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The Volterra–Lyapunov matrix theory for global stability analysis of alcohol-related health risks model

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ABSTRACT

This paper studies the global stability analysis of a modified framework proposed by Mayengo et al. on alcohol-related health risks model. We first present the global stability analysis of risk-free equilibrium (RFE). Later, the global stability of the risk endemic equilibrium is studied. This goal was achieved by appropriate utilization of the symmetrical properties in the structure of Volterra–Lyapunov matrices. The analysis and results presented in this paper make building blocks towards a comprehensive study and deeper understanding of the fundamental mechanism in alcohol-related health risks and similar models. The numerical examples are simulated to validate the theoretical model results presented.

Introduction

Mathematical models describing the dynamics of human infectious diseases and other contagious conditions have played an essential role in understanding the patterns of disease transmission, control and prevention in epidemiology over the years. The usefulness of mathematical models in the field of epidemiology may not be overemphasized [1]. Epidemiological models are used to describe and predict the dynamics of infectious diseases, conditions and behaviors. They are also useful in improving scientific support for decision-making by incorporating and analyzing the factors responsible for the spread of the disease [1,2].

The global stability analysis of epidemiological models plays an essential role in predicting the state of infection and suggests methodology to control the disease [3–6]. Recently, the phenomenon of Volterra–Lyapunov matrix theory for global stability analysis has received impressive consideration in disease control [1,2,7]. For instance, Zahedi and Kargar [2] analyzed global stability of HIV/AIDS model using Volterra–Lyapunov matrix theory approach. In this study, a nonlinear mathematical model of the HIV/AIDS is analyzed with utilization of constant controls. In a similar vein, Chien and Shateyi [1] utilized Volterra–Lyapunov approach in the stability analysis of *Babesiosis* transmission dynamics model on bovines and ticks populations. Global stability analysis of epidemic model with nonlinear incidence rate function according to the Lyapunov functions and Volterra–Lyapunov matrices is studied in Shao and Shateyi [7]. On the other hand, Masoumnezhad et al. [8] discussed the global stability of mathematical model of an infectious disease. In this study, application of Volterra–Lyapunov matrix approach is used to prove the global stability of the endemic equilibrium point. This goal was reached through reducing the originally 4×4 matrix to matrices of lower dimensions under some conditions. Similarly, Parsaei et al. [9] applied Volterra–Lyapunov matrix properties in the proof of global stability of an epidemic model of computer viruses spreading over the internet.

The study of global stability analysis of the risk-endemic equilibrium is essential in predicting the evolution of the modeled condition after a long period of time so that prevention and intervention strategies can be effectively designed and executed. Despite the wide use of Lyapunov functions to study the stability analysis of various dynamical systems, this study applies Volterra–Lyapunov stable matrices in performing global analysis [1,2,7]. Volterra–Lyapunov approach is such a powerful tool as it reduces the burden of determining the coefficients of the Lyapunov functions [2]. We, therefore, apply the combined method of Lyapunov functions and the Volterra–Lyapunov matrix symmetric properties leading to the proof of the risk-endemic global stability [1,2,7]. Although the approach of Lyapunov functions has long been used, there is no well established systematic recruitment procedures of Lyapunov candidate [2]. This remains unsolved challenge particularly in the determination of the appropriate coefficients of the Lyapunov function which largely depends on trials and errors [2].

The structure of this paper is as follows. The model formulation is presented in Section "Model formulation"; basic properties of the model in Section "Basic properties"; stability analysis in Section "Stability analysis"; in Section "Numerical simulations" we demonstrate the

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Table 1
Description of the variables and parameters for model (1).

Parameters	Description
π	Recruitment rate
μ	Per capital death rate
α	Alcohol induced death rate
δ	Rate of becoming alcoholic for moderate drinkers
σ	Rate of becoming moderate risk drinkers for low risk
ν	Rate at which low risk drinkers join religious group
ψ	Rate at which an alcoholic joins religious group
ξ	Rate of quitting alcohol for moderate risk drinkers
η	Rate of quitting alcohol for alcoholic drinkers
ω	Rate at which recovered individuals re-join susceptible group
θ_1	Modification parameter
θ_2	Modification parameter
ρ	Proportion of the peer pressure used to recruit low risk drinkers
β	Chances of becoming an alcoholic drinker after a successful interaction with drinkers
c	Effective contact necessary for one to become a drinker

effectiveness of the proposed method by means of numerical examples, and the conclusion.

Model formulation

Alcohol consumption behavior is a well known risk factor for various health complications with social cultural beliefs providing partial immunity from engaging into alcohol drinking habit. This study modifies the framework of alcoholism model *SPLMAR* studied by Mayengo et al. [10,11] and Mayengo [12] into a considerable simple *SLMHR* model. The model considers time dependent total population N in five distinct "epidemiological" sub-populations, namely: Risk susceptible (S), Low risk drinkers (L), Medium risk drinkers (M), High risk drinkers (H) and Recovered (R). In classical epidemic models, effective rate of the divine intervention emanating from social cultural beliefs to the risk population is assumed to be proportional to the number of individuals exposed to such risks [13]. We establish model system (1) using the parameters described in Table 1.

$$\begin{cases} \frac{dS}{dt} &= -(\mu+\lambda)S+\omega R+\pi \\ \frac{dL}{dt} &= \rho\lambda S-(\mu+\sigma)L \\ \frac{dM}{dt} &= (1-\rho)\lambda S+\sigma L-(\mu+\delta+\xi)M \\ \frac{dH}{dt} &= \delta M-(\mu+\alpha+\eta)H \\ \frac{dR}{dt} &= \xi M+\eta H-(\omega+\mu)R \end{cases} \tag{1}$$

where $S(0) > 0, L(0) \geq 0, M(0) \geq 0, H(0) \geq 0, R(0) \geq 0$. Following Mayengo et al. [10], the force of peer influences λ is defined as follows:

$$\lambda = c\beta\left(\frac{L+\theta_1 M+\theta_2 H}{N}\right) \tag{2}$$

The total population size $N(t)$ is given by;

$$N = S+L+M+H+R \tag{3}$$

It is easy to verify that $\mathbf{R}^5$ is positive invariant for system Eq. (1) which satisfies the following equation:

$$\frac{dN}{dt} = \pi-\mu N-\alpha H \tag{4}$$

Basic properties

Invariant region

The model in (1) above represents human population proportions, it is therefore assumed that all the state variables and parameters of the model are non-negative for all time $t \geq 0$. We analyze the model (1) in a suitable feasible region, obtained as follows. We first show that all feasible solutions are uniformly-bounded in a proper subset Ω of $\mathbf{R}^5_+$. Let $(S, L, M, H, R) \in \mathbf{R}^5_+$ be any solution of system Eq. (1) with non-negative initial conditions. It is easy to verify that all solutions of system Eq. (1) initiating in set $\Omega = \{S, L, M, H, R\}$ eventually enter the set $\Omega = \left\{(S, L, M, H, R) : N = S+L+M+H+R \leq \frac{\pi}{\mu}\right\}$ where $0 < S \leq \frac{\pi}{\mu}, L \geq 0, M \geq 0, H \geq 0, R \geq 0$ showing that Ω is positively invariant under the flow induced by system (1). We can therefore, conclude that the system Eq. (1) is well posed mathematically and epidemiological relevant, sufficiently to consider the dynamics of the flow generated by system Eq. (1) in Ω [14].

Lemma 1. *The region $\Omega \subset \mathbf{R}^5_+$ is positively-invariant for the basic model (1) with non-negative initial conditions in $\mathbf{R}^5_+$.*

Existence of equilibria

The model system Eq. (1) has at least two critical points, which are the solution set to Eq. (5) below,

$$\begin{aligned} -(\mu+\lambda^*)S^*+\omega R^*+\pi &= 0 \\ \rho\lambda S^*-a_{11}L^* &= 0 \\ (1-\rho)\lambda^* S^*+\sigma L^*-a_{22}M^* &= 0 \\ \delta M^*-a_{33}H^* &= 0 \\ \xi M^*+\eta H^*-(\omega+\mu)R^* &= 0 \end{aligned} \tag{5}$$

with simplifying factors $a_{11} = \mu+\sigma$, $a_{22} = \mu+\delta+\xi$ and $a_{33} = \mu+\eta+\alpha$. The critical points are obtained by solving the system of Eq. (5) simultaneously, resulting into

$$E_1 = (S^*, L^*, M^*, H^*, R^*) \tag{6}$$

where

$$\begin{cases} S^* &= \frac{\omega\left(a_{33}\xi+\delta\eta\right)H^*+(\mu+\omega)\delta\pi}{\delta(\omega+\mu)(\mu+\lambda^*)}, \\ L^* &= \frac{\left(\omega\left(a_{33}\xi+\delta\eta\right)H^*+(\mu+\omega)\delta\pi\right)\lambda^*\rho}{\delta(\omega+\mu)(\mu+\lambda^*)a_{11}}, \\ M^* &= \frac{a_{33}H^*}{\delta}, \\ H^* &= H^* \\ R^* &= \frac{H^*\left(a_{33}\xi+\delta\eta\right)}{\delta(\omega+\mu)}. \end{cases} \tag{7}$$

The risk-free equilibrium point (E_0) is obtained by evaluating Eq. (7) at $L^* = M^* = H^* = 0$, and $\lambda^* = 0$ resulting into

$$E_0 = (S_0^*, L_0^*, M_0^*, H_0^*, R_0^*) = \left(\frac{\pi}{\mu}, 0, 0, 0, 0\right). \tag{8}$$

It is clear that, the risk-free equilibrium point (E_0) satisfies the equation

$$-\mu N_0+\pi = 0 \tag{9}$$

where N_0 is an initial population. Otherwise, when $L^* \neq 0, M^* \neq 0, H^* \neq 0$ and $\lambda^* \neq 0$ we have a risk endemic equilibrium point (E_1) described in Eq. (6).

Basic risk reproduction number

To establish conditions for the linear stability of the risk-free equilibrium point (E_0), the basic risk reproduction number, $\mathcal{R}_0$ is obtained by using the next generation operator method [10,15–17] to the system and established as:

$$\mathcal{R}_0 = \frac{c\beta\,\left((1-\rho)\left(a_{33}\,\theta_1+\delta\,\theta_2\right)a_{11}+\rho\,\left(\left(\sigma\,\theta_1+a_{22}\right)a_{33}+\theta_2\,\delta\,\sigma\right)\right)}{a_{11}a_{22}a_{33}} \tag{10}$$

By definition, the basic risk reproduction number $\mathcal{R}_0$ is the threshold quantity defining the average number of secondary health risks caused by a single alcoholic during his alcoholic life time in an entirely risk susceptible population [10,15–18].

At risk-endemic equilibrium, we have

$$\lambda^* = c\beta\left(\frac{L^*+\theta_1 M^*+\theta_2 H^*}{N^*}\right) \tag{11}$$

such that $L^* = \frac{\rho\lambda^* S^*}{a_{11}}$, $M^* = \frac{(1-\rho)\lambda^* S^* + \sigma L^*}{a_{22}}$ and $H^* = \frac{\delta M^*}{a_{33}}$. Upon substituting these values into (11) and appropriate simplifications the following result is established

$$\lambda^*\left(1-\frac{S^*}{N^*}\mathcal{R}_0\right) = 0. \tag{12}$$

The solution to this equation suggests, either $\lambda^* = 0$ or $1-\frac{S^*}{N^*}\mathcal{R}_0 = 0$. The solution $\lambda^* = 0$ occurs at the risk-free equilibrium point, that is $\lambda^* = \lambda_0^*$, consequently we have $1-\frac{S^*}{N^*}\mathcal{R}_0 \geq 0$. At this point, the ratio $\frac{S^*}{N^*} = \frac{S_0^*}{N_0^*} = 1$ suggesting that $\mathcal{R}_0 \leq 1$ and hence the establishment of the following results

$$\mathcal{R}_0 \leq 1. \tag{13}$$

On the other hand, the solution $1-\frac{S^*}{N^*}\mathcal{R}_0 = 0$ occurs at the risk-endemic equilibrium point where $\lambda^* > 0$ and the ratio $\frac{S^*}{N^*} < 1$ suggesting

$$\mathcal{R}_0 > 1. \tag{14}$$

Stability analysis

Global stability of risk-free equilibrium

Theorem 2. *The risk-free equilibrium (E_0) of the model system Eq. (1) is globally asymptotically stable when $\mathcal{R}_0 \leq 1$.*

Proof. We adopt the approach used in [19–21] to study the global stability of the risk-free equilibrium. In this method, we split the differential equations presented in Eq. (1) into two subsystems as follows

$$\begin{aligned}\frac{dX_s}{dt} &= B_1\left(X_s - X_{E_0}\right) + B_2 X_i\\ \frac{dX_i}{dt} &= B_3 X_i\end{aligned} \tag{15}$$

where X_s and X_i are respectively, the non-transmitting and transmitting classes, E_0 is the risk-free equilibrium, whereas B_1, B_2 and B_3 are the matrices to be computed. Therefore, for a non-transmitting subsystem we have

$$\begin{aligned}\frac{dX_s}{dt} &= \begin{cases}-(\mu+\lambda)S+\omega R+\pi\\ \xi M+\eta H-(\omega+\mu)R\end{cases}\\ &= B_1\left(X_s - X_{sE_0}\right)+B_2X_i\end{aligned}$$

where, $B_1 = \begin{pmatrix}-\mu & \omega\\ 0 & -(\mu+\omega)\end{pmatrix}$, and $B_2 = \begin{pmatrix}-c\beta\frac{S}{N} & -c\beta\theta_1\frac{S}{N} & -c\beta\theta_2\frac{S}{N}\\ 0 & \xi & \eta\end{pmatrix}$. Similarly, for risk transmitting subsystem the following results are established

$$\begin{aligned}\frac{dX_i}{dt} &= \begin{cases}\rho\lambda S - a_{11}L\\ (1-\rho)\lambda S+\sigma L - a_{22}M\\ \delta M - a_{33}H\end{cases}\\ &= B_3X_i\end{aligned}$$

where

$$B_3 = \begin{pmatrix}-\left(a_{11}-\frac{c\beta\rho S}{N}\right) & \frac{c\beta\rho\theta_1 S}{N} & \frac{c\beta\rho\theta_2 S}{N}\\ \frac{c\beta(1-\rho)S}{N}+\sigma & -\left(a_{22}-\frac{c\beta(1-\rho)\theta_1 S}{N}\right) & \frac{c\beta(1-\rho)\theta_2 S}{N}\\ 0 & \delta & -a_{33}\end{pmatrix}.$$

It can be observed that matrix B_1 has real and negative eigenvalues. Thus, the system Eq. (1) is globally asymptotically stable at E_0. To prove the stability of B_3, we adopt the idea of stable Metzler matrix and apply the Lemma utilized by Dumont et al. [20] in which B_3 is a Metzler matrix since $B_3(i,j) \geq 0, \forall i \neq j$.

Lemma 3. *Let D be a square Metzler matrix written in block form:*

$$D = \begin{pmatrix}T & U\\ V & W\end{pmatrix}$$

where T and W are square matrices, D is Metzler stable if and only if matrices T and $W - VT^{-1}U$ are Metzler stable.

Proof. Utilizing Lemma 3, we write the Metzler matrix B_3 as a square Metzler matrix

$$B_3 = \begin{pmatrix}T & U\\ V & W\end{pmatrix},$$

such that the matrices T, V and W are defined as

$$T_{1\times1} = -\left(a_{11}-\frac{c\beta\rho S}{N}\right),\; U_{1\times2} = \begin{pmatrix}\frac{c\beta\rho\theta_1 S}{N} & \frac{c\beta\rho\theta_2 S}{N}\end{pmatrix},$$

$$V_{2\times1} = \begin{pmatrix}\frac{c\beta(1-\rho)S}{N}+\sigma\\ 0\end{pmatrix},$$

$$W_{2\times2} = \begin{pmatrix}-\left(a_{22}-\frac{c\beta(1-\rho)\theta_1 S}{N}\right) & \frac{c\beta(1-\rho)\theta_2 S}{N}\\ \delta & -a_{33}\end{pmatrix}.$$

It is clear that, T has a negative real eigenvalue making it a stable Metzler matrix and

$$W - VT^{-1}U = \begin{pmatrix}\frac{\left(\theta_1\,(1-\rho)\,a_{11}+\rho\,\left(\sigma\,\theta_1+a_{22}\right)\right)\frac{c\beta S}{N}-a_{11}a_{22}}{a_{11}-\frac{c\beta S}{N}} & \frac{\frac{c\beta\theta_2 S}{N}\left(a_{11}(1-\rho)+\rho\,\sigma\right)}{a_{11}-\frac{c\beta\rho S}{N}}\\ \delta & -a_{33}\end{pmatrix}$$

whose determinant, with proper substitution and simplification gives

$$\det(W - VT^{-1}U) = \frac{a_{11}a_{22}a_{33}\left(1-\mathcal{R}_0\frac{S}{N}\right)}{a_{11}-c\beta\rho\frac{S}{N}} > 0$$

suggesting that $W - VT^{-1}U$ is Metzler stable.

Global stability of the endemic equilibrium

The system Eq. (1) is studied in epidemiologically feasible region $\Omega = \left\{(S,L,M,H,R) \in \mathbf{R}_+^5 : N(t) = S(t)+L(t)+M(t)+H(t)+R(t) \leq \frac{\pi}{\mu}\right\}$ defined in Section "invariant", as $t \to \infty$, $N \to N^*$, implying that $\lambda = c\beta\left(\frac{L+\theta_1 M+\theta_2 H}{N^*}\right)$.

Lemma 4. *Let V be the Lyapunov function such that,*

$$\begin{aligned}V &= \frac{1}{2}\left(\kappa_1(S-S^*)^2+\kappa_2(L-L^*)^2+\kappa_3(M-M^*)^2+\kappa_4(H-H^*)^2\right.\\ &\quad\left.+\kappa_5(R-R^*)^2\right)\end{aligned} \tag{16}$$

where κ_is are positive constants.

The time derivative of V is given by

$$\begin{aligned} \frac{dV}{dt} &= \kappa_1(S-S^*)\frac{dS}{dt} + \kappa_2(L-L^*)\frac{dL}{dt} + \kappa_3(M-M^*)\frac{dM}{dt} \\ &\quad + \kappa_4(H-H^*)\frac{dH}{dt} + \kappa_5(R-R^*)\frac{dR}{dt} \end{aligned} \tag{17}$$

performing appropriate substitutions and some algebraic manipulation and simplifications we establish that

$$\frac{dV}{dt} = \mathcal{Y}\left(\mathcal{B}\mathcal{A} + \mathcal{A}^T\mathcal{B}^T\right)\mathcal{Y}^T \tag{18}$$

where $\mathcal{Y} = (S-S^*, L-L^*, M-M^*, H-H^*, R-R^*)$, $\mathcal{B} = diag\left(\kappa_1, \ldots, \kappa_5\right)$ and

$$\mathcal{A} = \begin{pmatrix} -(\mu+\lambda^*) & -\frac{c\beta S^*}{N^*} & -\frac{c\beta\theta_1 S^*}{N^*} & -\frac{c\beta\theta_2 S^*}{N^*} & \omega \\ \rho\lambda^* & -a_{11} + \frac{c\beta\rho S^*}{N^*} & \frac{c\beta\rho\theta_1 S^*}{N^*} & \frac{c\beta\rho\theta_2 S^*}{N^*} & 0 \\ (1-\rho)\lambda^* & \frac{c\beta(1-\rho)S^*}{N^*} + \sigma & -a_{22} + \frac{c\beta(1-\rho)\theta_1 S^*}{N^*} & \frac{c\beta(1-\rho)\theta_2 S^*}{N^*} & 0 \\ 0 & 0 & \delta & -a_{33} & 0 \\ 0 & 0 & \xi & \eta & -(\omega+\mu) \end{pmatrix}. \tag{19}$$

To establish the global stability of the risk endemic equilibrium, E_1, we investigate the Volterra–Lyapunov stability of matrix $\mathcal{A}$ defined in Eq. (19). Thus the following notations and preliminaries are the prerequisites [1,2,8,9,22,23]:

Definition 1 (*[1,2,7,8]*). Let $\mathcal{M}$ be a square matrix with symmetry property and is a positive (negative) definite, in this case $\mathcal{M}$ is written as $\mathcal{M} > 0(\mathcal{M} < 0)$.

Definition 2 ([1,2,7,8]). We write a matrix $A_{n\times n} > 0(A_{n\times n} < 0)$ if $A_{n\times n}$ is symmetric positive (negative) definite.

Definition 3 ([1,2,7,8]). If there exists a positive diagonal matrix $\mathcal{H}_{n\times n}$ such that $\mathcal{H}A + A^T\mathcal{H}^T < 0$ then, a nonsingular matrix $A_{n\times n}$ is Volterra–Lyapunov stable.

Definition 4 ([1,2,7,8]). If there exists a positive diagonal matrix $\mathcal{H}_{n\times n}$ such that $\mathcal{H}A + A^T\mathcal{H}^T < 0(> 0)$ then, a nonsingular matrix $A_{n\times n}$ is diagonally stable.

The following lemma determines all 2×2 Volterra–Lyapunov stable matrices.

Lemma 5 ([1,2,7,8]). *Let* $\mathcal{B} = \begin{pmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ \mathcal{B}_{21} & \mathcal{B}_{22} \end{pmatrix}$ *be a* 2×2 *matrix, then* $\mathcal{B}$ *is Volterra–Lyapunov stable if and only if* $\mathcal{B}_{11} < 0$, $\mathcal{B}_{22} < 0$ *and* $\det(\mathcal{B}) > 0$.

Lemma 6 ([1,2,7,8]). *Consider the nonsingular* $A_{n\times n} = [A_{ij}]$, $(n \geq 2)$, *the positive diagonal matrix* $\mathcal{H}_{n\times n} = diag(\mathcal{H}_1, \ldots, \mathcal{H}_n)$ *and* $C = A^{-1}$ *such that*

$$\begin{cases} A_{nn} > 0, \\ \widetilde{\mathcal{H}}\widetilde{A} + (\widetilde{\mathcal{H}}\widetilde{A})^T > 0, \\ \widetilde{\mathcal{H}}\widetilde{C} + (\widetilde{\mathcal{H}}\widetilde{C})^T > 0 \end{cases}$$

then, there is $\mathcal{H}_n > 0$ *such that* $\mathcal{H}A + A^T\mathcal{H}^T > 0$.

It is worthy noting that a matrix $\widetilde{A}_{(n-1)\times(n-1)}$ is the resulting matrix made by deleting the last row and column of the matrix A. Following Masoumnezhad et al. [8], Shao and Shateyi [7], Zahedi and Kargar [2], Chien and Shateyi [1] and Parsaei et al. [9], we establish the following Lemmas and Theorems to investigate the global stability of the endemic equilibrium E_1.

Theorem 7. *The matrix* $\mathcal{A}_{5\times 5}$ *defined in Eq. (19) is Volterra–Lyapunov stable.*

Proof. It is clear that $-\mathcal{A}_{55} > 0$, utilizing Lemma 6 we need to prove that the matrices $C = \widetilde{-\mathcal{A}}$ and C^{-1} are diagonally stable. We begin this proof by considering matrix $C = \widetilde{-\mathcal{A}}$, from Eq. (19) we obtain

$$C = \begin{pmatrix} (\mu+\lambda^*) & \frac{c\beta S^*}{N^*} & \frac{c\beta\theta_1 S^*}{N^*} & \frac{c\beta\theta_2 S^*}{N^*} \\ -\rho\lambda^* & a_{11} - \frac{c\beta\rho S^*}{N^*} & -\frac{c\beta\rho\theta_1 S^*}{N^*} & -\frac{c\beta\rho\theta_2 S^*}{N^*} \\ -(1-\rho)\lambda^* & -\frac{c\beta(1-\rho)S^*}{N^*} - \sigma & a_{22} - \frac{c\beta(1-\rho)\theta_1 S^*}{N^*} & -\frac{c\beta(1-\rho)\theta_2 S^*}{N^*} \\ 0 & 0 & -\delta & a_{33} \end{pmatrix}. \tag{20}$$

Based on Lemma 6, we state and prove that $C = \widetilde{-\mathcal{A}}$ and C^{-1} are diagonally stable which prove that the matrix $\mathcal{A}$ is Volterra–Lyapunov stable.

Lemma 8. *The matrix* C *defined in Eq. (20), is diagonal stable.*

Proof. It is obvious that $C_{44} = a_{33} > 0$, utilizing Lemma 6 we need to show that a reduced matrix $\mathcal{D} = \widetilde{C}$ and its inverse matrix $(\mathcal{D}^{-1})$ are diagonally stable, which accomplishes the proof.

Thus, from Eq. (20) we have

$$\mathcal{D} = \begin{pmatrix} (\mu+\lambda^*) & \frac{c\beta S^*}{N^*} & \frac{c\beta\theta_1 S^*}{N^*} \\ -\rho\lambda^* & a_{11} - \frac{c\beta\rho S^*}{N^*} & -\frac{c\beta\rho\theta_1 S^*}{N^*} \\ -(1-\rho)\lambda^* & -\frac{c\beta(1-\rho)S^*}{N^*} - \sigma & a_{22} - \frac{c\beta(1-\rho)\theta_1 S^*}{N^*} \end{pmatrix} \tag{21}$$

Lemma 9. *The matrix* $\mathcal{D}$ *defined in Eq. (21), is diagonal stable.*

Proof. In this regards, we firstly need to show that $\mathcal{D}_{33} > 0$. From the system Eq. (5) we have

$$(1-\rho)\lambda^* S^* + \sigma L^* - a_{22}M^* = 0$$

which, upon substitution of λ^* and some algebraic manipulations, it is easy to show that, $\mathcal{D}_{33} = a_{22} - \frac{c\beta(1-\rho)\theta_1 S^*}{N^*} > 0$. It follows that the proof will be accomplished by proving that $\widetilde{\mathcal{D}}$ is diagonally stable.

The matrix $\mathcal{D}^{-1}$ is formulate the as

$$\frac{1}{\det(\mathcal{D})}\begin{pmatrix} \mathcal{D}_{11} & -\frac{c\beta S^*}{N^*}\left(\sigma\theta_1 + a_{22}\right) & -\frac{c\beta\theta_1 S^*}{N^*}a_{11} \\ a_{22}\rho\lambda^* & -\left(\frac{c\beta(1-\rho)\theta_1 S^*}{N^*} - a_{22}\right)\mu + a_{22}\lambda^* & \frac{c\beta\rho\theta_1\mu S^*}{N^*} \\ \left((1-\rho)a_{11} + \rho\sigma\right)\lambda^* & \left(\sigma + \frac{c\beta(1-\rho)S^*}{N^*}\right)\mu + \sigma\lambda^* & \mathcal{D}_{33} \end{pmatrix} \tag{22}$$

where

$$\mathcal{D}_{11} = \left(\left((1-\rho)a_{11} + \rho\sigma\right)\theta_1 + a_{22}\rho\right)\frac{c\beta S^*}{N^*} + a_{11}a_{22},$$

$$\mathcal{D}_{33} = \mu\left(a_{11} - \frac{c\beta\rho S^*}{N^*}\right) + a_{11}\lambda^*,$$

$$\det(\mathcal{D}) = \left(a_{22} - \frac{c\beta(1-\rho)\theta_1 S^*}{N^*}\right)\mu a_{11}$$

$$+\, a_{11} a_{22}\, \lambda^* - \frac{c\beta\rho S^*}{N^*} \left(\sigma\, \theta_1 + a_{22}\right) \mu > 0.$$

Lemma 10. *The matrix $\mathcal{D}^{-1}$ defined in Eq. (22), is diagonal stable.*

Proof. From the proof of $\mathcal{D}_{33} > 0$ in Lemma 9 it is clear that $\det(\mathcal{D}) > 0$. We, next need to show that $\mathcal{D}^{-1}_{33} > 0$. From the second equation of system Eq. (5) we have $\rho\lambda^* S^* - a_{11} L^* = 0$, when manipulated algebraically it can be shown that $\mathcal{D}^{-1}_{33} = \mu \left(a_{11} - \frac{c\beta\rho S^*}{N^*} \right) + \lambda^* a_{11} > 0$. The proof will be accomplished by the proof of diagonal stability of $\widetilde{\mathcal{D}^{-1}}$ and $\mathcal{G}^{-1}$ where $\mathcal{G} = \mathcal{D}^{-1}$

Considering $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{D}^{-1}}$, such that

$$\widetilde{\mathcal{D}} = \begin{pmatrix} (\mu + \lambda^*) & \frac{c\beta S^*}{N^*} \\ -\rho\lambda^* & a_{11} - \frac{c\beta\rho S^*}{N^*} \end{pmatrix} \quad \text{and}$$

$$\widetilde{\mathcal{D}}^{-1} = \frac{1}{\det(\widetilde{\mathcal{D}})} \begin{pmatrix} a_{11} - \frac{c\beta\rho S^*}{N^*} & -\frac{c\beta S^*}{N^*} \\ \rho\, \lambda^* & \mu + \lambda^* \end{pmatrix} \tag{23}$$

where $\det(\widetilde{\mathcal{D}}) = \left(a_{11} - \frac{c\beta\rho S^*}{N^*} \right) \mu + a_{11}\lambda^*$.

Lemma 11. *The matrices $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{D}}^{-1}$ defined in Eq. (23), are Volterra–Lyapunov stable.*

Proof. Since both $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{D}}^{-1}$ have the dimension of 2×2, we utilize Lemma 5. The matrix $\widetilde{\mathcal{D}}$ is Volterra–Lyapunov stable if and only if $\widetilde{\mathcal{D}}_{11} > 0$, $\widetilde{\mathcal{D}}_{22} > 0$ and $\det(\widetilde{\mathcal{D}}) > 0$. In this regard, it is clear that $\widetilde{\mathcal{D}}_{11} > 0$, it suffices to show that $\widetilde{\mathcal{D}}_{22} > 0$ and $\det(\widetilde{\mathcal{D}}) > 0$. The relation $a_{11} - \frac{c\beta\rho S^*}{N^*} > 0$ is established in the preceding proofs, affirming that both $\widetilde{\mathcal{D}}_{22} > 0$ and $\det(\widetilde{\mathcal{D}}) > 0$. In a similar vein, we observe that $\widetilde{\mathcal{D}}^{-1}_{11} > 0$, $\widetilde{\mathcal{D}}^{-1}_{22} > 0$ and $\det(\widetilde{\mathcal{D}}^{-1}) > 0$, hence $\widetilde{\mathcal{D}}^{-1}$ Volterra–Lyapunov stable.

On the other hand, $\mathcal{D}^{-1}$ is Volterra–Lyapunov stable if and only if $\mathcal{D}^{-1}_{33} > 0$ and $\widetilde{\mathcal{D}^{-1}}$ is diagonally stable. Since $\mathcal{D}^{-1}_{33} > 0$ we verify that $\widetilde{\mathcal{D}^{-1}}$ is diagonally stable. Thus, the matrix $\widetilde{\mathcal{D}^{-1}}$ is given by

$$\frac{1}{\det(\mathcal{D})} \times \begin{pmatrix} \left(\left((1-\rho)\, a_{11} + \rho\,\sigma\right) \theta_1 + a_{22}\, \rho\right) \frac{c\beta S^*}{N^*} + a_{11} a_{22} & -\frac{c\beta S^*}{N^*} \left(\sigma\, \theta_1 + a_{22}\right) \\ a_{22} \rho\, \lambda^* & a_{22}\, \lambda^* + \left(a_{22} - \frac{c\beta\,(1-\rho)\, \theta_1 S^*}{N^*} \right) \mu \end{pmatrix}. \tag{24}$$

Lemma 12. *The matrix $\widetilde{\mathcal{D}^{-1}}$ defined in Eq. (24), is Volterra–Lyapunov stable.*

Proof. From the preceding proofs, it is now clear that $\widetilde{\mathcal{D}^{-1}}_{11} > 0$, $\widetilde{\mathcal{D}^{-1}}_{22} > 0$ and $\det(\widetilde{\mathcal{D}^{-1}}) > 0$ which prove that, the matrix $\widetilde{\mathcal{D}^{-1}}$ is Volterra–Lyapunov stable.

Let $\mathcal{G}_{2\times2} = \widetilde{\mathcal{D}^{-1}}$, the matrix $\mathcal{G}^{-1}$ is formulated as

$$\frac{1}{\det(\mathcal{G})} \times \begin{pmatrix} a_{22}\, \lambda^* + \left(a_{22} - \frac{c\beta\,(1-\rho)\, \theta_1 S^*}{N^*} \right) \mu & \frac{c\beta S^*}{N^*} \left(\sigma\, \theta_1 + a_{22}\right) \\ -a_{22} \rho\, \lambda^* & \left(\left((1-\rho)\, a_{11} + \rho\,\sigma\right) \theta_1 + a_{22}\, \rho\right) \frac{c\beta S^*}{N^*} + a_{11} a_{22} \end{pmatrix} \tag{25}$$

where $\det(\mathcal{G}) = (\det(\mathcal{D}))^2 \times \det(\mathcal{G}_1)$ and

$$\det(\mathcal{G}_1) = \left(\left(\left((1-\rho)\, a_{11} + \rho\,\sigma\right) \theta_1 + a_{22}\, \rho\right) \frac{c\beta S^*}{N^*} + a_{11} a_{22} \right) \left(a_{22}\, \lambda^* \right.$$

$$\left. + \left(a_{22} - \frac{c\beta\,(1-\rho)\, \theta_1 S^*}{N^*} \right) \mu \right) + a_{22}\rho\lambda^* \left(\frac{c\beta S^*}{N^*} \left(\sigma\, \theta_1 + a_{22}\right) \right) > 0 \tag{26}$$

Lemma 13. *The matrix $\mathcal{G}^{-1}$ defined in Eq. (25), is Volterra–Lyapunov stable.*

Proof. From matrix Eq. (25), it is clear that, $\mathcal{G}^{-1}_{11} > 0$, $\mathcal{G}^{-1}_{22} > 0$ and $\det(\mathcal{G}_1) > 0$ consequently, $\det(\mathcal{G}^{-1}) > 0$. Using Lemma 5 we conclude that $\mathcal{G}^{-1}$ is Volterra–Lyapunov stable.

Considering the second part of the proof, the matrix $\mathcal{C}^{-1}$ is established as follows:

$$\mathcal{C}^{-1} = \frac{1}{\det(\mathcal{C})} \begin{pmatrix} \mathcal{C}^{-1}_{11} & \mathcal{C}^{-1}_{12} & \mathcal{C}^{-1}_{13} & \mathcal{C}^{-1}_{14} \\ \mathcal{C}^{-1}_{21} & \mathcal{C}^{-1}_{22} & \mathcal{C}^{-1}_{23} & \mathcal{C}^{-1}_{24} \\ \mathcal{C}^{-1}_{31} & \mathcal{C}^{-1}_{32} & \mathcal{C}^{-1}_{33} & \mathcal{C}^{-1}_{34} \\ \mathcal{C}^{-1}_{41} & \mathcal{C}^{-1}_{42} & \mathcal{C}^{-1}_{43} & \mathcal{C}^{-1}_{44} \end{pmatrix} \tag{27}$$

where

$$\mathcal{C}^{-1}_{11} = a_{11} a_{22} a_{33} \left(1 - \mathcal{R}_0 \frac{S^*}{N^*} \right) > 0,$$

$$\mathcal{C}^{-1}_{12} = -\left(\left(\sigma\, \theta_1 + a_{22}\right) a_{33} + \delta\,\sigma\,\theta_2 \right) \frac{c\beta S^*}{N^*} < 0,$$

$$\mathcal{C}^{-1}_{13} = -\left(a_{33}\, \theta_1 + \delta\, \theta_2 \right) a_{11} \frac{c\beta S^*}{N^*} < 0,$$

$$\mathcal{C}^{-1}_{14} = -a_{22} a_{11} \frac{c\beta S^*}{N^*} \theta_2 < 0,$$

$$\mathcal{C}^{-1}_{21} = \rho a_{33} a_{22} \lambda^* > 0,$$

$$\mathcal{C}^{-1}_{22} = a_{22} a_{33} \left(\mu + \lambda^*\right) - \mu \left(a_{33}\theta_1 + \delta\theta_2\right) \frac{c\beta\,(1-\rho)\, S^*}{N^*} > 0,$$

$$\mathcal{C}^{-1}_{23} = \left(a_{33}\, \theta_1 + \delta\, \theta_2 \right) \frac{c\beta S^*}{N^*} \rho\, \mu > 0,$$

$$\mathcal{C}^{-1}_{24} = a_{22} \rho\, \theta_2\, \mu \frac{c\beta S^*}{N^*} > 0,$$

$$\mathcal{C}^{-1}_{31} = \left((1-\rho)\, a_{11} + \rho\,\sigma \right) a_{33}\, \lambda^* > 0,$$

$$\mathcal{C}^{-1}_{32} = \left(\left(\frac{c\beta\,(1-\rho)\, S^*}{N^*} + \sigma \right) \mu + \sigma\, \lambda^* \right) a_{33} > 0,$$

$$\mathcal{C}^{-1}_{33} = \left(\lambda^* a_{11} + \mu \left(a_{11} - \rho \frac{c\beta S^*}{N^*} \right) \right) a_{33} > 0,$$

$$\mathcal{C}^{-1}_{34} = \left((1-\rho)\, a_{11} + \rho\,\sigma \right) \mu\, \theta_2 \frac{c\beta S^*}{N^*} > 0,$$

$$\mathcal{C}^{-1}_{41} = \left((1-\rho)\, a_{11} + \rho\,\sigma \right) \lambda^*\, \delta > 0,$$

$$\mathcal{C}^{-1}_{42} = \left(\left(\frac{c\beta\,(1-\rho)\, S^*}{N^*} + \sigma \right) \mu + \sigma\, \lambda^* \right) \delta > 0,$$

$$\mathcal{C}^{-1}_{43} = \left(\lambda^* a_{11} + \mu \left(a_{11} - \frac{c\beta\rho S^*}{N^*} \right) \right) \delta > 0,$$

$$\mathcal{C}^{-1}_{44} = \left(a_{22} - \frac{c\beta\,(1-\rho)\, \theta_1 S^*}{N^*} \right) \mu a_{11} + \left(a_{11} a_{22}\, \lambda^* - \frac{c\beta\rho S^*}{N^*} \left(\sigma\, \theta_1 + a_{22}\right) \mu \right) > 0,$$

and $\det(\mathcal{C}) = a_{11} a_{22} a_{33} \left(\left(1 - \frac{S^*}{N^*} \mathcal{R}_0 \right) \mu + \lambda^* \right) > 0$ are the simplifying factors. The following Lemma is established

Lemma 14. *The matrix $\mathcal{C}^{-1}$ defined in Eq. (27), is diagonal stable.*

Proof. We need to prove that $\det(\mathcal{C}^{-1}) > 0$, which is given by $\det(\mathcal{C}^{-1}) = \det(\mathcal{C})^2 \times \det(\mathcal{C}^*)$. It suffices to show that $\det(\mathcal{C}^*) > 0$ (See Appendix "The proof of $\det(\mathcal{C}^*) > 0$"), consequently, $\det(\mathcal{C}^{-1}) > 0$ and also $C_{44}^{-1} > 0$. Based on Lemma 6 the proof of diagonal stability of the matrices $\mathcal{J}$ and $\mathcal{J}^{-1}$ where $\mathcal{J} = \widetilde{\mathcal{C}^{-1}}$ complete this proof.

The matrices $\mathcal{J}$ and $\mathcal{J}^{-1}$ are formulated such that

$$\mathcal{J} = \frac{1}{\det(\mathcal{C})}\begin{pmatrix} C_{11}^{-1} & C_{12}^{-1} & C_{13}^{-1} \\ C_{21}^{-1} & C_{22}^{-1} & C_{23}^{-1} \\ C_{31}^{-1} & C_{32}^{-1} & C_{33}^{-1} \end{pmatrix} \tag{28}$$

and

$$\mathcal{J}^{-1} = \frac{1}{\det(\mathcal{J}^{-1})} \times \begin{pmatrix} C_{22}^{-1}C_{33}^{-1} - C_{23}^{-1}C_{32}^{-1} & -C_{12}^{-1}C_{33}^{-1} + C_{13}^{-1}C_{32}^{-1} & C_{12}^{-1}C_{23}^{-1} - C_{13}^{-1}C_{22}^{-1} \\ -C_{21}^{-1}C_{33}^{-1} + C_{23}^{-1}C_{31}^{-1} & C_{11}^{-1}C_{33}^{-1} - C_{13}^{-1}C_{31}^{-1} & -C_{11}^{-1}C_{23}^{-1} + C_{13}^{-1}C_{21}^{-1} \\ C_{21}^{-1}C_{32}^{-1} - C_{22}^{-1}C_{31}^{-1} & -C_{11}^{-1}C_{32}^{-1} + C_{12}^{-1}C_{31}^{-1} & C_{11}^{-1}C_{22}^{-1} - C_{12}^{-1}C_{21}^{-1} \end{pmatrix} \tag{29}$$

where $\det(\mathcal{J}^{-1}) = (\det(\mathcal{C}))^2 \times \det(\mathcal{J}^*)$. In this case, we know that $\det(\mathcal{J}^{-1}) > 0$ since $\det(\mathcal{J}^*) > 0$ (See Appendix "The proof of $\det(\mathcal{J}^*) > 0$").

Lemma 15. *The matrix $\mathcal{J}$ defined in Eq. (28), is diagonal stable.*

Proof. From Eq. (28) we have $C_{33}^{-1} > 0$ suggesting that the matrix $\mathcal{J}$ is diagonal stable if and only if $\widetilde{\mathcal{J}}$ and $\widetilde{\mathcal{J}}^{-1}$ are Volterra–Lyapunov stable.

Lemma 16. *The matrix $\mathcal{K} = \mathcal{J}^{-1}$ defined in Eq. (29), is diagonal stable.*

Proof. From Eq. (29) we have $\det(\mathcal{K}) = (\det(\mathcal{J}))^2 \times \det(\mathcal{K}^*)$. It is clear that $\det(\mathcal{K}) > 0$ if and only if $\det(\mathcal{K}^*) > 0$. After tedious work of algebraic computations we prove that $\det(\mathcal{K}) > 0$, $\mathcal{J}_{11}^{-1} > 0$ and $\mathcal{J}_{33}^{-1} > 0$ (See Appendices "The proof of $\mathcal{J}_{11}^{-1} > 0$" and "The proof of $\mathcal{J}_{33}^{-1} > 0$"). This suggests that the matrix $\mathcal{K} = \mathcal{J}^{-1}$ is diagonal stable if and only if matrices $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{K}}^{-1}$ are Volterra–Lyapunov stable.

The reduced matrix $\widetilde{\mathcal{J}}$ and its inverse $\widetilde{\mathcal{J}}^{-1}$ may be presented as follows:

$$\widetilde{\mathcal{J}} = \frac{1}{\det(\mathcal{C})}\begin{pmatrix} C_{11}^{-1} & C_{12}^{-1} \\ C_{21}^{-1} & C_{22}^{-1} \end{pmatrix} \text{ and } \widetilde{\mathcal{J}}^{-1} = \frac{1}{\det(\widetilde{\mathcal{J}})}\begin{pmatrix} C_{22}^{-1} & -C_{12}^{-1} \\ -C_{21}^{-1} & C_{11}^{-1} \end{pmatrix} \tag{30}$$

where $\det(\widetilde{\mathcal{J}}) = (\det(\mathcal{C}))^2 \times \det(\widetilde{\mathcal{J}}^*)$.

Lemma 17. *The matrices $\widetilde{\mathcal{J}}$ and $\widetilde{\mathcal{J}}^{-1}$ defined in Eq. (29), are Volterra–Lyapunov stable.*

Proof. In this regards, using Lemma 5 we examine the signs of entries C_{11}^{-1}, C_{22}^{-1} and the determinant $\det(\widetilde{\mathcal{J}})$ whereas $C_{11}^{-1} > 0$, $C_{22}^{-1} > 0$ and $\det(\widetilde{\mathcal{J}}) = (\det(\mathcal{C}))^2 \times (C_{11}^{-1}C_{22}^{-1} - C_{12}^{-1}C_{21}^{-1})$. The entry $C_{12}^{-1} < 0$ making $\det(\widetilde{\mathcal{J}}) > 0$, clearly both matrices $\widetilde{\mathcal{J}}$ and $\widetilde{\mathcal{J}}^{-1}$ Volterra–Lyapunov stable.

On the other hand, considering the matrices $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{K}}^{-1}$ below, the following Lemma is established

$$\widetilde{\mathcal{K}} = \frac{1}{\det(\mathcal{J})}\begin{pmatrix} C_{22}^{-1}C_{33}^{-1} - C_{23}^{-1}C_{32}^{-1} & -C_{12}^{-1}C_{33}^{-1} + C_{13}^{-1}C_{32}^{-1} \\ -C_{21}^{-1}C_{33}^{-1} + C_{23}^{-1}C_{31}^{-1} & C_{11}^{-1}C_{33}^{-1} - C_{13}^{-1}C_{31}^{-1} \end{pmatrix} \text{ and}$$

$$\widetilde{\mathcal{K}}^{-1} = \frac{1}{\det(\widetilde{\mathcal{K}})}\begin{pmatrix} C_{11}^{-1}C_{33}^{-1} - C_{13}^{-1}C_{31}^{-1} & C_{12}^{-1}C_{33}^{-1} - C_{13}^{-1}C_{32}^{-1} \\ C_{21}^{-1}C_{33}^{-1} - C_{23}^{-1}C_{31}^{-1} & C_{22}^{-1}C_{33}^{-1} - C_{23}^{-1}C_{32}^{-1} \end{pmatrix}. \tag{31}$$

where $\det(\widetilde{\mathcal{K}}^{-1}) = (\det(\mathcal{J}))^2 \times \det(\widetilde{\mathcal{K}}^*)$.

Lemma 18. *The matrices $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{K}}^{-1}$ defined in Eq. (31), are Volterra–Lyapunov stable.*

Proof. Utilizing Lemma 5 it can be shown that the values $C_{22}^{-1}C_{33}^{-1} - C_{23}^{-1}C_{32}^{-1} > 0$, $C_{11}^{-1}C_{33}^{-1} - C_{13}^{-1}C_{31}^{-1} > 0$, $\det(\widetilde{\mathcal{K}}) = (\det(\mathcal{J}))^2 \times \det(\widetilde{\mathcal{K}}^*) > 0$ and $\det(\widetilde{\mathcal{K}}^{-1}) > 0$. It can be concluded that, the matrices $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{K}}^{-1}$ defined in Eq. (31), are Volterra–Lyapunov stable, hence $\mathcal{A}_{5\times5}$ is indeed Volterra–Lyapunov stable.

Summarizing the above discussions, we have the following conclusions for the globally asymptotically stability of the endemic equilibrium.

Theorem 19. *When $\mathcal{R}_0 > 1$ the endemic equilibrium $E_1 = (S^*, L^*, M^*, H^*, R^*)$ of Model Eq. (1) is globally asymptotically stable, in Ω.*

Proof. Theorem 7 with the aid Lemmas 8 and 14 guarantee that the endemic equilibrium of the model System (1) is globally asymptotically stable.

Numerical simulations and discussions

In this section, we present some examples of numerical simulations of system Eq. (1) using the basic risk reproduction number $\mathcal{R}_0$ to support the analytical results.

Simulations

Example 1. Consider system Eq. (1) with the parameters $\mu = 0.002$, $\pi = 0.31$, $\alpha = 0.005$, $\delta = 0.075$, $\sigma = 0.01$ [24], $\xi = 0.0025$ [25], $\eta = 0.005$ [25], $\omega = 0.001$ [24], $\theta_1 = 0.02$; $\theta_2 = 0.05$, $\rho = 0.065$, $\beta = 0.025$, $c = 2.4$.

The selected set of values produce the basic risk reproduction number less than one ($\mathcal{R}_0 = 0.5732 < 1$), in this case, the system Eq. (1) has only the risk-free equilibrium of $E_0 = (155, 0, 0, 0, 0)$. The numerical simulation results of system Eq. (1) presented in Fig. 1, we observe five distinct solution curves by phase portraits of $S(t)$, $H(t)$ and $R(t)$ emanating from setting up five different initial values. While we maintained the values for $L(0) = M(0) = 20$, the initial values for $S(t)$, $H(t)$ and $R(t)$ were varied to validate the stability of the risk-free equilibrium. In the long run, the phase portrait converged to the risk-free equilibrium point where $S_0^* = 155$, $H_0^* = 0$, $R_0^* = 0$. In Fig. 2, we observe that the five orbits converge to the E_0 at $L_0^* = M_0^* = H_0^* = 0$, with five different initial conditions.

Example 2. Consider system Eq. (1) with the parameters $\mu = 0.002$, $\pi = 0.31$, $\alpha = 0.005$, $\delta = 0.075$, $\sigma = 0.01$ [24], $\xi = 0.0025$ [25], $\eta = 0.005$ [25], $\omega = 0.001$ [24], $\theta_1 = 0.02$; $\theta_2 = 0.05$, $\rho = 0.065$, $\beta = 0.025$, $c = 24$.

In this case, the value of c was allowed to increase ten times of its initial value, from $c = 2.4$ to $c = 24$ leading to a basic risk reproduction number greater than one ($\mathcal{R}_0 = 5.7322 > 1$), consequently, the system Eq. (1) gives two equilibria; the risk-free equilibrium of E_0 and the risk endemic E_1. The phase diagram of system Eq. (1) at different initial values of $S(t)$, $H(t)$ and $R(t)$, shown in Fig. 3, shows that all system responses converge to point of E^*. In Fig. 4, we see that five orbits converge to the E^*, at different initial conditions of $L(t)$, $M(t)$ and $H(t)$.

Conclusions

This paper considered the dynamics of alcohol-related risks in the epidemic $SLMHR$ model with the nonlinear force of influence λ. The conditions for the global stability of the endemic equilibrium were established using the theory of Volterra–Lyapunov stable matrices. This strategy simplifies the calculations and the proofs. The numerical examples are simulated to validate the theoretical model results.

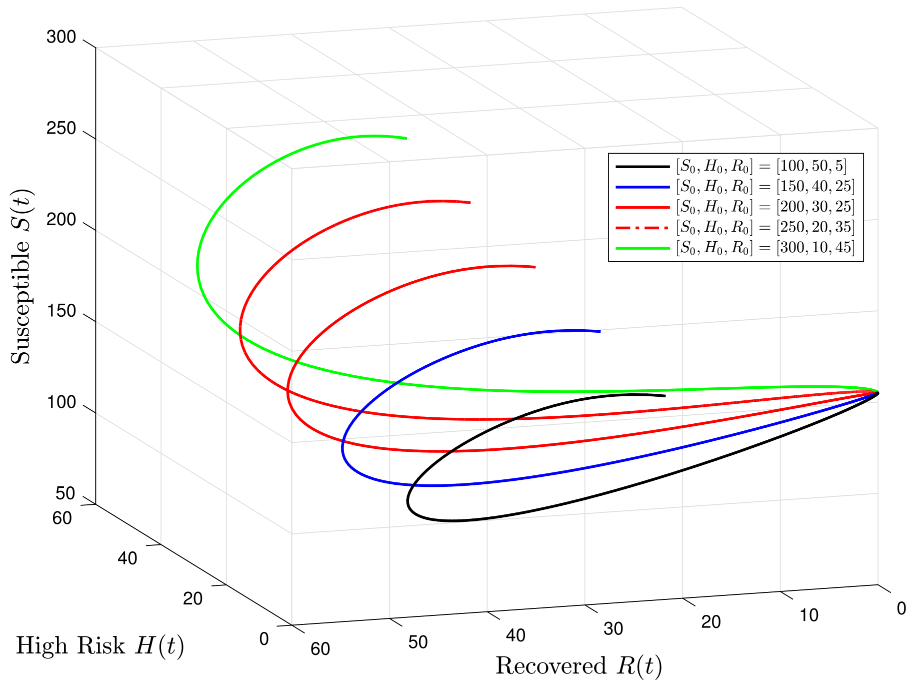

Fig. 1. The evolution dynamics of $S(t), H(t), R(t)$ populations over time when $\mathcal{R}_0 = 0.5732 < 1$.

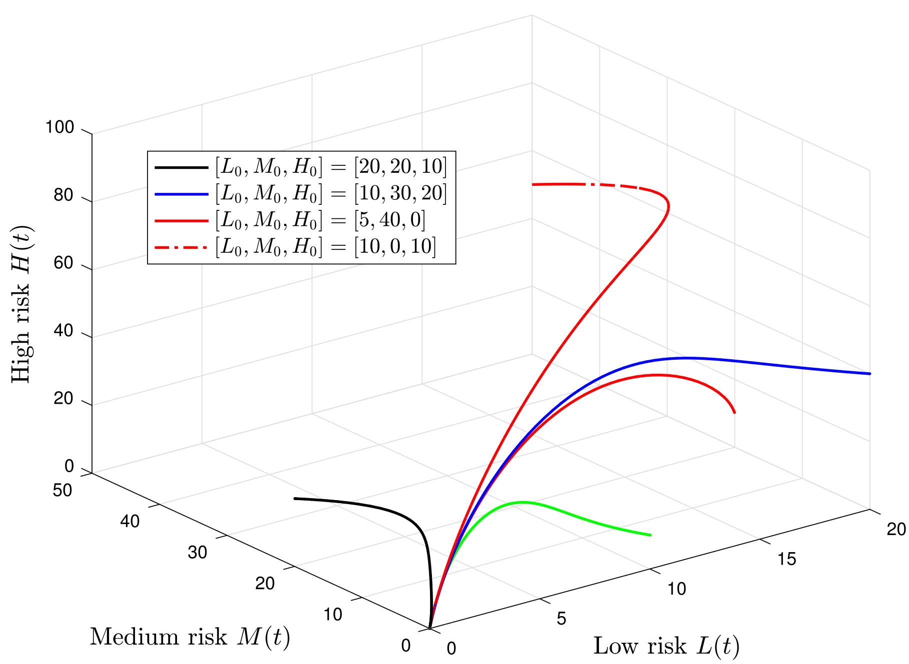

Fig. 2. The evolution dynamics of $L(t)$, $M(t)$, $H(t)$ populations over time when $\mathcal{R}_0 = 0.5732 < 1$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix. Some Proofs

The proof of $\det(C^*) > 0$

Consider the matrix C^* below

$$C^* = \begin{pmatrix} C_{11}^{-1} & C_{12}^{-1} & C_{13}^{-1} & C_{14}^{-1} \\ C_{21}^{-1} & C_{22}^{-1} & C_{23}^{-1} & C_{24}^{-1} \\ C_{31}^{-1} & C_{32}^{-1} & C_{33}^{-1} & C_{34}^{-1} \\ C_{41}^{-1} & C_{42}^{-1} & C_{43}^{-1} & C_{44}^{-1} \end{pmatrix} \tag{A.1}$$

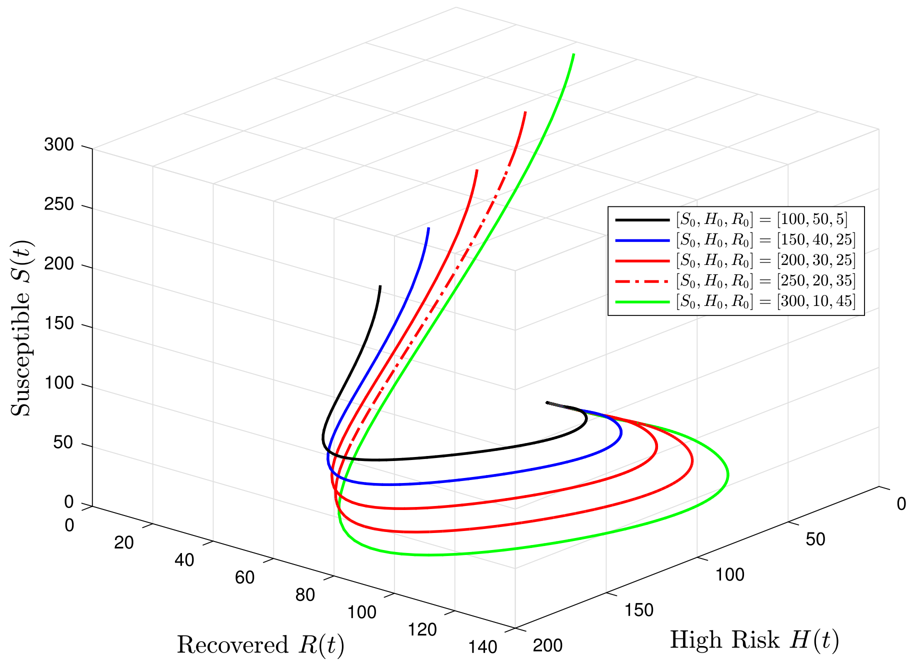

Fig. 3. The evolution dynamics of $S(t), H(t), R(t)$ populations over time when $\mathcal{R}_0 = 5.7322 > 1$.

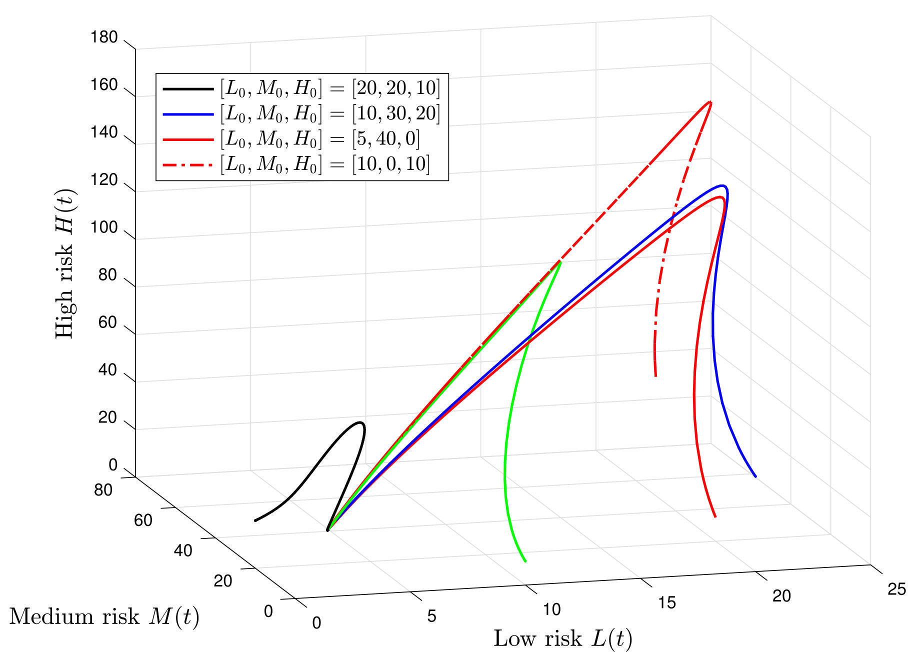

Fig. 4. The evolution dynamics of $L(t), M(t), H(t)$ populations over time when $\mathcal{R}_0 = 5.7322 > 1$.

whose determinant, upon simplification, is given by

$$\det(C^*) = \frac{1}{N^{*3}}\Big[\left(a_{11}a_{22}a_{33}\left(\mathcal{R}_0 S^* + \lambda^* N^* + \mu\left(N^* - \mathcal{R}_0 S^*\right)\right)\right) \left(\mu\left(N^* - \mathcal{R}_0 S^*\right) + \lambda^* N^*\right)^2 a_{11}^2 a_{22}^2 a_{33}^2\Big].$$

Utilizing Eq. (12) at risk endemic, we have $N^* - \mathcal{R}_0 S^* = 0$. Therefore, we have

$$\det(C^*) \quad = \quad \frac{1}{N^*}\left[\left(\mathcal{R}_0 S^* + \lambda^* N^*\right)\lambda^{*2} a_{11}^3 a_{22}^3 a_{33}^3\right]$$

which guarantees $\det(C^*) > 0$.

The proof of $\det(\mathcal{J}^*) > 0$

Consider the matrix $\mathcal{J}^*$ below

$$\mathcal{J}^* = \begin{pmatrix} C_{11}^{-1} & C_{12}^{-1} & C_{13}^{-1} \\ C_{21}^{-1} & C_{22}^{-1} & C_{23}^{-1} \\ C_{31}^{-1} & C_{32}^{-1} & C_{33}^{-1} \end{pmatrix} \tag{A.2}$$

whose determinant, upon simplification, is given by

$$\det(\mathcal{J}^*) \quad = \quad \frac{1}{N^2}\left(a_{33}\ \left(a_{11}a_{22}a_{33}\mu\left(N^* - \mathcal{R}_0 S^*\right) + N^* a_{11}a_{22}a_{33}\lambda^*\right)\ldots \ldots\left(a_{11}a_{22}a_{33}\left(\mu\left(N^* - \mathcal{R}_0 S^*\right) + \lambda^* N^*\right)\right)\right).$$

At risk endemic, $N^* - \mathcal{R}_0 S^* = 0$, consequently

$\det(\mathcal{J}^*) = a_{11}^2 a_{22}^2 a_{33}^3 \lambda^{*2} > 0.$

The proof of $\mathcal{J}_{11}^{-1} > 0$

Given that $\mathcal{J}_{11}^{-1} = C_{22}^{-1} C_{33}^{-1} - C_{23}^{-1} C_{32}^{-1}$ which expands into

$$\begin{aligned}\mathcal{J}_{11}^{-1} = \frac{1}{N} \Big(\big(\big(-S^* c\beta \left((1-\rho)\left(\theta_1 a_{33} + \delta\theta_2\right) a_{11} \right.\\ \left. +\rho \left(\left(\sigma\theta_1 + a_{22}\right) a_{33} + \sigma\delta\theta_2 \right) \right)\\ +N^* a_{11} a_{22} a_{33} \big) \mu + N^* a_{11} a_{22} a_{33}\, \lambda^* \big) \left(\mu + \lambda^*\right) a_{33} \Big)\end{aligned}$$

which can be simplified into

$$\mathcal{J}_{11}^{-1} = \frac{1}{N} \left(\left(\left(N^* - \mathcal{R}_0 S^*\right) \mu + N^* \lambda^* \right) (\mu + \lambda^*)\, a_{11} a_{22} a_{33}^2 \right) .$$

Utilizing Eq. (12) at risk endemic, we have $N^* - \mathcal{R}_0 S^* = 0$ which implies that $\mathcal{J}_{11}^{-1} > 0$.

The proof of $\mathcal{J}_{33}^{-1} > 0$

Given that $\mathcal{J}_{33}^{-1} = C_{11}^{-1} C_{22}^{-1} - C_{12}^{-1} C_{21}^{-1}$ which expands into

$$\begin{aligned}\mathcal{J}_{33}^{-1} = \frac{1}{N^2} \left(a_{11} a_{22} a_{33} \left(N^* - \mathcal{R}_0 S^*\right) \left(a_{22} a_{33} \left(\mu + \lambda^*\right) N^* \right.\right.\\ \left.\left. - S^* c\mu\, \beta\, (1-\rho) \left(\theta_1 a_{33} + \delta\,\theta_2\right) \right) \right) +\\ \frac{1}{N} \left(\left(\left(\sigma\,\theta_1 + a_{22}\right) a_{33} + \delta\,\sigma\,\theta_2 \right) c\beta\, S^* \rho a_{33} a_{22} \lambda^* \right)\end{aligned}$$

which, upon substituting $N^* - \mathcal{R}_0 S^* = 0$, can be simplified into

$$\mathcal{J}_{33}^{-1} = \frac{1}{N} \left(\left(\left(\sigma\,\theta_1 + a_{22}\right) a_{33} + \delta\,\sigma\,\theta_2 \right) c\beta\, S^* \rho a_{33} a_{22} \lambda^* \right) > 0 .$$

The proof of $\det(\widetilde{\mathcal{K}}^*) > 0$

Consider the matrix $\mathcal{J}^*$ below

$$\widetilde{\mathcal{K}}^* = \begin{pmatrix} C_{22}^{-1} C_{33}^{-1} - C_{23}^{-1} C_{32}^{-1} & -C_{12}^{-1} C_{33}^{-1} + C_{13}^{-1} C_{32}^{-1} \\ -C_{21}^{-1} C_{33}^{-1} + C_{23}^{-1} C_{31}^{-1} & C_{11}^{-1} C_{33}^{-1} - C_{13}^{-1} C_{31}^{-1} \end{pmatrix} \tag{A.3}$$

whose determinant, upon simplification, is given by

$$\begin{aligned}\det(\widetilde{\mathcal{K}}^*) = \frac{1}{N^3} \left({a_{33}}^2 \left(a_{11} a_{22} a_{33} \left(\mu \left(N^* - \mathcal{R}_0 S^*\right) \right.\right.\right.\\ \left.\left. +\ \lambda^* N^* \right) \right) \left(\left(\mu + \lambda^*\right) N^* \right.\\ \left. - \mu \mathcal{R}_0 S^* \right) \left(a_{11}\ \left(\mu + \lambda^*\right) N^* - S^* \beta\, c\mu\, \rho \right) a_{11} a_{22} a_{33} \Big) .\end{aligned}$$

Utilizing Eq. (12) at risk endemic, we substitute $N^* - \mathcal{R}_0 S^* = 0$ and simplification gives,

$$\det(\widetilde{\mathcal{K}}^*) = \frac{1}{N} \left(a_{11}^2 a_{22}^2 a_{33}^4 \lambda^{*2} \left(\mu \left(a_{11} N^* - \beta\, c\rho S^* \right) + \lambda^* N^* \right) \right) .$$

We know that $a_{11} - c\beta\rho \left(\dfrac{S^*}{N^*} \right) > 0$ implying that $a_{11} N^* - \beta\, c\rho S^* > 0$ confirming the positiveness of $\det(\widetilde{\mathcal{K}}^*)$.

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