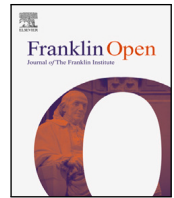2023-09

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Fanuel, Ibrahim

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Fuzzy modelling on the depletion of forest biomass and forest-dependent wildlife population

Ibrahim M. Fanuel [a,c,*], Silas Mirau [a], Maranya Mayengo [a], Francis Moyo [b]

[a] *Department of Applied Mathematics and Computational Science, Nelson Mandela African Institution of Science and Technology, Arusha, Tanzania*
[b] *Department of Biodiversity Conservation and Ecosystem Management, Nelson Mandela African Institution of Science and Technology, Arusha, Tanzania*
[c] *Department of ICT and Mathematics, College of Business Education, Tanzania*

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ABSTRACT

This paper presents a system of non-linear differential equations describing the depletion of forest biomass and forest-dependent wildlife population caused by human population and its associated activities. The model incorporates the imprecise nature of the parameters, which are treated as triangular fuzzy numbers to reflect the inherent uncertainty. We utilised α cut to transform these imprecise parameters into intervals. Subsequently, employing the principles of interval mathematics, we effectively converted the related differential equation into a pair of distinct differential equations. By leveraging the signed distance of the fuzzy numbers, we further simplified the equations, resulting in a single differential equation, which led to the formulation of a defuzzified model. The existence of equilibrium points with their stability behaviour is presented. Furthermore, the existence of trans-critical bifurcation is analysed. Through numerical simulations, we observe significant differences between the solutions of system in crisp and fuzzy environments. These findings highlight the importance of using fuzzy models to accurately represent the dynamics of complex natural systems. Consequently, we conclude that fuzzy models provide a trustworthy representation of the dynamics of complex natural systems.

1. Introduction

Forests provide a variety of significant benefits, including ecological, economic, and social advantages [1]. Ecologically, they are crucial for preserving the world's biodiversity, providing habitat for an extensive range of plant and wildlife species [2]. To human, forests are the source of wood, energy, medicine, and fodder [3]. Furthermore, they provide vital ecosystem services such as carbon sequestration, soil conservation, and watershed protection. Forests also have cultural and spiritual significance for local communities and offer opportunities for recreation and tourism. Despite the benefits that forests provide, they are threatened by a wide range of human activities [1]. These include unsustainable harvesting; human settlement encroachment; conversion of the forest land; tourism and recreational pressure; forest fires; as well as mining and fossil fuel exploitation [4].

Studies show that, around 1.6 billion people rely totally or partially on forest resources for their livelihood [5], attempts to completely limit people's utilisation of the forest resources is not possible. To ensure sustainable forest utilisation, understanding of the dynamics of human–forest interaction is of utmost important. There is considerable number

of theoretical and experimental studies in the pursuit of understanding the interactions between human and forest. In the realm of theoretical studies which is the focus of this paper, mathematical models have become important tools for analysing the impacts of human population and its associated activities on forest and forest-dependent wildlife species [6].

In 1996, Shukla et al. [7] developed and analysed a non-linear mathematical model to study the effect of changing habitat on survival of wildlife species. According to the study, uncontrolled population pressure leads to a decline in forest biomass density, which in turn leads to a reduction in wildlife species density and, ultimately, extinction. Similar studies developed and analysed models to study the effects of forest depletion on the survival of wildlife species [8–10]. The findings were consistence with Shukla et al. [7], that uncontrolled human population and its associated activities have directly and indirect negative effects on the survival of wildlife species. Furthermore, studies identify parameters threshold for the systems to remain permanent. Recently, Sinha et al. [11] examined the negative impact of industrial activities on forests and wildlife. The results showed that the density of forest
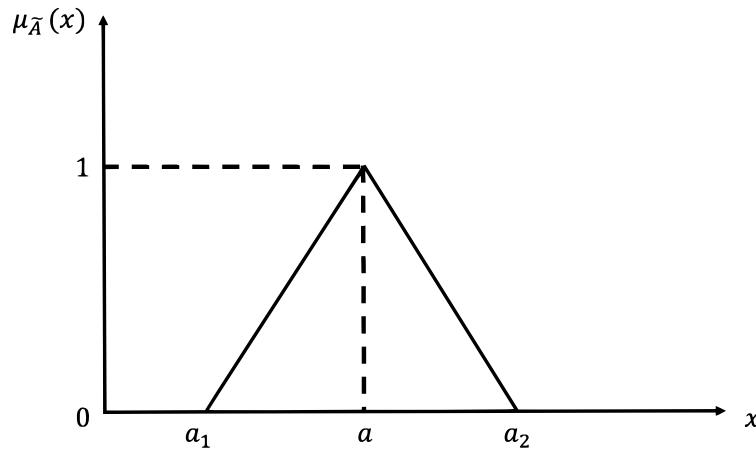
Fig. 1. Triangular fuzzy number.

biomass and wildlife population decreases as industrial activities in forest regions increase. That is in the long run, to avoid the extinction of wildlife species, the expansion of forests biomass should be very high as compared to their uses by industries to fulfil the needs of human population.

It can be noted that, most of works modelled human–forest interactions in a crisp environment under the assumption that all parameters are certain. But, in reality, biological parameters are not always fixed as they vary with the surrounding environment. For instance, the growth rate of forest biomass can differ based on factors such as soil fertility, presence of pollinators, rainfall, and other variables, which are not uniform throughout. Given this perspective, the model developed in fuzzy environment is considered to be meaningful than the crisp model [12]. In doing so, the parameters can be treated as fuzzy numbers, and the formulated model will take into account the variability and uncertainty of the parameters that influence the dynamics of forest–human interactions.

The theory of fuzzy sets was first introduced by Zadeh [13]. Since then, fuzzy sets have been applied to a variety of fields, including mathematical modelling. Barros et al. [14] utilised the concept of fuzzy differential equations in population dynamics to account for uncertainty in the model system. However, there is a limited amount of research available that incorporates uncertainty in the parameters of models that analyse human–forest interactions. This is a notable gap in the literature, as many of the parameters that affect forest ecosystems are uncertain due to the complexity and variability of the natural systems involved. For example, Pal et al. [15,16] applied fuzzy differential equation and Utility Function Method (UFM) to analyse the stability and bifurcation of harvesting predator–prey of ecological systems. The findings reveal that the model in fuzzy environment are good representation of the biological interactions. However, it is worth noting that while the UFM is suitable for representing complex relationships between variables, the signed distance approach is the preferred method for optimising models and analysing their behaviour [17]. Furthermore, Panja [18] developed a fuzzy parameters based model on depletion and conservation of forest biomass. The study includes three compartments, forest biomass, human population and technological efforts. The findings revealed that the fuzzy parameters have effects on the developed model as the equilibrium levels between crisp and fuzzy models were significantly different.

Motivated by these works, we developed the model to study the impact of human activities on forest biomass and forest-dependent wildlife population. The model consists of four compartments: forest biomass, forest-dependent wildlife population, human population, and human activities. We represented the corresponding model parameters as triangular fuzzy numbers to account for parameter uncertainty. The use of triangular fuzzy membership offers advantages such as intuitive

nature, computational simplicity, and the ability to facilitate representation [16]. Furthermore, we employed the Signed Distance (SD) measure to evaluate the degree of similarity or dissimilarity between two fuzzy sets. This optimisation approach enables us to gain insights into the behaviour of the system within a fuzzy environment.

The rest of the paper is organised as follows: Section 2 introduces some fundamental mathematical concepts related to fuzzy sets. Section 3 outlines the development of the proposed model. Equilibrium analysis of the model is presented in Section 4, while stability analysis is presented in Section 5. Section 6 presents the numerical solution, and Section 7 concludes the paper.

2. Preliminaries

This section provides an overview of some fundamental mathematical definitions that have been employed to investigate the depletion of forest biomass in a fuzzy environment. The definitions are presented and discussed in brief, the details can be obtained from the cited references.

Definition 1 (*Fuzzy Set [13,16]*). A fuzzy set $\widetilde{A}$ in a universe of discourse set X is defined as the set of pairs $\widetilde{A} = \left\{ \left(x, \mu_{\widetilde{A}}(x)\right) : x \in X \right\}$. The mapping $\mu_{\widetilde{A}}(x) : X \mapsto [0,1]$ is the membership function of fuzzy set A, and $\mu_{\widetilde{A}}(x)$ is the membership value or degree of membership of $x \in X$ in a fuzzy set $\widetilde{A}$.

Definition 2 (*Triangular Fuzzy Number (TFN) [19]*). A Triangular Fuzzy Number $\widetilde{A} = \left(a_1, a, a_2\right)$ (see Fig. 1) is fuzzy set of the real number $\mathbb{R}$ characterised by its continuous membership function $\mu_{\widetilde{A}}(x) : X \mapsto [0,1]$ as follows

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a - a_1}, & \text{if } a_1 \leq x \leq a, \\ \dfrac{a_2 - x}{a_2 - a}, & \text{if } a \leq x \leq a_2, \\ 0 & \text{Otherwise.} \end{cases}$$

Definition 3 (*α–Cut of a Fuzzy Number [19]*). The α–cut of a fuzzy number $\widetilde{A}$ in X (see Fig. 2) is the crisp set

$$A(\alpha) = \left\{ x \in X : \mu_{\widetilde{A}}(x) \geq \alpha \right\} \forall \alpha \in [0,1],$$

$A(\alpha)$ is a non empty bounded closed interval contained in X defined as $A(\alpha) = \left[A_l(\alpha), A_r(\alpha)\right]$, such that $A_l(\alpha)$ and $A_r(\alpha)$ are lower and upper bounds of the closed interval, respectively. Thus, the α–cut of a TFN $\widetilde{A}$ is the closed and bounded interval $\left[A_l(\alpha), A_r(\alpha)\right]$, such that $A_l(\alpha) = \inf \left\{ x : \mu_{\widetilde{A}}(x) \geq \alpha \right\} = a_1 + \alpha(a - a_1)$, and $A_r(\alpha) = \sup \left\{ x : \mu_{\widetilde{A}}(x) \geq \alpha \right\} = a_2 - \alpha(a_2 - a)$.

Table 1
Description of fuzzy parameters.

| Param	Description
$\widetilde{s}$	Fuzzy intrinsic growth rate coefficients of forest biomass
$\widetilde{\beta}_0$	Fuzzy depletion rate coefficients of forest biomass due to wildlife population under natural processes
$\widetilde{\eta}$	Fuzzy proportionality constants which represent the growth of wildlife population in the presence of forest biomass
$\widetilde{\beta}_1$	Fuzzy depletion rate coefficients of forest biomass due to human population
$\widetilde{\beta}_2$	Fuzzy depletion rate coefficients of forest biomass due to human activities
$\widetilde{v}_1$	Fuzzy depletion rate coefficients of wildlife population due to human population
$\widetilde{v}_2$	Fuzzy depletion rate coefficients of wildlife population due to human activities
$\widetilde{\theta}$	Fuzzy intrinsic growth rate coefficients of human population
$\widetilde{\lambda}$	Fuzzy proportionality constants which represent the growth of human population in the presence of forest biomass
$\widetilde{\sigma}$	Fuzzy depletion rate coefficients of human population due to wild animals
$\widetilde{\gamma}$	Fuzzy growth rate coefficients of human activities due to human population
$\widetilde{\gamma}_1$	Fuzzy natural depletion rate coefficients of human activities
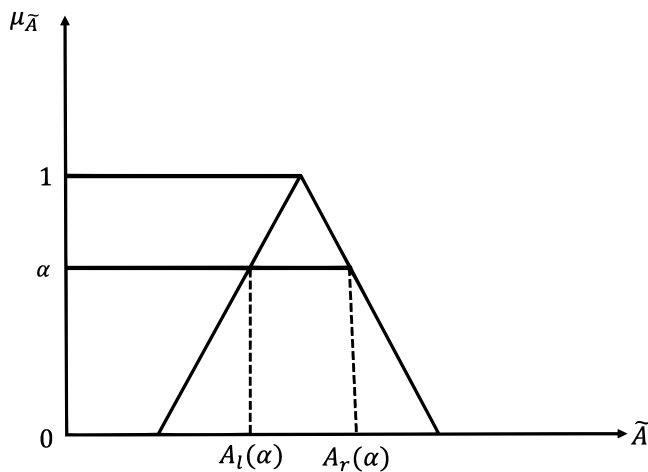
Fig. 2. The α–cut of a triangular fuzzy number.

Definition 4 (*Interval Arithmetic [20]*). Let $[P_l, P_r]$ and $[Q_l, Q_r]$ be two interval numbers, the addition and subtraction of the interval numbers is given as follows:

$$[P_l, P_r] \pm [Q_l, Q_r] = [P_l \pm Q_l, P_r \pm Q_r].$$

Definition 5 (*Signed Distance of Fuzzy Numbers [12,21]*). Assume $\widetilde{G}$ is a fuzzy number, then the signed distance of $\widetilde{G}$ measured from $\widetilde{0}$ denoted by $d\left(\widetilde{G}, \widetilde{0}\right)$ is given as

$$d\left(\widetilde{G}, \widetilde{0}\right) = \frac{1}{2} \int_0^1 \left(\widetilde{G}_l\left(\alpha\right) + \widetilde{G}_r\left(\alpha\right)\right) d\alpha.$$

3. Model formulation

We considered the crisp model developed by Fanuel et al. [22], presented in Eq. (1), as the foundation of our study. The model assumes that $B(t)$ is the density of forest biomass in the region under study at any given time, while $W(t)$ represents the density of forest-dependent wildlife population. Furthermore, $N(t)$ represents the human population, and $H(t)$ represents the associated human activities within the same region at time t.

$$\begin{cases} \dot{B} = sB\left(1 - \frac{B}{L}\right) - \beta_0 BW - \beta_1 BN - \beta_2 B^2 H, \\ \dot{W} = r(B)W\left(1 - \frac{W}{K(B)}\right) - v_1 WN - v_2 WH, \\ \dot{N} = \theta N\left(1 - \frac{N}{M}\right) + \lambda \beta_1 BN - \sigma NW, \\ \dot{H} = \gamma N - \gamma_1 H, \end{cases} \quad (1)$$

with conditions $B(0) \geq 0$, $W(0) \geq 0$, $N(0) \geq 0$, $H(0) \geq 0$, $0 \leq \lambda \leq 1$.

To delve deeper into uncertainties associated with the parameters of the model, we extended our analysis by fuzzifying the model (1). This extension allowed us to account for the inherent imprecision that often exists in real-world scenarios [12]. Suppose $\varDelta_i$ is the parameter responsible for controlling the degree of fuzziness in a fuzzy set, it follows, if $\widetilde{p}_i$ is the fuzzy parameter in the set of parameters (see Table 1) and p_i is its corresponding crisp parameter, then the parameter $\widetilde{p}_i$ is given by $\widetilde{p}_i = \left(p_i - \varDelta_i, p_i + \varDelta_{i+1}\right)$, such that, $0 < \varDelta_i < p_i$ and $\varDelta_{i+1} > 0$. Thus, on utilising Hukuhara derivative concept [23], the system of fuzzy differential Eqs. (2) describes the model in fuzzy environment.

$$\begin{cases} \frac{d\widetilde{B}}{dt} = \widetilde{s}B\left(1 - \frac{B}{L}\right) - \widetilde{\beta}_0 BW - \widetilde{\beta}_1 BN - \widetilde{\beta}_2 B^2 H, \\ \frac{d\widetilde{W}}{dt} = \widetilde{\beta}_0 \widetilde{\eta} BW\left(1 - \frac{W}{K(B)}\right) - \widetilde{v}_1 WN - \widetilde{v}_2 WH, \\ \frac{d\widetilde{N}}{dt} = \widetilde{\theta} N\left(1 - \frac{N}{M}\right) + \widetilde{\lambda}\widetilde{\beta}_1 BN - \widetilde{\sigma} NW, \\ \frac{d\widetilde{H}}{dt} = \widetilde{\gamma} N - \widetilde{\gamma}_1 H, \end{cases} \quad (2)$$

with initial conditions $B(0) \geq 0$, $W(0) \geq 0$, $N(0) \geq 0$, $H(0) \geq 0$.

The first equation characterises the forest biomass dynamics, incorporating logistic growth with an intrinsic growth rate $(\widetilde{s})$ and a carrying capacity (L). Simultaneously, the second term in the equation represents the depletion of forest biomass due to the wildlife population under natural processes through a bilinear interaction $(\widetilde{\beta}_0 BW)$. Human population exert a significant influence on forest biomass growth by utilising forest resources, this aspect is captured by the term $(\widetilde{\beta}_1 BN)$ [10]. These activities include tree cutting for wood and medicinal purposes, as well as grazing. Moreover, the model explicitly incorporates the detrimental impact of deforestation for agricultural, residential, and industrial purposes on the carrying capacity of forest resources. This decline in forest land, which cannot be replenished, is quantified through the fourth term $(\widetilde{\beta}_2 B^2 H)$ [10], highlighting the irreversible nature of this degradation process.

The second equation in the model describes the dynamics of the forest-dependent wildlife population, following a logistic growth pattern. The growth rate $(\widetilde{\beta}_0 \widetilde{\eta} B)$ and carrying capacity $(K(B))$ are influenced by the available forest biomass, reflecting the species' reliance on the forest ecosystem. The term $\widetilde{v}_1 WN$ accounts for direct wildlife depletion caused by illegal activities like poaching or smuggling [10]. However, we assume that human utilisation of wildlife has negligible effects on the human population's growth, as the human population under consideration do not depend on wildlife for livelihoods. Furthermore, the term $\widetilde{v}_2 WH$ captures the indirect decline in wildlife population due to human activities, such as habitat destruction and fragmentation.

The third equation in the model represents the dynamics of the human population, assuming logistic growth with an intrinsic rate of $\widetilde{\theta}$ and a carrying capacity of M. The term $\widetilde{\lambda}\widetilde{\beta}_1 BN$ accounts for the human

population growth due to their utilisation of forest resources [24]. Moreover, interactions with wildlife populations impact the human population as they expand and encroach upon natural habitats, leading to resource competition and safety concerns [25]. This interaction is accounted for by the term $\widetilde{\sigma} NW$. The fourth equation in the model encapsulates the dynamics of human activities, demonstrating a positive correlation with the density of the human population ($\widetilde{\gamma} N$) while exhibiting a natural decline governed by the rate $\widetilde{\gamma}_1$.

To analyse the model system (2), we transformed the fuzzy parameters into intervals by taking the α–cut on both side of the model equation (see Definition 3). This transformation allowed us to obtain intervals that represent the range of possible values for each parameter, considering their fuzzy nature. Subsequently, we leveraged the principles of interval mathematics to tackle the interval-based differential equations derived from the model. By treating the intervals as mathematical objects, we converted the original differential equation into a pair of distinct differential equations that encapsulate the possible behaviours of the system within the defined intervals. To further simplify the analysis and gain deeper insights into the system's dynamics, we utilised the concept of signed distance for the fuzzy numbers as defined in Definition 5. This approach allowed us to obtain a system of defuzzified differential Eqs. (3), which captured the essential characteristics of the system's behaviour under the influence of fuzzy parameters.

$$\begin{cases} \dfrac{dB}{dt} = S_1 B \left(1 - \dfrac{B}{L}\right) - S_2 BW - S_3 BN - S_4 B^2 H, \\ \dfrac{dW}{dt} = S_5 BW \left(1 - \dfrac{W}{K(B)}\right) - S_6 WN - S_7 WH, \\ \dfrac{dN}{dt} = S_8 N \left(1 - \dfrac{N}{M}\right) + S_9 BN - S_{10} NW, \\ \dfrac{dH}{dt} = S_{11} N - S_{12} H, \end{cases} \quad (3)$$

with initial condition $B(0) \geq 0$, $W(0) \geq 0$, $N(0) \geq 0$, $H(0) \geq 0$.

The parameters of the deffuzified model (3) are given as follows: for $d\left(\widetilde{p}_i, \widetilde{0}\right)$,

$$S_\kappa = p_i + \frac{1}{4}\left(\Delta_{i+1} - \Delta_i\right),$$

while for $d\left(\widetilde{p}_i \otimes \widetilde{p}_j, \widetilde{0}\right)$ such that $j \neq i$

$$S_\kappa = \frac{p_i p_j}{2} + \frac{p_j}{4}\left(\Delta_{i+1} - \Delta_i\right) + \frac{p_i}{4}\left(\Delta_{j+1} - \Delta_j\right) + \frac{1}{4}\left(\Delta_i \Delta_j + \Delta_{i+1}\Delta_{j+1}\right).$$

The details of defuzzification process is given in Appendix A.

3.1. Boundedness and permanence of the system

Following Chen [26], the region of attraction that describes the boundedness of the system (3) is stated in Lemma 1, while, the conditions for the system to remain permanent are stated in Theorem 2.

Lemma 1. *The set*

$$\Omega = \left\{(B, W, N, H): \; 0 \leq B \leq L, \; 0 \leq W \leq K(L), \; 0 \leq N \leq N_m, \; 0 \leq H \leq H_m\right\}$$

is the region of attraction for model system (3) and attract all solutions initiating in the interior of positive orthant, where,

$$N_m = \frac{M}{S_8}\left(S_8 + S_9 L\right), and \; H_m = \frac{S_{11} N_m}{S_{12}}.$$

Furthermore, suppose the initial condition of the system is given by

$$y_0 = \left(B_0, W_0, N_0, H_0\right),$$

then following Chaudhary [27], the system (3) is uniformly persistent if there exists positive constants τ_1 and τ_2 such that each non-negative solution

$B\left(t, y_0\right), W\left(t, y_0\right), N\left(t, y_0\right),$ and $H\left(t, y_0\right)$

of the system with initial conditions $y_0 \in \mathbb{R}_+^4$ satisfies

$$\min\left\{\liminf_{t\to\infty} B\left(t, y_0\right), \liminf_{t\to\infty} W\left(t, y_0\right), \liminf_{t\to\infty} N\left(t, y_0\right), \liminf_{t\to\infty} H\left(t, y_0\right)\right\} \geq \tau_1$$

$$\max\left\{\limsup_{t\to\infty} B\left(t, y_0\right), \limsup_{t\to\infty} W\left(t, y_0\right), \limsup_{t\to\infty} N\left(t, y_0\right), \limsup_{t\to\infty} H\left(t, y_0\right)\right\} \leq \tau_2.$$

Theorem 2. *The fuzzy model system (3) with initial condition y_0 is uniformly persistent provided inequalities $S_1 > S_2 K(L) + S_3 N_m$, $S_5 > S_6 N_m + S_7 H_m$, and $S_8 > S_{10} F_2$ hold.*

Proof. From Lemma 1, it is important to note that $0 \leq B(t) \leq L$, $0 \leq W(t) \leq K(L)$, $0 \leq N(t) \leq N_m$ and $0 \leq H(t) \leq H_m$ for sufficient large t. Similarly, from the equations of the model system (3) we obtain the following:

$$\liminf_{t\to\infty} B(t) \geq \frac{L\left(S_1 - S_2 K(L) - S_3 N_m\right)}{S_1 + LS_4 H_m} = F_1 \text{ (say)},$$

such that $S_1 > S_2 K(L) + S_3 N_m$,

$$\liminf_{t\to\infty} W(t) \geq \frac{K(F_1)\left(S_5 - S_6 N_m - S_7 H_m\right)}{S_5} = F_2 \text{ (say)},$$

such that $S_5 > S_6 N_m + S_7 H_m$,

$$\liminf_{t\to\infty} N(t) \geq \frac{M(S_8 - S_{10} F_2)}{S_8} = F_3 \text{ (say)},$$

such that $S_8 > S_{10} F_2$, and,

$$\liminf_{t\to\infty} H(t) \geq \frac{S_{11} F_3}{S_{12}} = F_4 \text{ (say)},$$

respectively. Thus, choosing

$\tau_1 = \min\left(F_1, F_2, F_3, F_4\right)$ and $\tau_2 = \max\left(L, K(L), N_m, H_m\right)$

we have the system which is permanent.

Based on Lemma 1 and Theorem 2, the solutions of the model (3) are bounded and permanent. This allows for the use of the model to investigate the effect of human–forest interaction on forest biomass. The next subsections explore the presence of equilibrium points and their stabilities.

4. Equilibrium analysis

The equilibrium points were obtained by solving the equations of the model (3) when equated to zero. Given our assumption that the wildlife population entirely depends on forest biomass, we observed that the model has a total of six equilibrium points: five boundary equilibria denoted as $E^{(0)}, E^{(1)}, E^{(2)}, E^{(3)}, E^{(4)}$, and one interior equilibrium point denoted as E^*.

$$E^{(0)}(0,0,0,0), \quad E^{(1)}(L,0,0,0), \quad E^{(2)}\left(B^{(2)}, W^{(2)}, 0, 0\right),$$

$$E^{(3)}\left(0, 0, M, \frac{MS_{11}}{S_{12}}\right), \quad E^{(4)}\left(B^{(4)}, 0, N^{(4)}, H^{(4)}\right), \quad E^*\left(B^*, W^*, N^*, H^*\right).$$

It can be observed that the existence of $E^{(0)}, E^{(1)}$ and $E^{(3)}$ is trivial, hence, the proof for their existence is omitted. Thus, we show the existence of $E^{(2)}, E^{(4)}$ and E^*.

4.1. Existence of boundary equilibria $E^{(2)}$ and $E^{(4)}$

Defining

$$K(B) = K_0 + K_1 B, \quad (4)$$

such that $K_1 > 0$, $K(0) = K_0 > 0$, and $K'(B) > 0$ for $B > 0$. Theorem 3 which is stated without proof provides sufficient conditions for the existence of $E^{(2)}$ and $E^{(4)}$.

Theorem 3. *Equilibrium points $E^{(2)}$ and $E^{(4)}$ exist provided condition (5) holds.*

$$S_1 > \max\{S_2 K_0, S_3 M\}. \tag{5}$$

The presence of $E^{(2)}$ signifies an undisturbed forest ecosystem, unaffected by the human population and its associated activities. Although this scenario may be unrealistic, it can only be achieved when the intrinsic growth rate of forest biomass surpasses the rate of depletion caused by the forest-dependent wildlife population. On the other hand, the existence of $E^{(4)}$ indicates the coexistence of forest biomass, the human population, and human activities in the absence of wildlife population. This occurs when the depletion of the forest by the human population, at its maximum carrying capacity, is lower than its intrinsic growth rate.

4.2. Existence of interior equilibrium point E^*

The points B^*, W^*, N^* and H^* are positive solution of the algebraic Eqs. (6)–(9).

$$S_1\left(1 - \frac{B^*}{L}\right) - S_2 W^* - S_3 N^* - S_4 B^* H^* = 0, \tag{6}$$

$$S_5\left(1 - \frac{W^*}{K(B^*)}\right) - S_6 N^* - S_7 H^* = 0, \tag{7}$$

$$S_8\left(1 - \frac{N^*}{M^*}\right) + S_9 B^* - S_{10} W^* = 0, \tag{8}$$

$$S_{11} N^* - S_{12} H^* = 0. \tag{9}$$

Taking Eqs. (8) and (9), after some algebraic manipulations we obtain Eqs. (10) and (11), respectively.

$$N^* = \frac{M}{S_8}\left(S_8 + S_9 B^* - S_{10} W^*\right) = h_1(B^*, W^*) \quad (Say), \tag{10}$$

$h_1(B^*, W^*)$ is non–negative provided $S_8 + S_9 B^* > S_{10} W^*$.

$$H^* = \frac{S_{11} h_1(B^*, W^*)}{S_{12}} = h_2(B^*, W^*) \quad (Say), \tag{11}$$

Upon substitution of Eqs. (10) and (11) into Eqs. (6) and (7), we obtained isoclines (12) and (13), respectively.

$$S_1\left(1 - \frac{B^*}{L}\right) - S_2 W^* - S_3 h_1(B^*, W^*) - S_4 B^* h_2(B^*, W^*) = 0 = H_1(B^*, W^*) \quad (Say), \tag{12}$$

$$S_5\left(1 - \frac{W^*}{K(B^*)}\right) - S_6 h_1(B^*, W^*) - S_7 h_2(B^*, W^*) = 0 = H_2(B^*, W^*) \quad (Say). \tag{13}$$

The isocline (12) leads us to the following inferences:

(a) When $W^* = 0$, gives

$$H_1(B^*, 0) = S_1\left(1 - \frac{B^*}{L}\right) - \frac{S_3 M\left(S_8 + S_9 B^*\right)}{S_8}$$
$$- \frac{M S_4 S_{11} B^*(S_8 + S_9 B^*)}{S_8 S_{12}} = R_1(B^*) \quad (Say). \tag{14}$$

From Eq. (12), we infer the following:

(i) When $B^* = 0$,

$$R_1(0) = S_1 - S_3 M,$$

which is positive provided condition (5) holds.

(ii) When $B^* = L$,

$$R_1(L) = -\frac{S_3 M(S_8 + S_9 L)}{S_8} - \frac{S_4 S_{11} L M(S_8 + S_9 L)}{S_8 S_{12}} < 0.$$

(iii) The derivative of $R(B^*)$ with respect to B^* gives,

$$R_1'(B^*) = -\frac{S_1}{L} - \frac{S_3 S_9 M}{S_8} - \frac{S_4 S_{11} M\left(2 S_9 B^* + S_8\right)}{S_8 S_{12}} < 0.$$

Thus, inferences (i)–(iii) lead to the conclusion that Eq. (14) has a unique positive root in the interval $(0, L)$.

(b) When $W^* \to \infty$, $B^* < 0$,

(c) If we define $\left(\dfrac{dW^*}{dB^*}\right)_1$ as the derivative of W^* with respect to B^* from the isocline (14), then we have

$$\left(\frac{dW^*}{dB^*}\right)_1 < 0.$$

Further, the isocline (13), leads us to the following conclusions:

(a) When $W^* = 0$, gives $R_2(B^*)$,

$$H_2(B^*, 0) = S_5 B^* - S_6 h_1(B^*, 0) - \frac{S_7 S_{11} h_1(B^*, 0)}{S_{12}} = R_2(B^*) \quad (Say). \tag{15}$$

We derive the following inferences form Eq. (15),

(i) When $B^* = 0$, gives

$$R_2(0) = -S_6 M - \frac{M S_{11} S_7}{S_{12}} < 0.$$

(ii) When $B^* = L$, gives

$$R_2(L) = S_5 L - \frac{S_6 M\left(L S_9 + S_8\right)}{S_8} - \frac{M S_7 S_{11}\left(L S_9 + S_8\right)}{S_8 S_{12}},$$

$R_2(L)$ is positive provided

$$S_5 L > \frac{S_6 M\left(L S_9 + S_8\right)}{S_8} + \frac{M S_7 S_{11}\left(L S_9 + S_8\right)}{S_8 S_{12}}.$$

(iii) Further, the derivative of $R_2(B^*)$ with respect to B^* gives

$$R_2'(B^*) = S_5 - \frac{v_1 S_9 M}{S_8} - \frac{S_7 S_9 S_{11} M}{S_8 S_{12}},$$

$R_2'(B^*)$ is positive, provided

$$S_5 > \frac{v_1 S_9 M}{S_8} + \frac{S_7 S_{11} S_9 M}{S_8 S_{12}}.$$

With these considerations (i–iii), $R_2(B^*) = 0$ has a positive root B^* which is unique in the interval $(0, L)$.

(b) $W^* > 0$ as $B^* \to \infty$,

(c) If we define $\left(\dfrac{dW^*}{dB^*}\right)_2$ as the derivative of W^* with respect to B^* from the isocline (15), then we have

$$\left(dW^*/dB^*\right)_2 > 0.$$

These considerations lead to the conclusion that (B^*, W^*) is unique in the regions of $0 < B^* < L$ and $0 < W^* < K(L)$, if and only if

$$\left(dW^*/dB^*\right)_1 < 0, \text{ and } \left(dW^*/dB^*\right)_2 > 0.$$

The expressions of both $(dW^*/dB^*)_1$ and $(dW^*/dB^*)_2$ are determined from Eqs. (14) and (15), respectively. To verify the conditions for uniqueness of the points (B^*, W^*), we used the following counter example; assume that human population (N) and human activities (H) have no effect on forest biomass (B), that is, $S_3 = 0$, $S_4 = 0$, then we noted that,

$$\left(dW^*/dB^*\right)_1 = -S_1/S_2 L < 0.$$

Similarly, taking Eq. (4), under maximum values of B^* and W^*, when forest-dependent wildlife population (W) is unaffected by human population (N) and human activities (H), that is, $S_6 = 0$, $S_7 = 0$, gives

$$\left(dW^*/dB^*\right)_2 = K_1 > 0.$$

Under this counter example, uniqueness conditions are satisfied. Thus, this concludes the existence proof for B^* and W^*. Once the values of B^* and W^* are known, the values of N^* and H^* can be evaluated from Eqs. (11) and (12), respectively.

5. Stability analysis

Stability analysis is a key tool to understand system dynamics. Local asymptotic stability requires solutions to converge to an equilibrium point when initial conditions are close to that point, while global asymptotic stability convergence to an equilibrium point regardless of initial conditions [28,29].

5.1. Local stability analysis

The local stability behaviour of the boundary equilibria is determined by examining the sign of the eigenvalues of the corresponding Jacobian matrix. The general Jacobian matrix of the system (3) is given as $\mathcal{J}$. Theorem 4 provides the summary of the stability analysis of the boundary equilibria $E^{(0)}$, $E^{(1)}$, $E^{(2)}$, $E^{(3)}$, and $E^{(4)}$.

$$\mathcal{J} = \begin{pmatrix} a_1 & -S_2 B & -S_3 B & -S_4 B^2 \\ a_2 & a_3 & -S_6 W & -S_7 W \\ S_9 N & -S_{10} N & a_4 & 0 \\ 0 & 0 & S_{11} & -S_{12} \end{pmatrix}$$

where

$$a_1 = S_1\left(1 - \frac{B}{L}\right) - \frac{S_1 B}{L} - S_2 W - S_3 N - 2S_4 BH,$$

$$a_2 = S_5 W\left(1 - \frac{W}{K(B)}\right) + \frac{K_1 S_5 W^2}{(K(B))^2},$$

$$a_3 = S_5 B\left(1 - \frac{W}{K(B)}\right) - \frac{S_5 BW}{K(B)} - S_6 N - S_7 H,$$

$$a_4 = S_8\left(1 - \frac{N}{M}\right) - \frac{S_8 N}{M} + S_9 B - S_{10} W.$$

Theorem 4. *Stability analysis of the equilibria* $E^{(0)}$, $E^{(1)}$, $E^{(2)}$, $E^{(3)}$, *and* $E^{(4)}$.

   i. *The equilibrium point* $E^{(0)}$ *is always unstable.*
   ii. *The equilibrium point* $E^{(1)}$ *is always unstable.*
   iii. *The equilibrium point* $E^{(2)}$ *is unstable provided inequality* $S_8 + S_9 B^{(2)} > S_{10} W^{(2)}$ *hold.*
   iv. *The equilibrium point* $E^{(3)}$ *is unstable provided either* $E^{(4)}$ *or* E^* *exists, that is,* $S_1 > S_3 M$.
   v. *The equilibrium point* $E^{(4)}$ *is unstable if* $S_5 B^{(4)} > S_6 N^{(4)} + S_7 H^{(4)}$.

On the other hand, assessing the stability of an interior equilibrium point E^* using eigenvalue inferences is very tricky. We used the Lyapunov direct method after linearising the system (3). Theorem

Theorem 5. *The model system* (3) *is locally asymptotically stable around the equilibrium point* E^* *provided conditions* (16)–(20) *hold.*

$$\left(\frac{W^*}{K(B^*)} + \frac{K'(B^*) B^* W^*}{(K(B^*))^2}\right)^2 < \left(\frac{2B^*(S_1 + S_4 H^* L)}{3S_2 LK(B^*)}\right), \tag{16}$$

$$\left(\frac{S_2 S_6}{S_5} + \frac{S_3 S_{10}}{S_9}\right)^2 < \left(\frac{2S_2 B^*}{3K(B^*)}\right)\left(\frac{S_3 S_8}{S_9 M}\right), \tag{17}$$

$$(S_{11})^2 < \frac{2S_3 S_8 S_{12}}{3S_9 M}, \tag{18}$$

$$\left(\frac{S_7}{S_5}\right)^2 < \frac{4S_{12} B^*}{9S_2 K(B^*)}, \tag{19}$$

$$S_4^2 < 2S_{12}\left(\frac{S_1 + S_3 H^* L}{3B^{*2} L}\right). \tag{20}$$

(The proof of this theorem is given in *Appendix* B).

5.2. Global stability analysis

The global stability behaviour of the system (3) at E^* is evaluated using Lyapunov's stability theory [22]. Theorem 6 establishes the conditions for the system's global stability behaviour at E^*.

Theorem 6. *The interior equilibrium* E^* *if exists is globally stable inside the region of attraction* Ω *provided conditions* (20)–(25) *hold.*

$$\left(\frac{\phi LK(L)}{K_0^2} + \frac{W^*}{K(B^*)}\right) < \left(\frac{S_1 + S_4 H^* L}{3S_2 K(B^*)}\right), \tag{21}$$

$$\left(\frac{S_2 S_6}{S_5} + \frac{S_3 S_{10}}{S_9}\right)^2 < \left(\frac{2S_2 L}{3K(B^*)}\right)\left(\frac{S_3 S_8}{S_9 M}\right), \tag{22}$$

$$(S_{11})^2 < \frac{2S_3 S_8 S_{12}}{3S_9 M}, \tag{23}$$

$$\left(\frac{S_7}{S_5}\right)^2 < \frac{4S_{12} L}{9S_2 K(B^*)}, \tag{24}$$

$$S_4^2 < 2S_{12}\left(\frac{S_1 + S_3 H^* L}{3L^3}\right). \tag{25}$$

(The proof of this theorem is given in *Appendix* C).

5.3. Existence of trans-critical bifurcation

The inferences on stability analysis of equilibrium point $E^{(3)}$ shows that the point is stable whenever $S_3 > S_3^* = S_1/M$ and when S_3 decreases to S_3^*, $E^{(3)}$ loses its stability and the interior equilibrium point E^* emerges. Thus, if S_3 is bifurcation parameter where an exchange of stability between $E^{(3)}$ and E^* takes place at $S_3 = S_3^*$, then, there is a trans–critical bifurcation between these two equilibrium points. Thus, the matrix of the linearised system (3) around the equilibrium point $E^{(3)}$ with S_3 evaluated at $S_3^* = S_1/M$ is given as

$$\mathcal{J}\left(E^{(3)}, S_3^*\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\left(\dfrac{S_6 S_{12} M + S_7 S_{11} M}{S_{12}}\right) & 0 & 0 \\ S_9 M & -S_{10} M & -S_8 & 0 \\ 0 & 0 & S_{11} & -S_{12} \end{pmatrix}$$

It can be noted that the matrix $\mathcal{J}\left(E^{(3)}, S_3^*\right)$ has a simple zero eigenvalue and three negative eigenvalues. This property allows the use of Centre Manifold Theory [30] to study the dynamics of the system (3) near S_3^*. It follows, the right eigenvector of $\mathcal{J}\left(E^{(3)}, S_3^*\right)$ corresponding to 0 eigenvalues denoted by $\mathbf{w} = [w_1, w_2, w_3, w_4]^T$ is given as

$$\mathbf{w} = \left[\frac{S_3 S_8 S_{12}}{S_1 S_9 S_{11}}, 0, \frac{S_{12}}{S_{11}}, 1\right]^T,$$

consequently, the left eigenvector $\mathbf{v}$ is given by

$$\mathbf{v} = \left[\frac{S_1 S_9 S_{11}}{S_3 S_8 S_{12}}, 0, 0, 0\right],$$

such that $\mathbf{v}.\mathbf{w} = 1$. Therefore, on utilising [31, Theorem 4.1], the coefficients of a and b are given as follows:

$$a = \sum_{k,i,j=1}^{4} v_k w_i w_j \frac{\partial^2 f_k}{\partial X_i \partial X_j}\left(E^{(3)}, S_3^*\right),$$

$$= v_1 w_1^2 \frac{\partial^2 f_1}{\partial B^2}\left(E^{(3)}, S_3^*\right) + 2v_1 w_1 w_3 \frac{\partial^2 f_1}{\partial B \partial N}\left(E^{(3)}, S_3^*\right),$$

$$a = -2v_1\left[w_1 w_3 \frac{S_1}{M} + w_1^2\left(\frac{S_1}{L} + \frac{S_4 S_{11} M}{S_{12}}\right)\right] < 0,$$

$$b = \sum_{k,i=1}^{4} v_k w_i \frac{\partial^2 f_k}{\partial X_i \partial S_3}\left(E^{(3)}, S_3^*\right) = v_1 w_1 \frac{\partial^2 f_1}{\partial B \partial S_3},$$
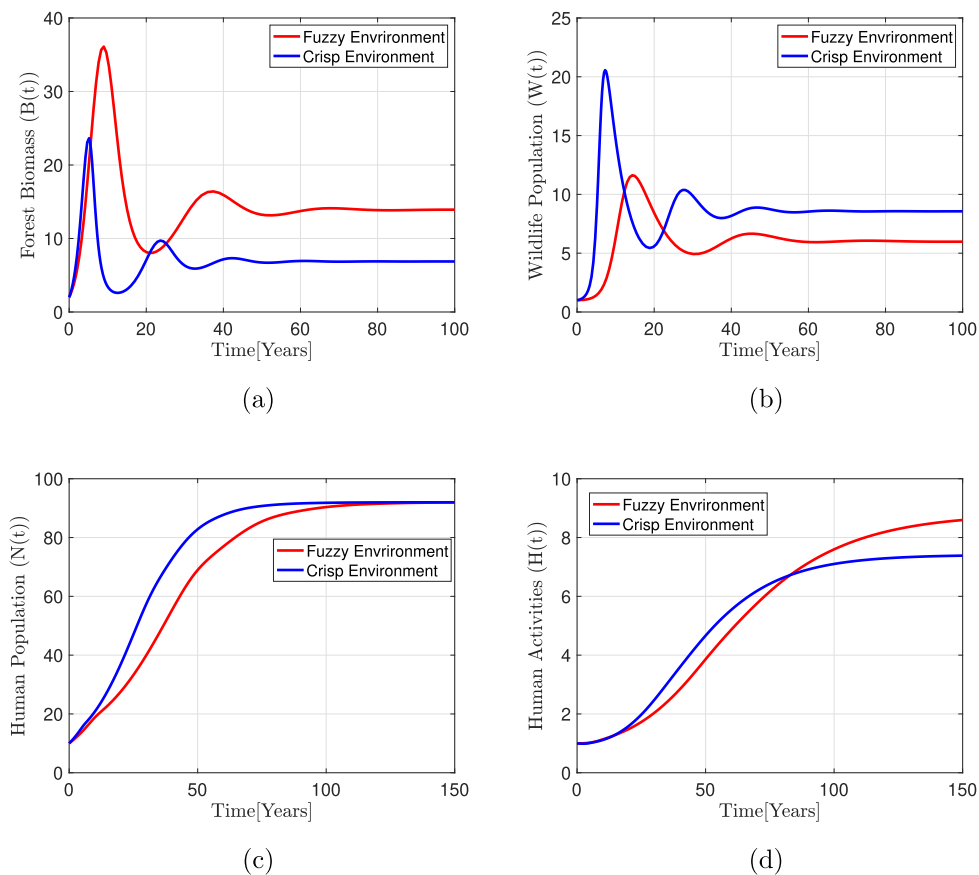
$$b = -M < 0.$$

Fig. 3. Time evolution of model solutions (a) Forest biomass (b) Wildlife population (c) Human population and (d) Human activities in crisp and fuzzy environment.

We observe that $a < 0$ and $b < 0$, when $S_3 < S_3^*$, the boundary equilibrium $E^{(3)}$ is unstable and there exists a positive stable interior equilibrium point E^*. When $S_3 > S_3^*$, $E^{(3)}$ becomes stable while positive interior equilibrium loses its feasibility and negative unstable interior equilibrium emerges. Thus, there is exists a trans–critical bifurcation between $E^{(3)}$ and E^* with bifurcation parameter S_3.

6. Numerical simulations

We carried out numerical simulation to study the effects of fuzzy parameters on the dynamics of the system. Since the problem is not a case study for a specific species, the parameter values in Table 2 and some hypothetical parametric values of Δ_i are used. Upon using these parametric values, numerical solution were computed by taking advantage of MATLAB built in ODE solver ode45. Starting with initial condition $B(0) = 2$, $W(0) = 1$, $N(0) = 10$, $H(0) = 1$, the dynamics of forest biomass, wildlife population, human population and human activities converge to the equilibrium levels (see Figs. 3(a)–3(d)). From that figure, it can be noted that equilibrium levels of four compartments are different in two different environments (crisp and fuzzy) suggesting that fuzzy parameters have impact on the dynamics of the model, thus adjusting the values of Δ_i to control fuzziness of parameter will leads to decisions based on uncertain or imprecise information.

Furthermore, taking different initial conditions, we plotted the phase diagrams in $B - W - N$ and $B - W - H$ which are shown in Figs. 4(a)–4(b), we observed that regardless of the starting points, the solutions converge to the interior equilibrium point suggesting that the interior equilibrium point is globally asymptotically stable for the fuzzy model. In other planes, non–linear stability can be demonstrated in similar way. To verify the existence of a transcritical bifurcation between $E^{(3)}$ and E^* numerically we obtained the critical value of

the bifurcation parameter S_3 as $S_3^* = 0.00812$. Moreover, we plotted the evolution of the model's solutions as the bifurcation parameter S_3 varies while other parameters kept fixed, as shown in Figs. 5(a)–5(b). We observed that when the value of S_3 crosses the critical value, a change in feasibility and stability occurs between $E^{(3)}$ and E^*, suggesting the presence of a transcritical bifurcation as shown in analytical analysis.

Similarly, we investigated how the system's behaviour changes with parametric variations of some critical parameters. These critical parameters were determined from the stability analysis of the interior equilibrium points E^*. Figs. 6(a)–6(d), 7(a)–7(d), and 8–8(d) demonstrate the change in the system's stability due to changes in parameter responsible for depletion of forest biomass due to human activities (S_4), human activities growth rate (S_{11}) and depletion rate of forest-dependent wildlife population due to human activities (S_7), respectively. The figures clearly demonstrate that ensuring the long-term stability of the system necessitates keeping these parameters below critical values. Specifically, the depletion rate of forest biomass due to human activities should not exceed 0.01, while the growth rate of human activities must be limited to 0.1. Furthermore, the depletion rate of the wildlife population resulting from human activities should be constrained to 0.07, assuming all other parameters remain unchanged. These findings offer practical guidance for sustainable forest management and wildlife conservation efforts in the studied region.

7. Discussion and conclusion

Depletion of forest biomass poses a significant threat to global ecosystems and biodiversity. Forests play a crucial role in maintaining Earth's ecological balance, making their degradation have far-reaching consequences. The loss of forest biomass is primarily driven by human

Table 2
Parameter values and sources.

Parameter	Value	Source	Parameter	Value	Source
s	0.8	[10,32,33]	M	100	[10,34]
L	100	[10,34,35]	K_0	10	Assumed
β_0	0.05	[9]	K_1	3	[9]
η	0.9	[9]	λ	0.05	[33]
β_1	0.003	[33]	σ	0.001	[33]
β_2	0.0004	Assumed	θ	0.5	[10,33]
v_1	0.002	[33]	γ	0.004	Assumed
v_2	0.0001	[33]	γ_1	0.05	[33]

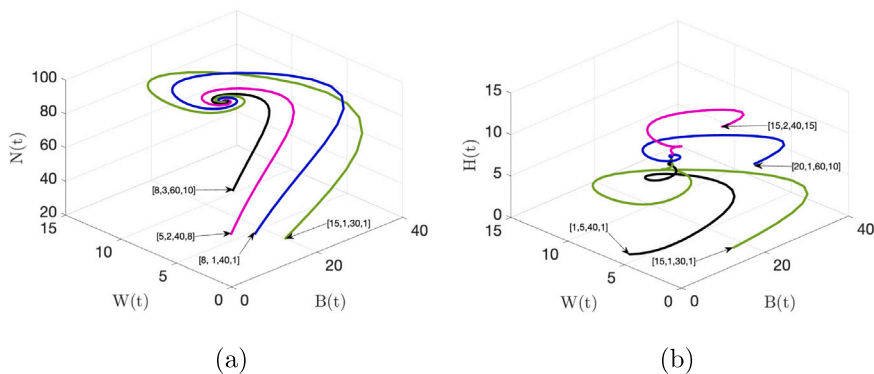|			
$\Delta_1 = 0.79,$	$\Delta_2 = 0.05,$	$\Delta_3 = 0.05,$	$\Delta_4 = 0.01,$
$\Delta_5 = 0.0025,$	$\Delta_6 = 0.001,$	$\Delta_7 = 0.00039,$	$\Delta_8 = 0.0001,$
$\Delta_9 = 0.89,$	$\Delta_{10} = 0.005,$	$\Delta_{11} = 0.0019,$	$\Delta_{12} = 0.0004,$
$\Delta_{13} = 0.00009,$	$\Delta_{14} = 0.00005,$	$\Delta_{15} = 0.09,$	$\Delta_{16} = 0.005,$
$\Delta_{17} = 0.045,$	$\Delta_{18} = 0.006,$	$\Delta_{19} = 0.0008,$	$\Delta_{20} = 0.0009,$
$\Delta_{21} = 0.0078,$	$\Delta_{22} = 0.0002,$	$\Delta_{23} = 0.0038,$	$\Delta_{24} = 0.0005,$
$\Delta_{25} = 0.045,$	$\Delta_{26} = 0.006.$		

Fig. 4. Non–linear stability of the interior equilibrium point E^* in (a) $B - W - N$ and (b) $B - W - H$ planes.
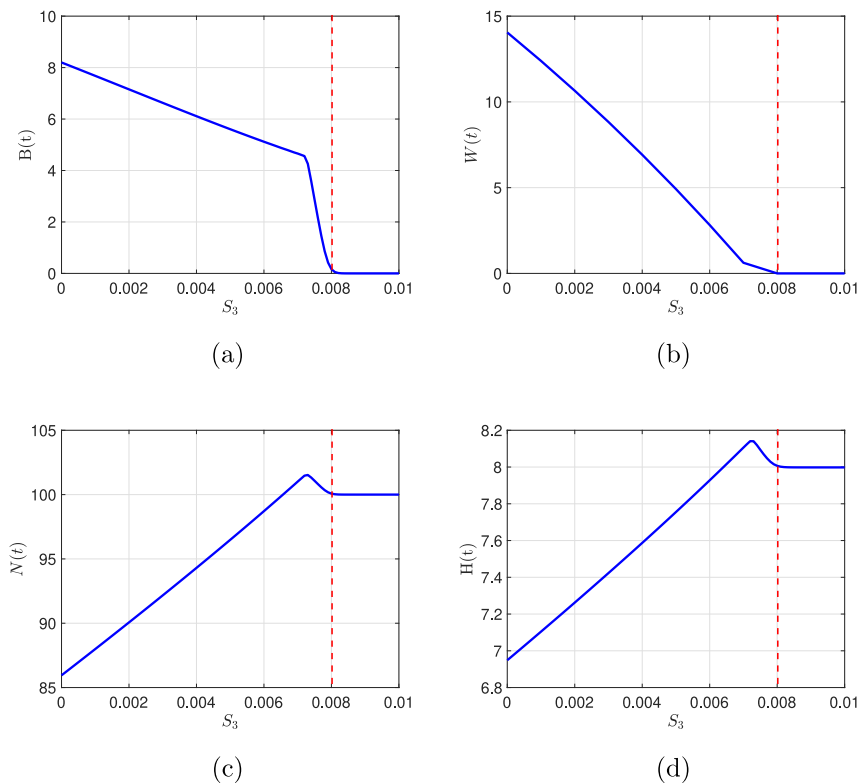

Fig. 5. Trans–critical bifurcation between $E^{(3)}$ and E^* with respect to the depletion rate of forest biomass due to human population (S_3), (a) Forest biomass (b) Wildlife population (c) Human population and (d) Human activities. Other parameter values were kept fixed.
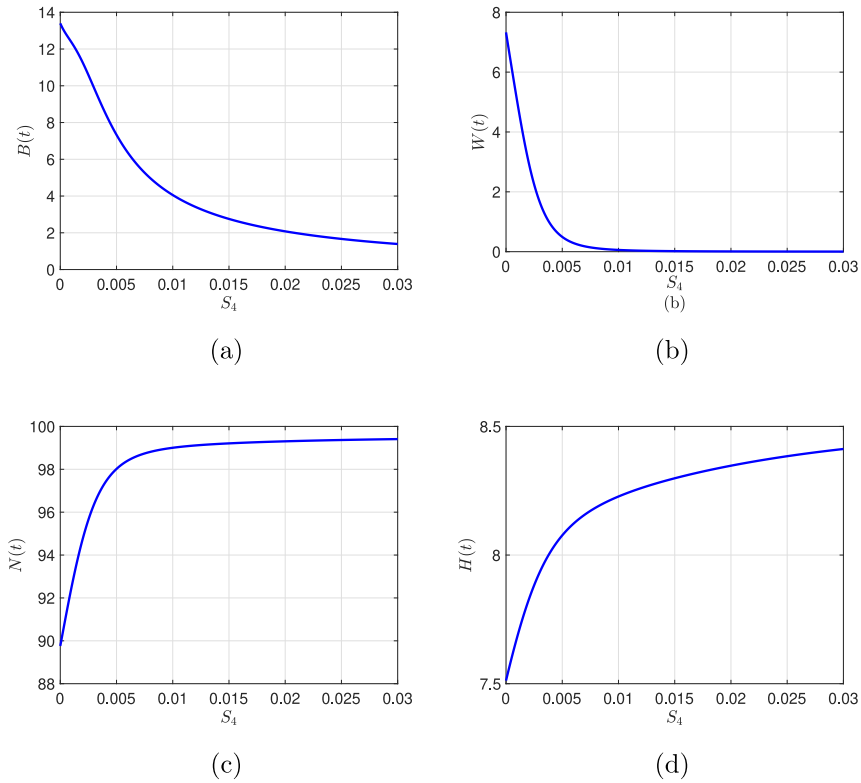
Fig. 6. Evolution of the solution of (a) Forest biomass (b) Wildlife population (c) Human population and (d) Human activities for different values of S_4. Other parameter values are kept fixed.
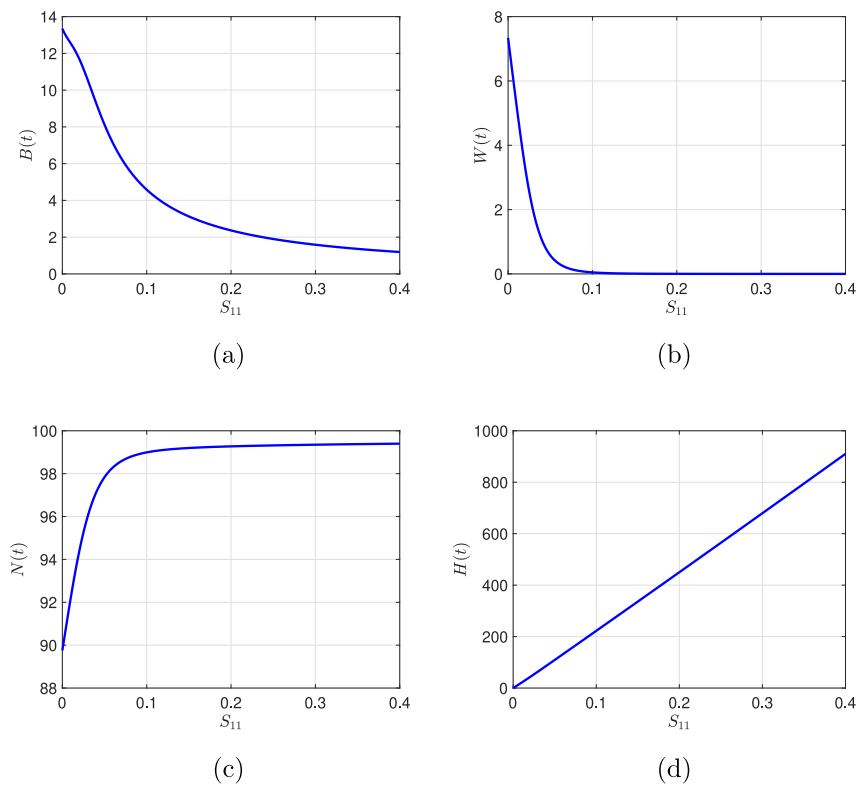

Fig. 7. Evolution of the solution of (a) Forest biomass (b) Wildlife population (c) Human population and (d) Human activities for different values of S_{11}. Other parameter values were kept fixed.
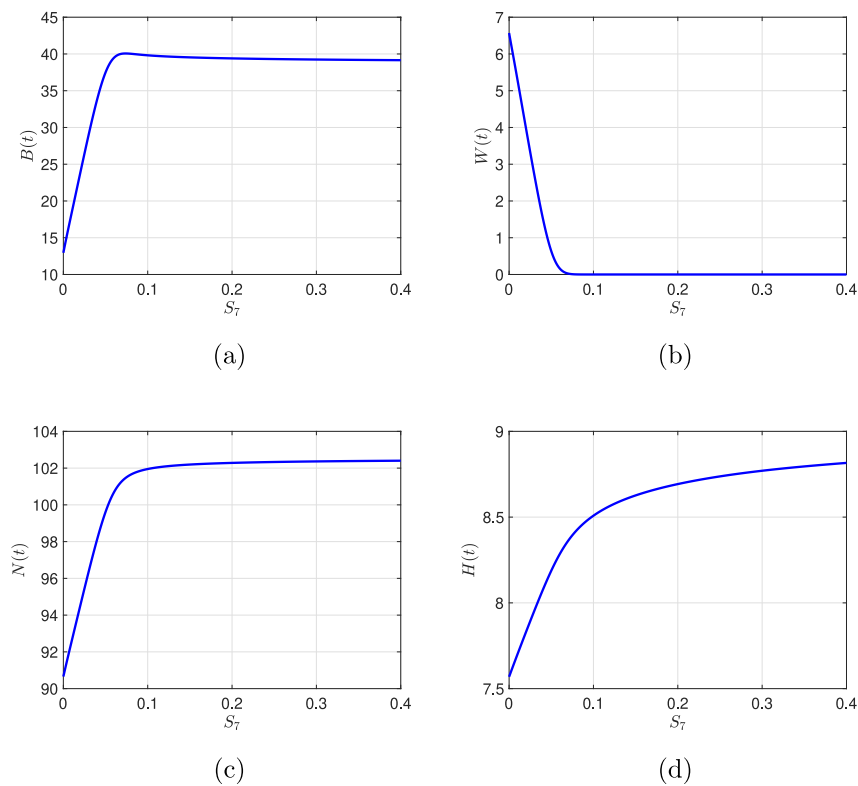
Fig. 8. Evolution of the solution of (a) Forest biomass (b) Wildlife population (c) Human population and (d) Human activities for different values of S_4. Other parameter values are kept fixed.

activities, including deforestation, illegal logging, and unsustainable land-use practices. The consequences of forest biomass depletion are profound, leading to habitat loss for countless plant and animal species, increasing the risk of extinction, and disrupting entire ecosystems. To establish sustainable strategies for forest conservation, understanding the dynamics of human–forest interactions is of paramount importance.

Thus, this paper presents a system of non-linear differential equations that describes the depletion of forest biomass and forest–dependent wildlife population caused by human population and its associated activities in a fuzzy environment. The model assumes that the parameters are triangular fuzzy numbers due to their intuitive nature, ease of use, computational simplicity, and ability to facilitate representation. The fuzzy model was defuzzified using the signed distance of the fuzzy numbers. The bounded space of the model solutions and the conditions for the system to remain uniformly persistent are presented. The paper also identifies the potential equilibrium points of the fuzzy model whereby one non-negative interior equilibrium point and five non-negative boundary equilibria were found, and their stabilities were analysed. The paper also presents the Trans-critical bifurcation concerning the forest depletion rate resulting from human population.

The simulation results presented demonstrate significant differences in the equilibrium levels of all variables between the crisp and fuzzy models. These findings underscore the considerable impact of imprecise parameter values on the stability of the model system, particularly affecting the equilibrium levels of forest biomass and forest-dependent wildlife. This highlights the crucial role of fuzzy parameters in understanding and predicting the behaviour of the system under uncertain conditions. Furthermore, regardless of the initial conditions, all solutions converge to the interior equilibrium points, signifying the global asymptotic stability of these points. This stability is essential for ensuring the long-term viability of the system. In addition to analysing the model's stability, we conducted a study on the parametric variations of critical parameters. Our investigation identified threshold values of

0.01, 0.1, and 0.07 for the depletion rate of forest biomass due to human activities, the natural growth rate of human activities, and the depletion rate of wildlife population due to human activities, respectively. Maintaining these parameters below the identified thresholds is vital for preserving the system's stability and ecological balance.

These findings shed light on the significance of fuzzy parameters in modelling complex systems and provide valuable insights into the behaviour and sustainability of the studied ecosystem. Thus, it can be concluded that fuzzy model is more trustworthy this is because crisp models are a particular case of fuzzy models, further, the use of fuzzy parameters in the model may leads to a more accurate representation of the dynamics of forest depletion. Nonetheless, future studies could emphasise the incorporation of spatial dynamics into the model and the integration of real-world data to improve prediction precision. Furthermore, inclusion of time delay in ecological interactions to account the temporal dynamics that influence the overall stability and sustainability of the system could be an area of interest.

CRediT authorship contribution statement

Ibrahim M. Fanuel: Conceptualization, Methodology, Writing – original draft, Editing. **Silas Mirau:** Conceptualization, Methodology, Writing – original draft, Review, Editing, Supervision. **Maranya Mayengo:** Conceptualization, Methodology, Writing – original draft, Review, Editing, Supervision. **Francis Moyo:** Conceptualization, Methodology, Writing – original draft, Review, Editing, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Defuzzification process

Taking the α cut on both sides of model system (see Definition 3) gives

$$
\begin{cases}
\widetilde{\dfrac{dB}{dt}}(\alpha) = (\widetilde{s})(\alpha)\, B\left(1 - \dfrac{B}{L}\right) - \left(\widetilde{\beta_0}\right)(\alpha)\, BW - \left(\widetilde{\beta_1}\right)(\alpha)\, BN - \left(\widetilde{\beta_2}\right)(\alpha)\, B^2 H, \\[2mm]
\widetilde{\dfrac{dW}{dt}}(\alpha) = \left(\widetilde{\beta_0} \otimes \widetilde{\eta}\right)(\alpha)\, BW\left(1 - \dfrac{W}{K(B)}\right) - (\widetilde{\nu_1})(\alpha)\, WN - (\widetilde{\nu_2})(\alpha)\, WH, \\[2mm]
\widetilde{\dfrac{dN}{dt}}(\alpha) = \left(\widetilde{\theta}\right)(\alpha)\, N\left(1 - \dfrac{N}{M}\right) + \left(\widetilde{\lambda} \otimes \widetilde{\beta_1}\right)(\alpha)\, BN - (\widetilde{\sigma})(\alpha)\, NW, \\[2mm]
\widetilde{\dfrac{dH}{dt}}(\alpha) = (\widetilde{\gamma})(\alpha)\, N - (\widetilde{\gamma_1})(\alpha)\, H.
\end{cases}
\tag{A.1}
$$

More precisely, system (A.1) can be written as

$$
\left[\left(\dfrac{dB}{dt}\right)_l(\alpha), \left(\dfrac{dB}{dt}\right)_r(\alpha)\right] = \left[s_l(\alpha), s_r(\alpha)\right] B\left(1 - \dfrac{B}{L}\right) - \left[(\beta_0)_l(\alpha), (\beta_0)_r(\alpha)\right] BW
$$
$$
- \left[(\beta_1)_l(\alpha), (\beta_1)_r(\alpha)\right] BN - \left[(\beta_2)_l(\alpha), (\beta_2)_r(\alpha)\right] B^2 H,
$$

$$
\left[\left(\dfrac{dW}{dt}\right)_l(\alpha), \left(\dfrac{dW}{dt}\right)_r(\alpha)\right] = \left[(\beta_0)_l\eta_l(\alpha), (\beta_0)_r\eta_r(\alpha)\right] BW\left(1 - \dfrac{W}{K(B)}\right)
$$
$$
- \left[(\nu_1)_l(\alpha), (\nu_1)_r(\alpha)\right] WN - \left[(\nu_2)_l(\alpha), (\nu_2)_r(\alpha)\right] WH,
$$

$$
\left[\left(\dfrac{dN}{dt}\right)_l(\alpha), \left(\dfrac{dN}{dt}\right)_r(\alpha)\right] = \left[\theta_l(\alpha), \theta_r(\alpha)\right] N\left(1 - \dfrac{N}{M}\right) - \left[\sigma_l(\alpha), \sigma_r(\alpha)\right] NW
$$
$$
+ \left[\lambda_l(\beta_1)_l(\alpha), \lambda_r(\beta_1)_r(\alpha)\right] BN,
$$

$$
\left[\left(\dfrac{dH}{dt}\right)_l(\alpha), \left(\dfrac{dH}{dt}\right)_r(\alpha)\right] = \left[\gamma_l(\alpha), \gamma_r(\alpha)\right] N - \left[(\gamma_1)_l(\alpha), (\gamma_1)_r(\alpha)\right] H.
$$

On utilising the properties of the interval theory (see Definition 4) gives

$$
\left(\dfrac{dB}{dt}\right)_l(\alpha) = s_l(\alpha)\, B\left(1 - \dfrac{B}{L}\right) - (\beta_0)_l(\alpha)\, BW - (\beta_1)_l(\alpha)\, BN - (\beta_2)_l(\alpha)\, B^2 H,
$$

$$
\left(\dfrac{dB}{dt}\right)_r(\alpha) = s_r(\alpha)\, B\left(1 - \dfrac{B}{L}\right) - (\beta_0)_r(\alpha)\, BW - (\beta_1)_r(\alpha)\, BN - (\beta_2)_r(\alpha)\, B^2 H,
$$

$$
\left(\dfrac{dW}{dt}\right)_l(\alpha) = (\beta_0)_l\eta_l(\alpha)\, BW\left(1 - \dfrac{W}{K(B)}\right) - (\nu_1)_l(\alpha)\, WN - (\nu_2)_l(\alpha)\, WH,
$$

$$
\left(\dfrac{dW}{dt}\right)_r(\alpha) = (\beta_0)_r\eta_r(\alpha)\, BW\left(1 - \dfrac{W}{K(B)}\right) - (\nu_1)_r(\alpha)\, WN - (\nu_2)_r(\alpha)\, WH,
$$

$$
\left(\dfrac{dN}{dt}\right)_l(\alpha) = \theta_l(\alpha)\, N\left(1 - \dfrac{N}{M}\right) + \lambda_l(\beta_1)_l(\alpha)\, BN - \sigma_l(\alpha)\, NW,
$$

$$
\left(\dfrac{dN}{dt}\right)_r(\alpha) = \theta_r(\alpha)\, N\left(1 - \dfrac{N}{M}\right) + \lambda_r(\beta_1)_r(\alpha)\, BN - \sigma_r(\alpha)\, NW,
$$

$$
\left(\dfrac{dH}{dt}\right)_l(\alpha) = \gamma_l(\alpha)\, N - (\gamma_1)_l(\alpha)\, H,
$$

$$
\left(\dfrac{dH}{dt}\right)_r(\alpha) = \gamma_r(\alpha)\, N - (\gamma_1)_r(\alpha)\, H,
$$

where

$$
(p_i)_l(\alpha) = p_i - \Delta_i + \alpha\Delta_i, \quad (p_i)_r(\alpha) = p_i - \Delta_{i+1} + \alpha\Delta_{i+1}.
$$

Thereafter, we used the signed distance (see Definition 5) to defuzzify the system,

$$
d\left(\widetilde{\dfrac{dB}{dt}}, \widetilde{0}\right) = d\left(\widetilde{s}, \widetilde{0}\right) B\left(1 - \dfrac{B}{L}\right) - d\left(\widetilde{\beta_0}, \widetilde{0}\right) BW - d\left(\widetilde{\beta_1}, \widetilde{0}\right) BN - d\left(\widetilde{\beta_2}, \widetilde{0}\right) B^2 H,
$$

$$
d\left(\widetilde{\dfrac{dW}{dt}}, \widetilde{0}\right) = d\left(\widetilde{\beta_0} \otimes \widetilde{\eta}, \widetilde{0}\right) BW\left(1 - \dfrac{W}{K(B)}\right) - d\left(\widetilde{\nu_1}, \widetilde{0}\right) WN - d\left(\widetilde{\nu_2}, \widetilde{0}\right) WH,
$$

$$
d\left(\widetilde{\dfrac{dN}{dt}}, \widetilde{0}\right) = d\left(\widetilde{\theta}, \widetilde{0}\right) N\left(1 - \dfrac{N}{M}\right) + d\left(\widetilde{\lambda} \otimes \widetilde{\beta_1}, \widetilde{0}\right) BN - d\left(\widetilde{\sigma}, \widetilde{0}\right) NW,
$$

$$
d\left(\widetilde{\dfrac{dH}{dt}}, \widetilde{0}\right) = d\left(\widetilde{\gamma}, \widetilde{0}\right) N - d\left(\widetilde{\gamma_1}, \widetilde{0}\right) H.
$$

Upon evaluating and substituting the values of the respective signed distance gives the defuzzified system (3) in the form of crisp model.

Appendix B. Proof of Theorem 5

Proof. To prove Theorem 5, we start by linearising the system (3), about E^* by using the following transformations: $B = B^* + b$, $W = W^* + w$, $N = N^* + n$, $H = H^* + h$, where b, w, n and h are small perturbations around the equilibrium E^*. The following linearised system is obtained:

$$
\begin{cases}
\dfrac{db}{dt} = -\left(\dfrac{S_1 B^*}{L} + S_4 B^* H^*\right) b - S_3 B^* w - S_3 B^* n - S_4 B^{*2} h, \\[2mm]
\dfrac{dw}{dt} = \left(S_5 W^*\left(1 - \dfrac{W^*}{K(B^*)}\right) + \dfrac{S_5 K_1 B^* W^{*2}}{(K(B^*))^2}\right) b - \left(\dfrac{S_5 B^* W^*}{K(B^*)}\right) w \\[2mm]
\qquad - S_6 W^* n - S_7 W^* h, \\[2mm]
\dfrac{dn}{dt} = S_9 N^* b - S_{10} N^* w - \dfrac{S_8 N^*}{M} n, \\[2mm]
\dfrac{dh}{dt} = S_{11} n - S_{12} h.
\end{cases}
\tag{B.1}
$$

Following Lata [10], to apply the Lyapunov direct method we considered the positive definite function (B.2),

$$
V = \frac{1}{2}\left(\frac{1}{B^*}b^2 + \frac{k_1}{W^*}w^2 + k_2 n^2 + k_3 h^2\right),
\tag{B.2}
$$

where k_1, k_2 and k_3 are positive constants. Upon differentiating function (B.2) with respect to t along the solutions of linearised system (B.1), we obtain

$$
\frac{dV}{dt} = -\left(\frac{S_1}{L} + S_4 H^*\right) b^2 - k_1\left(\frac{S_5 B^*}{K(B^*)}\right) w^2 - k_2\frac{S_8 N^*}{M} n^2 - k_3 S_{12} h^2
$$
$$
+ \left(-S_2 + k_1 S_5\right) bw - k_1\left(\frac{S_5 W^*}{K(B^*)} - \frac{S_5 K_1 B^* W^*}{K(B^*)^2}\right) bw
$$
$$
+ \left(-S_3 + k_2 S_9 N^*\right) bn + -S_4 bh - \left(k_1 S_6 + k_2 S_{10} N^*\right) wn
$$
$$
- k_1 S_7 wh + k_3 S_{11} nh.
\tag{B.3}
$$

Choosing arbitrary $k_1 = S_2/S_5$, $k_2 = S_3/S_9 N^*$, and $k_3 = 1$, which are all positive constants, it is worth noting that dV/dt is negative definite provided conditions (16) to (20) hold. This completes the proof of Theorem 5.

Appendix C. Proof of Theorem 6

Proof. We consider the following positive definite function [36] to prove the theorem

$$
U = \left(B - B^* - B^* \ln\frac{B}{B^*}\right) + l_1\left(W - W^* - W^* \ln\frac{W}{W^*}\right)
$$
$$
+ l_2\left(N - N^* - N^* \ln\frac{N}{N^*}\right)
$$
$$
+ \frac{l_3}{2}\left(H - H^*\right)^2,
\tag{C.1}
$$

where l_1, l_2 and l_3 are positive constants. The time derivative of (C.1) and upon substitution of the values of dB/dt, dW/dt, dN/dt and

dH/dt gives

$$\frac{dU}{dt} = (B - B^*)\left[-\left(\frac{S_1}{L} + S_4 H^*\right)(B - B^*) - S_2(W - W^*) - S_3(N - N^*)\right.$$
$$\left. -S_4 B(H - H^*)\right] + l_1(W - W^*)\left[S_5(B - B^*) - \frac{S_5 B}{K(B^*)}(W - W^*)\right.$$
$$\left. -S_6(N - N^*) - S_7(H - H^*)\right] + l_1(W - W^*)\left[-\left(S_5 B W \Gamma(B)\right)(B - B^*)\right.$$
$$\left. -\left(\frac{S_5 W^*}{K(B^*)}\right)(B - B^*)\right] + l_2(N - N^*)\left[-\frac{S_8}{M}(N - N^*) + S_9(B - B^*)\right.$$
$$\left. -S_{10}(W - W^*)\right] + l_3(H - H^*)\left[-(S_{13} - S_{12}B^2)(H - H^*) + S_{11}(N - N^*)\right.$$
$$\left. +S_{12}(BH^* + B^* H^*)(B - B^*)\right].$$

$$\frac{dU}{dt} = (B - B^*)\left[-\left(\frac{S_1}{L} + S_4 H^*\right)(B - B^*) - S_2(W - W^*)\right.$$
$$\left. - S_3(N - N^*) - S_4 B(H - H^*)\right]$$
$$+l_1\left(W - W^*\right)\left[S_5\left(B - B^*\right) - \frac{S_5 B}{K(B^*)}(W - W^*)\right.$$
$$\left. - S_6(N - N^*) - S_7(H - H^*)\right]$$
$$+l_1\left(W - W^*\right)\left[-\left(S_5 B W \Gamma(B)\right)\left(B - B^*\right) - \left(\frac{S_5 W^*}{K(B^*)}\right)\left(B - B^*\right)\right]$$
$$+l_2(N - N^*)\left[-\frac{S_8}{M}(N - N^*) + S_9(B - B^*) - S_{10}(W - W^*)\right]$$
$$+l_3(H - H^*)\left[-S_{12}(H - H^*) + S_{11}(N - N^*)\right].$$

where,

$$\Gamma(B) = \begin{cases} \dfrac{\dfrac{1}{K(B)} - \dfrac{1}{K(B^*)}}{B - B^*}, & B \neq B^* \\ \dfrac{-K'(B^*)}{(K(B^*))^2}, & B = B^*. \end{cases}$$

On utilising mean value theorem we have

$$|\Gamma(B)| = \frac{\phi}{K_0^2},$$

such that $0 < K'(B) \leq \phi$. Choosing arbitrary, $l_1 = S_2/S_5$, $l_2 = S_3/S_9$ and $l_3 = 1$. We observe that dU/dt is negative definite provided conditions (21) to (25) hold. This completes the proof of Theorem 6.

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