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Mwanga, Mohamed

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Posterior distribution of the unknown parameter of Poisson distribution under different priors: an application to under-five mortality data

Mohamed Khalfani Mwanga[1], Silas Steven Mirau[1], Jean Michel Tchuenche [1], Isambi Sailon Mbalawata[2]

[1]School of Computational and Communication Science and Engineering, The Nelson Mandela African Institution of Science and Technology, P.O. Box 447, Arusha, Tanzania

[2]African Institute for Mathematical Sciences, NEI Globla Secretariat, Rue KG590 ST, Kigali, Rwanda

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Abstract

Under-five mortality rate is one of the most essential indicator of a country's socio-economic well-being and public health status. Poisson distribution under different priors such as conjugate/Gamma prior, uniform and Jeffrey's prior is used to obtain posterior distribution of the unknown parameter with an application to under-five mortality data in six East Africa countries from 1960 to 2020. The estimates are examined through a Bayesian analysis while all the calculations are carried out the R-statistical software and MS Excel. Among all priors used in this study, conjugate prior was found to be compatible for the unknown parameters of the Poisson distribution. The cases of under-five mortality are found to reduce over time. East Africa countries through East Africa Community (EAC) should build strong and resilient health systems, identify and prioritize interventions to mitigate under-five mortality among Member States.

keywords: Posterior distribution, Bayesian, Prior, Under-five Mortality, Poisson Disribution

1 Introduction

The under-five mortality (U5M) rate, that is the probability of dying before 5 years of age (per 1000 live births) is one important and essential indicator of a country's socio-economic well-being and public health status [7]. Millions of children under the age of 5 die each year, mostly from avoidable causes such as pneumonia, diarrhea and malaria. In fact, socioeconomic factors such as place of residence, mother's educational level, or household wealth have been strongly associated with risk factors of U5M such as health behavior or exposure to diseases and injuries [21]. In almost half of all cases of U5M, major factors are malnutrition, unsafe water, sanitation and hygiene. U5M is an important indicator not only of child health and well-being, but of the general progress towards the Sustainable Development Goals (SDGs) [11].

Though there has been a decline in global under five mortality rate (U5MR) from 93 deaths per 1000 live births in 1990 to 39 in 2017, the highest rates are still seen in sub-Saharan Africa, with an U5MR of 76 deaths per 1000 live births in 2017, leading to 2.7 million deaths in the region [1]. The morbidity and mortality burden of children under 5 years of age remains unevenly distributed and high in both South Asia and sub-Saharan Africa, with about 80% of U5M commonly occurring in these two regions [11]. The under-five mortality rate (U5MR) is highest in sub-Saharan Africa where about 1 in 9 children dies [15], making sub-Saharan Africa (SSA) one of the most affected region by U5m worldwide [7], despite the U5MR declining globally from 69 deaths in 2000 to 38 deaths per 1000 live births in 2016, a 45% decreasing [23]. This decrease of U5MR is due to micro and macro-economic growth, low fertility rate, improved female education, and strengthened public health programmes [12]. One of the indicators of the socio-economic development of any country is child survival. Most developing nations are unable to address the root cause of child mortality due to the inadequate public health measures and poor health facilities [21].

As maps are used in public health to plan health interventions, monitor outbreaks, identify vulnerable populations, and communicate health data, mapping death rates provides invaluable visualization and analysis tools that scientists and researchers could use to address health problems [18]. Actuaries, demographers, mathematicians and statisticians have long been interested in the analysis of mortality statistics, not just in a country's current demographic structure, but also in estimating/projecting for the future [21]. The probability distribution that represents the uncertainty about a parameter before the current data are examined is known as prior distribution (or 'best guess') [6]. Because Bayesian analysis allows one to combine prior information about a population parameter with evidence from information contained in a sample to guide the statistical inference process, this methodology is applied herein to obtain posterior distribution of the unknown parameter with an application to under-five mortality data from six East Africa countries from 1960 to 2020.

2 Methods and results

2.1 Bayesian statistics

Posterior probability distribution, which describes the epistemic (statistical) uncertainty of parameters conditional on a collection of observed data [8] is best addressed using Bayesian statistics. The goal in Bayesian computation is therefore to obtain a set of independent draws from the posterior distribution to estimate quantities of interest with reasonable accuracy. That is to determine the likelihood function , then combine the prior distribution and likelihood function by applying Bayes' Theorem [24]. In most cases, the posterior distribution is used to obtain inferences and balancing of prior knowledge. The posterior have also been applied in making predictions about future events [20].
Bayesian statistics arises with a prior of probability distribution of the parameters, and therefore, the prior distribution commonly impact the posterior distribution [4]. Models in Bayesian analysis include likelihood function which indicates the probability of information (data) given the parameter values and the prior probability distribution [25]. The description of the likelihood function and prior include prescribed details expressed in mathematical way. Since the posterior distribution depends more on the choice of the prior distribution [13], Let θ be a parameter, y the data/evidence. Then, $P(\theta)$ is defined as prior, with $P(y/\theta)$ the likelihood and $P(\theta/y)$ the posterior [9]. Thus, Baye's theory is given as the posterior distribution is directly proportional to the product of prior and likelihood which is ;

$$P(\theta/y) \propto P(\theta) \times P(y/\theta). \tag{1}$$

In Bayesian statistics, prior distributions have a crucial role. Priors can be obtained in many forms such as a Poisson, normal or uniform distributions. Priors can be either an uninformative or informative prior [3]. Informative priors upturns the accuracy of the posterior distributions by updating previous information with new one, and thus, progressively accruing knowledge [17]. Informative priors which can provide explanations to computational issues and increase modeling efficiency are appropriate when prior information is available [10]. An un-informative/non-informative prior always states general information about a variable. Non-informative priors strongly impact posterior distributions. Bayesian studies with non-informative priors have been shown to provide the same results to computational and theoretical analysis [14]. This study aims to obtain posterior distribution of the unknown parameter of a Poisson distribution under different priors with the application to U5M data in six East Africa countries from 1960 to 2020.

2.2 Poisson distribution

Poisson distribution is suitable to describe the probability that a given occasion can happen within a given period [19]. The events that can be described by Poisson distribution are events which are independent from each other, that is, within a given interval the event may present from 0 to infinite times, and the possibility of an event to occur increases if the period of observation is longer [22]. The Poisson distribution with parameter $\lambda > 0$ of a discrete random variable x is given as

$$P(X = x) = \frac{\lambda^x e^{(-\lambda)}}{x!}, \tag{2}$$

where, e is a mathematical constant known as the base of the natural logarithm.

2.3 Posterior Distribution

2.3.1 Posterior distribution under Jeffrey's prior

Jeffrey's priors commonly used in Bayesian statistics are a non-informative prior distribution that works well with uni-dimensional parameter [16]. Jeffrey's prior defined in terms of the Fisher information is given by

$$P(X = x) = P_J(\lambda) \propto I(\lambda)^{1/2}, \tag{3}$$

where the Fisher's information is defined as

$$I(\lambda) = -E\left(\frac{\partial^2 L(\lambda/x)}{\partial \lambda^2}\right), \tag{4}$$

with the Log likelihood being the product of the individual Poisson probability density function (pdf)

$$L(\lambda/x) = \prod_{i=1}^{n} f_\lambda(x_i) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{(-\lambda)}}{x_i!} = \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod_{i=1}^{n} x_i!}. \tag{5}$$

Then,

$$\ln L(\lambda/x) = \sum_{i=1}^{n} x_i \ln \lambda - n\lambda - \sum_{i=1}^{n} log x_i, \tag{6}$$

and

$$\frac{\partial L(\lambda/x)}{\partial \lambda} = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n, \tag{7}$$

$$\frac{\partial^2 L(\lambda/x)}{\partial \lambda^2} = -\frac{\sum_{i=1}^{n} x_i}{\lambda^2}. \tag{8}$$

From equation (4)

$$I(\lambda) = \left(\frac{E(\sum_{i=1}^{n} x_i)}{\lambda^2} \right). \tag{9}$$

Also

$$E(\sum_{i=1}^{n} x_i) = E(n\overline{x}) = nE(\overline{x}) = n\lambda. \tag{10}$$

So that

$$I(\lambda) = \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda} \implies I(\lambda) \propto \lambda^{-1}, \tag{11}$$

and

$$P_I(\lambda) = \sqrt{I\ (\lambda)} \propto \lambda^{-1/2}. \tag{12}$$

Since *posterior* $\propto$ *prior* $\times$ *likelihood*,

$$P(\lambda/x) \propto P(\lambda) \times P(x/\lambda), \tag{13}$$

and

$$P(x/\lambda) = \prod_{i=1}^{n} Pr(X = x). \tag{14}$$

Thus,

$$P(\lambda/x) \propto \prod_{i=1}^{n} Pr(X = x) P(\lambda). \tag{15}$$

$$P(\lambda/x) \propto \lambda^{-1/2} \prod_{i=1}^{n} Pr(X = x) P(\lambda) = \lambda^{-1/2} \left(\prod_{i=1}^{n} e^{-\lambda} \lambda^{x} \right) = e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i} \lambda^{-1/2}. \tag{16}$$

After some little algebraic manipulation,

$$P(\lambda/x) \propto e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i - \frac{1}{2}} = \left(\lambda^{\sum_{i=1}^{n} x_i - \frac{1}{2}} \right) e^{-n\lambda}. \tag{17}$$

Therefore, the posterior distribution is a Gamma with

$$\Gamma \left(\sum_{i=1}^{n} x_i + \frac{1}{2}, n \right). \tag{18}$$

The posterior mean is $E(\lambda/x)$

$$\mu_{post} = \frac{\sum_{i=1}^{n} x_i + \frac{1}{2}}{n}. \tag{19}$$

The posterior variance is var(λ/x)

$$\mu_{post} = \frac{\sum_{i=1}^{n} x_i + \frac{1}{2}}{n}. \tag{20}$$

$$\sigma^2_{post} = \frac{\sum_{i=1}^{n} x_i + \frac{1}{2}}{n^2}. \tag{21}$$

2.3.2 Posterior distribution under Gamma/conjugate prior

The Gamma distribution is widely used as a conjugate prior in Bayesian statistics. A conjugate prior is an algebraic convenience, giving a closed-form expression for the posterior; otherwise numerical integration may be necessary. Further, conjugate priors may give intuition, by more transparently showing how a likelihood function updates a prior distribution [5].
From equation (13), x are data or information given where $P(\lambda/x)$ is a posterior, $P(x\lambda)$ is a likelihood and $P(\lambda)$ is a prior.
Next, from equation (14),

$$P(x/\lambda) = \prod_{i=1}^{n} \frac{\lambda^x e^{-\lambda}}{x}. \tag{22}$$

Then, equation (15) becomes

$$P(x/\lambda) = \prod_{i=1}^{n} \frac{\lambda^x e^{-\lambda}}{x} Pr(\lambda), \tag{23}$$

and thus,

$$P(\lambda/x) \propto \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{x} Pr(\lambda). \tag{24}$$

The appropriate prior for a Poisson model/distribution is a Gamma distribution (α, β) given by

$$Pr(\lambda) = \frac{\lambda^{\alpha-1} e^{-\beta\lambda}(\beta^\alpha)}{\beta^\alpha \Gamma(\alpha)}, \tag{25}$$

where $\Gamma(\alpha)$ is a Gamma function.
The expected value of a Gamma distribution with parameter α and β is α/β, and its variance is α/β^2. So,

$$P(\lambda/x) \propto \lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda} \lambda^{\alpha-1} e^{-\beta\lambda}, \tag{26}$$

and after some algebraic manipulations, one obtains

$$P(\lambda/x) \propto \lambda^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-n\lambda - \beta\lambda}. \tag{27}$$

Therefore, the posterior distribution is a Gamma with

$$\Gamma\left(\sum_{i=1}^{n} x_i + \alpha, n + \beta\right). \tag{28}$$

The posterior mean is

$$E(\lambda/x) := \mu_{post} = \frac{\sum_{i=1}^{n} x_i + \alpha}{n + \beta}, \tag{29}$$

and the posterior variance is

$$var(\lambda/x) := \sigma^2_{post} = \frac{\sum_{i=1}^{n} x_i + \alpha}{(n + \beta)^2}. \tag{30}$$

2.3.3 Posterior distribution under uniform prior

Uniform priors are unlikely representations of our actual prior state of knowledge. Supplying prior distributions with some information allows one to fit models that cannot be otherwise fitted with frequentist methods (that regard the population value as a fixed, unvarying (but unknown) quantity, without a probability distribution) [20].
For the case of uniform prior, from equation (23), if $P(\lambda) \propto 1$, then

$$P(\lambda/X) = \lambda^{(\sum_{i=1}^{n} x_i + 1) - 1} e^{-n\lambda}. \tag{31}$$

Therefore, the posterior distribution is a Gamma with

$$\Gamma\left(\sum_{i=1}^{n} x_i + 1, n\right). \tag{32}$$

The posterior mean

$$E(\lambda/x) := \mu_{post} = \frac{\sum_{i=1}^{n} x_i + 1}{n}, \tag{33}$$

and the posterior variance is

$$var(\lambda/x) := \sigma^2_{post} = \frac{\sum_{i=1}^{n} x_i + 1}{n^2}. \tag{34}$$

2.4 Numerical Analysis

The data set for East African countries U5M from World Bank for the year 1960 to 2020 was considered for the study [2]. U5M rates were estimated using Bayesian method, with the calculations using the R-statistical software. The posterior distribution were obtained

using different priors such as conjugate/Gamma prior, uniform and Jeffrey's priors. Table 1 below shows posterior mean, variance and mode under different priors from East African Countries.
One notes that conjugate prior had smallest values of mean, variance and mode (Table 1). Therefore, conjugate prior is well-suited for the unknown parameters of the Poisson distribution using the East Africa countries U5M data. The posterior is estimated using

$$P(\lambda/x) \propto \lambda^{\sum_{i=1}^{n} x_i+\alpha-1} e^{-n\lambda-\beta\lambda}, \tag{35}$$

where $\alpha = 1$ and $\beta = 1$.

Table 1: Posterior Description

	Jaffrey's Prior	**Uniform Prior**	**Conjugate Prior**
Tanzania			
Mean	151.321	151.33	148.889
Variance	2.48068	2.48081	2.40143
Mode	151.305	151.313	148.873
Uganda			
Mean	156.4279	156.4361	153.9129
Variance	2.564391	2.564526	2.482466
Mode	156.4115	156.4197	153.8968
South Sudan			
Mean	228.8279	228.8197	225.1452
Variance	3.751277	3.751411	3.631374
Mode	228.8115	228.8197	225.129
Kenya			
Mean	105.8508	105.859	104.1516
Variance	1.735259	1.735394	1.679865
Mode	105.8344	105.8426	104.355
DRC			
Mean	104.3525	104.3607	102.6774
Variance	1.710696	1.71083	1.656087
Mode	104.3361	104.3443	102.6613
Burundi			
Mean	158.1607	158.1689	155.6177
Variance	2.592798	2.592932	2.509964
Mode	158.1443	158.1525	155.6016
Rwanda			
Mean	164.6311	164.6393	161.9839
Variance	2.698876	2.699006	2.612643
Mode	164.6149	164.623	161.9677

Figure 1 below shows the posterior mean under different priors for all of the East African countries. Results show that South Sudan has the highest U5M occurrence, followed by Rwanda, Burundi, Uganda, Tanzania and Kenya, while DRC has the lowest prevalence of under-five morality.

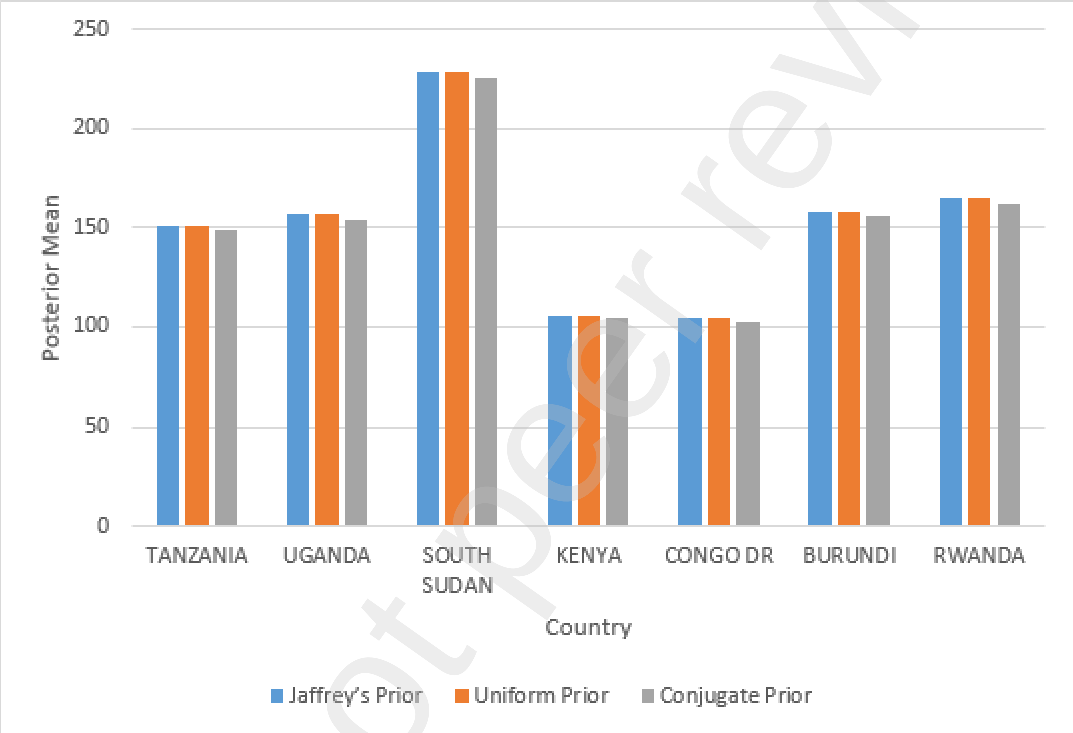

Figure 1: Posterior mean under different priors

Figure 2 below shows the posterior mode under different priors for all of the seven East African countries. South Sudan has the highest U5M incidence, followed by Rwanda, Burundi, Uganda, Tanzania and Kenya, while DRC had the lowest occurrence of under-five morality.

Figure 3 below illustrates the posterior variance under different priors for the East African countries. South Sudan has the highest U5M rate, followed by Rwanda, Burundi, Uganda, Tanzania and Kenya, while DRC had the lowest rate of U5M.

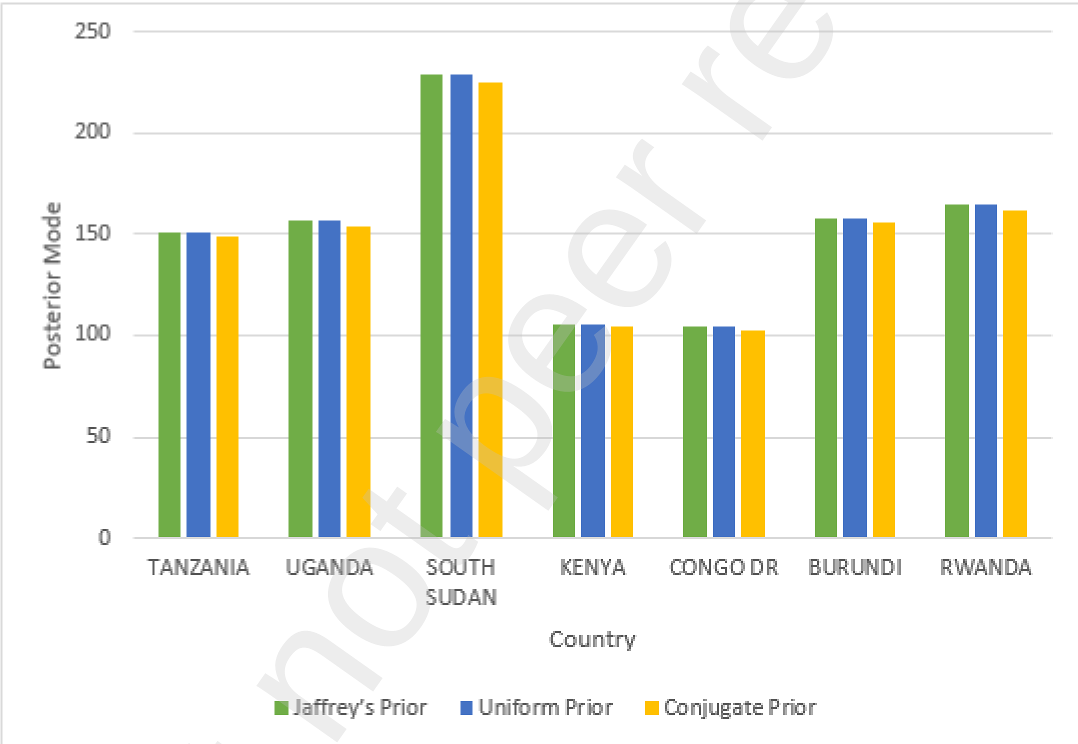

Figure 2: Posterior mode under different priors

Figure 3: Posterior variance under different priors

3 Conclusion

The U5MR is a key global indicator of child health and one of the most important measures of global health. Despite the global decline in U5MR, sub-Saharan Africa still bears the burden of the fatalities, which are amenable to poor health care and prevention. Using Poisson distribution under different priors such as conjugate/gamma prior, uniform and Jeffrey's prior, posterior distribution of the unknown parameter with an application to under-five mortality data in six East Africa countries from 1960 to 2020. Using Bayesian statistical analysis in addition to the R-statistical software and MS Excel, results show that conjugate prior is quite compatible for the unknown parameters of the Poisson distribution for the East Africa countries' data. South Sudan has the highest number of cases based on the posterior mean, mode and variance from all priors. Furthermore, DRC has the lowest number of cases based on posterior mean, mode and variance from the three priors used. To mitigate under-five mortality among its member states, East Africa countries through the East Africa Community (EAC) should build strong and resilient health systems, identify and prioritize interventions aiming at improving child survival in all subgroups of the population.This will require developing concrete national policy/strategy/action plan aimed at reducing under-five mortality. This study is not exhaustive. Bayesian analysis is used in this study, but one could also use Kalman filter, which is a special case of the former. Further studies could consider the alternative to Bayesian analysis, the framework of Null Hypothesis Statistical Testing; Dempster's rule of combination and compare the results when Jeffrey's prior is used.

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