; conversion of the forest land; tourism and recreational pressure; forest fires; as well as mining and fossil fuel exploitation [4].

Studies show that, around 1.6 billion people rely totally or partially on forest resources for their livelihood [5], attempts to completely limit people's utilisation of the forest resources is not possible. To ensure sustainable forest utilisation, understanding of the dynamics of human-forest interaction is of utmost important. There is considerable number

of theoretical and experimental studies in the pursuit of understanding the interactions between human and forest. In the realm of theoretical studies which is the focus of this paper, mathematical models have become important tools for analysing the impacts of human population and its associated activities on forest and forest-dependent wildlife species [6].

In 1996, Shukla et al. [7] developed and analysed a non-linear mathematical model to study the effect of changing habitat on survival of wildlife species. According to the study, uncontrolled population pressure leads to a decline in forest biomass density, which in turn leads to a reduction in wildlife species density and, ultimately, extinction. Similar studies developed and analysed models to study the effects of forest depletion on the survival of wildlife species [8–10]. The findings were consistence with Shukla et al. [7], that uncontrolled human population and its associated activities have directly and indirect negative effects on the survival of wildlife species. Furthermore, studies identify parameters threshold for the systems to remain permanent. Recently, Sinha et al. [11] examined the negative impact of industrial activities on forests and wildlife. The results showed that the density of forest

\* Corresponding author at: Department of Applied Mathematics and Computational Science, Nelson Mandela African Institution of Science and Technology, Arusha, Tanzania.

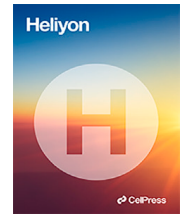
E-mail address: [ibrahimf@nm-aist.ac.tz](mailto:ibrahimf@nm-aist.ac.tz) (I.M. Fanuel).

<https://doi.org/10.1016/j.fraope.2023.100033>

Received 19 April 2023; Received in revised form 22 July 2023; Accepted 23 August 2023

Available online 23 August 2023

2773-1863/© 2023 The Author(s). Published by Elsevier Inc. on behalf of The Franklin Institute. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).



## Research article

# Conservation of forest biomass and forest-dependent wildlife population: Uncertainty quantification of the model parameters

Ibrahim M. Fanuel<sup>a,b,\*</sup>, Silas Mirau<sup>a</sup>, Damian Kajunguri<sup>c</sup>, Francis Moyo<sup>d</sup>

<sup>a</sup> Department of Applied Mathematics and Computational Science, Nelson Mandela African Institution of Science and Technology, Arusha, Tanzania

<sup>b</sup> Department of ICT and Mathematics, College of Business Education, Mwanza, Tanzania

<sup>c</sup> Department of Mathematics, Kabale University, Kabale, Uganda

<sup>d</sup> Department of Biodiversity Conservation and Ecosystem Management, Nelson Mandela African Institution of Science and Technology, Arusha, Tanzania

## ARTICLE INFO

## Keywords:

Uncertainty quantification  
Conservation  
Forest biomass  
Forest-dependent wildlife population  
Hypercube Latin sampling

## ABSTRACT

The ecosystem is confronted with numerous challenges as a consequence of the escalating human population and its corresponding activities. Among these challenges lies the degradation of forest biomass, which directly contributes to a reduction in forested areas and poses a significant threat to the survival of wildlife species through the intensification of intraspecific competition. In this paper, a non-linear mathematical model to study the conservation of forest and wildlife species that are reliant on forest ecosystem within the framework of human population dynamics and its related activities is developed and analysed. The study assessed the impacts of economic measures in the form of incentives on reducing population pressure on forest resources as well as the potential benefits of technological efforts to accelerate the rate of reforestation. Qualitative and quantitative analyses reveals that economic and technological factors have the potential to contribute to resource conservation efforts. However, these efforts can only be used to a limited extent, and contrary to that, the system will be destabilised. Sensitivity analysis identified the parameters pertaining to human population, human activities, economic measures, and technological efforts as the most influential factors in the model.

## 1. Introduction

The continuous depletion of forests is a pressing issue that requires attention and scientific investigation [1]. The rapid growth of human population and its associated activities are considered to be the main causes of its depletion [2]. The depletion of forest has a wide range of consequences such as environmental degradation and biodiversity loss [3]. Both human livelihoods and the survival of wildlife species depend on forests. For instance, forests are habitats for 80% – 90% of the terrestrial biodiversity [4], thus, wildlife species are in danger of extinction when forest degradation is uncontrolled. Further, about 1.6 billion people depend entirely or partially on forest resources for their livelihood, making it impossible to completely control access to and utilisation of forest resources [5]. In this situation, sustainable use of forest resources is the only way to ensure that resources are not in risk of extinction while human livelihoods are not jeopardised. To achieve that, information on how forests and human population interacts is very crucial. This information can be obtained in various ways, ranging from theoretical to experimental studies.

\* Corresponding author at: Department of Applied Mathematics and Computational Science, Nelson Mandela African Institution of Science and Technology, Arusha, Tanzania.

E-mail address: [ibrahimf@nm-aist.ac.tz](mailto:ibrahimf@nm-aist.ac.tz) (I.M. Fanuel).

<https://doi.org/10.1016/j.heliyon.2023.e16948>

Received 8 March 2023; Received in revised form 31 May 2023; Accepted 1 June 2023

2405-8440/© 2023 The Author(s). Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).



### Abstract

This study develops a mathematical model using the system of ODE's and Adaptive Metropolis-Hastings Markov Chain Monte Carlo (MCMC) to analyse the depletion of forest biomass and wildlife populations due to human activities. The model, based on a dynamic system of equations, provides insights into the significant factors affecting depletion, and the MCMC simulations offer detailed predictions on future trends.

### Introduction

In the face of escalating ecological challenges, understanding the intricate dynamics of forest and wildlife depletion has never been more crucial. This study introduces a comprehensive mathematical model designed to unravel the complex interplay between anthropogenic factors contributing to forest and wildlife depletion. Leveraging the robustness of Markov Chain Monte Carlo (MCMC) analysis, the approach offers a nuanced understanding of the current state of depletion but also provides predictive insights into future trends. This model aims to serve as a vital tool for policymakers, conservationists, and ecologists in strategising effective preservation and restoration efforts. Our analysis bridges the gap between theoretical ecology and practical conservation efforts, highlighting key areas of intervention and offering a data-driven foundation for sustainable environmental management.

### Methodology

Dynamical system is defined as

$$\dot{X} = g(X, \Lambda, t), \quad (1)$$

such that

$$X(0) = X_0, \quad X \in \mathbb{R}^n \quad \text{and} \quad \Lambda \in \mathbb{R}^m.$$

In this study, the Adaptive Metropolis-Hastings MCMC algorithm is used due to its ability to automatically tune the proposal distribution.

#### Algorithm 1 Adaptive Metropolis-Hastings

- 1: Generate the initial point  $\Lambda^{(0)}$  from the probability distribution  $p_{\Lambda}(\Lambda)$ . Set the non-adaptive period  $n_0$  and select the initial covariance matrix  $\Sigma_{\Lambda^{(0)}}$ .
- 2: **for**  $i = 1, 2, \dots$ , **do**
- 3: Sample a candidate point  $\Lambda^*$  from the proposal distribution  $q(\Lambda^* | \Lambda^{(i-1)})$ .
- 4: Compute the probability of accepting a proposed sample  $(\beta)$ 

$$\beta(\Lambda^{(i-1)}, \Lambda^*) = \min \left\{ 1, \frac{p(\Lambda^* | y_1, y_2, \dots, y_n) q(\Lambda^{(i-1)} | \Lambda^*)}{p(\Lambda^{(i-1)} | y_1, y_2, \dots, y_n) q(\Lambda^* | \Lambda^{(i-1)})} \right\}.$$
- 5: Establish acceptance/rejection rule: **set**

$$\Lambda^{(i)} = \begin{cases} \Lambda^* & \text{if } u < \beta(\Lambda^* | \Lambda^{(i-1)}), \\ \Lambda^{(i-1)} & \text{otherwise.} \end{cases} \quad \text{where } u \sim \mathcal{U}(0, 1),$$
- 6: Update the covariance matrix after every  $n_0^{th}$  iterations ( $i \geq n_0$ ) by using equation 2
$$\Sigma_i = \text{cov}(\Lambda^{(0)}, \Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}, \dots, \Lambda^{(i)}) + e I_m.$$
- 7:  $i = i + 1$ .

### Results & Discussion

The model to describe the depletion of forest biomass and forest dependent wildlife population depletion due to human activities is presented in equation 3. Where,  $B$  is forest biomass,  $W$  forest dependent wildlife population,  $N$  human population and  $H$  human activities.

$$\begin{cases} \dot{B} = sB \left(1 - \frac{B}{K}\right) - \beta_0 BW - \beta_1 BN - \beta_2 B^2 H, \\ \dot{W} = r(W)W \left(1 - \frac{W}{K(W)}\right) - v_1 W N - v_2 W H, \\ \dot{N} = \theta N \left(1 - \frac{N}{M}\right) + \lambda \beta_1 BN - \sigma NH, \\ \dot{H} = \gamma N - \gamma_1 H, \end{cases} \quad (3)$$

such that,

$$B(0) \geq 0, W(0) \geq 0, N(0) \geq 0, H(0) \geq 0, 0 \leq \lambda \leq 1.$$

On utilising Algorithm 1, the model parameters were identified as follows:

Table 1. Statistics of MCMC (Number of simulations = 100 000).

| Par.         | LS est. | Mean   | Median | CI               | std    | MCMC err. | ISI    | Geweke | Kurtosis | Skewness |
|--------------|---------|--------|--------|------------------|--------|-----------|--------|--------|----------|----------|
| $\beta_0$    | 0.0039  | 0.0039 | 0.0039 | [0.0038, 0.0039] | 0.0000 | 0.0000    | 0.0000 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_1$    | 0.79    | 0.7961 | 0.7963 | [0.7960, 0.7964] | 0.0004 | 0.0002    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_2$    | 0.05    | 0.0497 | 0.0498 | [0.0496, 0.0498] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.0267   | -0.0497  |
| $\beta_3$    | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_4$    | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_5$    | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_6$    | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_7$    | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_8$    | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_9$    | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_{10}$ | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_{11}$ | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_{12}$ | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_{13}$ | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_{14}$ | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_{15}$ | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_{16}$ | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_{17}$ | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_{18}$ | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_{19}$ | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |
| $\beta_{20}$ | 0.05    | 0.0549 | 0.0554 | [0.0548, 0.0554] | 0.0004 | 1.6e-5    | 0.0002 | 0.9995 | 3.1415   | 0.0002   |

Note: \*Parameter, CI=1st quartile estimate, CI=3rd quartile estimate, ISI=Standard deviation.

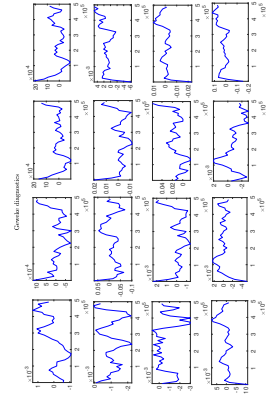


Figure 1. Geweke's diagnostics. The first row:  $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{15}, \beta_{16}, \beta_{17}, \beta_{18}, \beta_{19}, \beta_{20}$ , the second row:  $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{15}, \beta_{16}, \beta_{17}, \beta_{18}, \beta_{19}, \beta_{20}$ , the third row:  $K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8, K_9, K_{10}, K_{11}, K_{12}, K_{13}, K_{14}, K_{15}, K_{16}, K_{17}, K_{18}, K_{19}, K_{20}$ .

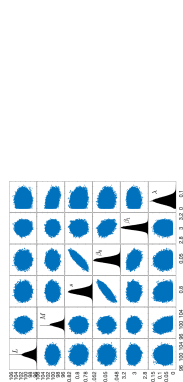


Figure 2. Histograms illustrating the probability density function estimation for model parameters, showcasing Gaussian distributions indicative of effective mixing.

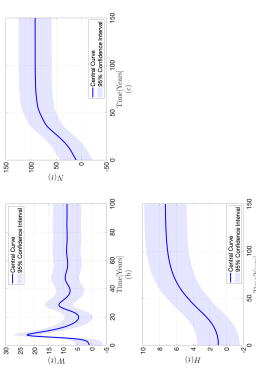


Figure 3. Solutions of the model system (3) using the parameter values in Table 1: (a) Forest biomass, (b) Wildlife population, (c) Human population and (d) Human activities.

### Conclusion

The study's findings underscore the significant impact of human activities on forest and wildlife depletion. The model's accurate predictions highlight the necessity for informed conservation strategies and demonstrate the effectiveness of using mathematical tools in forest management.

### References

- [1] Ibrahim M Faruqi, Silas Miru, Damian Kajungu, and Francis Moyo. Mathematical model to study the impact of anthropogenic activities on forest biomass and forest-dependent wildlife population. *International Journal of Dynamics and Control*, 11(4):1-18, 2023.
- [2] Heikki Haurio, Marko Laine, Antonietta Mira, and Eero Saksman. DREAM: Efficient Adaptive MCMC. *Statistics and computing*, 16(6):339-354, 2006.
- [3] Flavia Remo, Livingstone S Luboo, Iambi Salton Malahewa, and Betty K Nanyonga. A mathematical model for the dynamics and zoonotic transmission of bacterial wilt disease. *International Journal of Biomathematics*, 11(01):1850001, 2018.