INSURANCE COMPANIES PORTFOLIO OPTIMISATION WITH POSSIBILITIES OF RECOVERY AFTER RUIN

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ABSTRACT

This dissertation, is about a study on insurance companies that have experienced ruin but have a possibility of recovery from ruin. The study has proposed a perturbed mathematical model, analysed and used it for modelling the portfolio of insurance companies with possibilities of recovery after ruin. Return on investment and refinancing have been used as approaches for overcoming ruin. The model was analysed for various cases of possibilities of recovery after ruin in the closed interval [0, 1]. The basic perturbed classical risk process was later compounded by refinancing and return on investment. The Hamilton-Jacobi-Bellman and Integro-Differential Equation of Volterra type were obtained. The Volterra Integro-Differential Equation for survival function of an insurance company was converted to a third order ordinary differential equation and later converted into a system of first order ordinary differential equations which was solved numerically using the fourth order Runge-Kutta method. The results indicate that the return on investment plays a vital role in reducing ultimate ruin and that as the possibility of recovery for insurance companies increases, the return on investment reduces ruin much faster. Also, the survival function increases with the increasing intensity of the counting process but decreases with an increase in the instantaneous rate of stock return and return volatility. Because an insurance company faces more risks, these results also suggest that insurance companies should increase their counting process since doing so will help the insurance companies in servicing more customers.

DECLARATION

I, **Masoud Komunte**, do hereby declare to the Senate of Nelson Mandela African Institution of Science and Technology that this dissertation is my own original work and that it has neither been submitted nor being concurrently submitted for degree award in any other institution.

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Name and signature of supervisor 1 **Date**

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CERTIFICATION

The undersigned certify that they have read and hereby recommend for acceptance by the Nelson Mandela African Institution of Science and Technology the dissertation entitled: *Insurance Companies Portfolio Optimization with Possibilities of Recovery after Ruin*, in fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematical and Computer Sciences and Engineering of the Nelson Mandela African Institution of Science and Technology.

Prof. Dr.-Ing. Verdiana Grace Masanja 2 January, 2023

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DEDICATION

The dedication of this research report goes to my late father

Amri Mohamed Komunte

May the Almighty Allah rest your soul in eternal peace.

Amen.

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LIST OF ABBREVIATIONS AND NOTATION

Abbreviations

NM-AIST	Nelson Mandela African Institution of Science and Technology
TIRA	Tanzania Insurance Regulatory Authority
HJB	Hamilton-Jacob-Bellman
VIE	Volterra Integral Equation
ODE	Ordinary Differential Equation
SDE	Stochastic Differential Equation
VIDE	Volterra integro-differential equation
CLM	Cramér-Lundberg Model
i.i.d.	independent and identically distributed
a.s.	almost surely
NIC	National Insurance Corporation
SRM	Spectral Risk Measure
HAM	Homotopy Analysis Method
CEV	Constant Elasticity of Variance
PDE	Partial Differential Equation
IARA	Increasing Absolute Risk Aversion

Notation

$\mathbb{R}$	Real numbers
$\mathbb{R}_+$	Positive real numbers
$\mathbb{P}$	Probability measure
$\mathscr{F}$	Ω-field
Ω	Sample space
$\mathscr{L}$	Infinitesimal generator of an Itô process
$(\mathscr{F}_t)_{t\in[0,\infty)}$	Filtration
W_t	Standard 1-dimensional Brownian motion
σ	The volatility or diffusion coefficient
σ_X	Premium volatility
σ_R	Return Volatility
λ_X	Intensity of the counting process
c	Premium incomes per unit time for an insurer
p	Initial capital
γ	Possibility of recovery
r	Instantaneous rate of stock return
r_i	The value of the investment rate
M	Capital refinanced
Y	Capital risked
δ	Preference rate to refinancing
β	Mean of the exponential distribution

CHAPTER ONE

INTRODUCTION

1.1 General Introduction

This research focus on insurance portfolio optimisation on the assumption that the insurance companies have the possibility of recovery after suffering ruin. The aim of this research is to develop and solve the model that represents portfolios of insurance companies having possibility of recovery after ruin. Investing and refinancing techniques are used to explore the situations of these kind of insurance companies.

This chapter gives the background of the problem, problem statement, objectives of the study and research questions. Furthermore, important concepts are defined, it outlines the significance of this study, and the chapter ends with an outline of the entire dissertation.

1.1.1 Background

Insurance refers to a contract that is represented by using a policy where individuals or entities receives financial protections from a given insurance company against losses. Various types of life and non-life insurance are available aiming at giving protection to people and their property. There are insurance in health, cars, house and so on (Oyatoye & Arogundade, 2011).

The background of the insurance sector in Tanzania can be traced back to 1996 when the Insurance Act of Tanzania liberalised the Insurance market giving the direction and ways to private and new entrants to invest in the insurance market (Kamwambia, 2013). After nationalization of private insurance companies in 1967 before market liberalization in 1996, the insurance industry in Tanzania was previously monopolized by the government through National Insurance Corporation (NIC). These changes eventually attracted many domestic and foreign investors to invest in Tanzania's insurance sector with increased companies currently reaching 31 insurance companies (Abbas & Ning, 2016). Some of these companies in insurance business consist of joint ventures between local investors and foreign companies. This tends to bring a combination of external sectors knowledge and financing with the local insurance market expertise. Out of the total registered insurance companies, 23 are privately owned with at least one third local ownership, 2 (National Insurance Corporation Tanzania limited and Zanzibar Insurance Corporation) are 100% state owned by the government of the united republic of Tanzania and revolutionary government of Zanzibar respectively and 4 are 100% locally owned (TIRA, 2018).

A new Insurance Act was enacted in 2009 and provided a general framework for all companies operating in insurance industry. It also created Tanzania Insurance Regulatory Authority

(TIRA) that works as a regulatory and supervisory body. Since its approval and implementation, TIRA has pushed the insurance industry in Tanzania toward a risk based system where the greatest risks receive the highest attention (Kamwambia, 2013; TIRA, 2018). Tanzania Insurance Regulatory Authority prepares and distributes publications related to insurance to the public with the aim of improving insurance awareness and understanding (Kamwambia, 2013; TIRA, 2018). Also TIRA has been working very closely with insurances brokers in order to improve professional standards and efficiency of the insurance industry. In consequence, Tanzanians are aware of all the benefits of insurance and the insurance sector contributes to the country's growing economy (Nthenge, 2012; Abbas & Yushan, 2016).

In fulfilling its obligations, an insurance company will have a collection of investments that generate income to cover various clients' claims. This collection of investments for an insurance company is known as an insurance portfolio (Kozmenko & Oliynyk, 2015). An insurance company holding a portfolio with many investments that are liquid reduces the investor's risk since these investments can handle companies' claims whenever they arise.

Risks are everywhere in our daily lives: at home, job, on the road and so on. Generally, risk is associated with everything but we are interested with the risk that leads to financial losses. Therefore it is very important to secure expensive property that we own and insurance companies provide that security. Provision of insurance requires competent management as poor management may leads to the eventual ruin of the insurance company, resulting in its failure to fulfill its obligations. This happens when the insurer's level of surplus goes below zero, thus making the company bankrupt (Kasumo, 2019).

One of the very important measures that insurance companies should take is risk management. Several ways are available to ensure that risk is managed in an insurance company. Refinancing and investment are among other very good ways to overcome the risks of the insurance companies and give the optimal returns to the shareholders. Using investment, the insurer distributes part of those risks to the paying investments which in turn can save a company to cover clients' benefits (Hu *et al.*, 2018).

Through investment and refinancing strategies, insurers may themselves protect against any potentially big losses or at least ensure that their earnings will remains relatively stable when there is a possibility of recovery after ruin. In the literature such as Asanga *et al.* (2014), Asimit *et al.* (2015) and Björk *et al.* (2014), many optimisation problems have arisen as part of the risk management process to study how insurance companies can control ultimate ruin.

Research by Zhu *et al.* (2015) investigated insurers whose opportunities and set of investments contained the default security. They considered a proportional reinsurance and also investment optimisations problems for insurers existing in financial markets depending on risky stock assets, a savings account, and corporate bonds, while Loke and Thomann (2018) studied ruin probability based on a dual risk model having risk-free investments.

Sheng *et al.* (2014) and Hu *et al.* (2018) studied investment and reinsurance problems, Sheng *et al.* (2014) studied how to optimize the control in investments and reinsurance problems for an insurer using jump-diffusions risks process but with independence of the Brownian motions while Hu *et al.* (2018) studied optimal investments and reinsurance problems for re-insurer and insurer using jump-diffusion processes.

1.1.2 Investing and Refinancing

The insurance industry is currently undergoing fundamental transformation in terms of operations and competitiveness. Several disruptive factors in business have given rise to new players in the market with disruptive business models to out-perform their competitors. Investing and refinancing can be used as approaches when insurance players look at how they should react to this major shift. With investing and refinancing an insurance company can manage to operate much better even if it had suffered from ruin provided the investments are done properly and refinancing is done adequately and timely (Eisenberg, 2010).

1.2 Basic Mathematical Concepts and Definitions

Definition 1.1 ***(Sample Space)***
A sample space Ω can be defined as a set containing all possible outputs of a given probability or random experiment (Kijima, 2016).

Definition 1.2 ***(σ-field)***
A σ-field also known as σ-algebra can be defined as a set $\mathscr{F}$ that contains subsets of sample space that fulfils the following:

(i) $\Omega, \emptyset \in \mathscr{F}$.

(ii) If $B \subset \Omega \in \mathscr{F}$ then $B^c = \Omega \setminus B \in \mathscr{F}$.

(iii) If $B_1, B_2, ... \in \mathscr{F}$ then $\cup_{k=1}^{\infty} B_k \in \mathscr{F}, \cap_{k=1}^{\infty} B_k \in \mathscr{F}$ (Øksendal, 2003).

Definition 1.3 ***(Probability Measure)***
The probability measure can be defined as a set function $\mathbb{P} : \mathscr{F} \to [0,1]$ that is defined on $\mathscr{F}$ and that satisfies the following:

(i.) $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}(\emptyset) = 0$.

(ii.) Given an event $B \in \mathscr{F}$ then $\mathbb{P}(B) \in [0,1]$.

(iii.) For $B_1, B_2, ... \in \mathscr{F}$ then $\mathbb{P}\left(\cup_{k=1}^{\infty} B_k\right) \leq \sum_{k=1}^{\infty} \mathbb{P}(B_k)$.

(iv.) If $B_1, B_2, ...$ are disjoints sets in a set $\mathscr{F}$ then $\mathbb{P}\left(\cup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mathbb{P}(B_k)$. Therefore, if $B, C \in \mathscr{F}$ then $B \subseteq C$ implies that $\mathbb{P}(B) \leq \mathbb{P}(C)$ (Øksendal, 2003; Kijima, 2016).

Definition 1.4 ***(Probability Space)***
A probability space is the triple of the form $(\Omega, \mathscr{F}, \mathbb{P})$ in such a way that Ω is a non-empty set, $\mathbb{P}$ the probability measure on Ω and $\mathscr{F}$ is a σ-field containing subsets of Ω (Øksendal, 2003; Kaluszka & Krzeszowiec, 2012).

Definition 1.5 ***(Random Variable)***
For a given probability space of the form $(\Omega, \mathscr{F}, \mathbb{P})$ then a random variable Y is a function with real values defined on the sample space Ω whose value is determined by the random experiment (Liao, 2013).

Definition 1.6 ***(Stochastic Process)***
A stochastic process also known as a random process is a family of random variables $Y(t) : t \in [0, T]$ which is defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ in which $[0, T]$ is a subset of the real line $\mathbb{R}$. A stochastic process is said to be real if the all random variables are real valued and the random process is complex if all the random variables are complex valued (Todorovic, 2012).

Note: In most cases T represent an interval of time, a random variable $Y(t)$ describes the random process at time t. The set T is known as the domain of the stochastic process $Y(t)$, when T is discrete for example $T = \mathbb{N}_+ = \{0, 1, 2, ...\}$ then stochastic process is discrete and when T is continuous for example $T = \mathbb{R}_+ = [0, \infty)$ then the stochastic process is said to be continuous (Todorovic, 2012).

Definition 1.7 ***(Filtration)***
A filtration on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is an increasing family $\mathbb{F} = (\mathscr{F}_t)_{t \in [0, \infty)}$ of σ-fields such that $\mathscr{F}_s \subset \mathscr{F}_t \subset \mathscr{F}$, $\forall\ 0 \leq s \leq t$ in $[0, \infty)$. $\mathscr{F}_t$ is interpreted as the information known at time t, and increases as time elapses. The quadruple $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ is called the filtered probability space.

A filtration $\mathbb{F} = (\mathscr{F}_t)_{t \in [0, \infty)}$ is said to satisfy the usual conditions if it is right continuous, that is $\mathscr{F}_t^+ = \cap_{s \geq t} \mathscr{F}_s = \mathscr{F}_t, \quad \forall t \in [0, \infty)$ and it is complete, that is it even contains the smallest σ-field containing all $\mathscr{F}_t, t \in [0, \infty)$.

Definition 1.8 ***(Stopping Time)***
A stopping time is a random variable (random time) whose value permits a given stochastic process to exhibits a certain behaviour of interest. A stopping time is mostly defined by a stopping rule, a mechanism for deciding whether to continue or stop a process on the basis of the present position and past events. For a given filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t \in [0, \infty)}, \mathbb{P})$ then a stopping time is any non-negative random variable $\mathscr{T}$ that satisfies $\mathscr{T} \leq t \in \mathscr{F}$ for any

$t \geq 0$.

Definition 1.9 ***(Adapted Process)***
A process $(X_t)_{t\in[0,\infty)}$ is adapted with respect to filtration $\mathscr{F}$ if for all $t \in [0,\infty)$ then X_t is $\mathscr{F}_t$ measurable. This is to say the adapted process is a process whose value at any time t is revealed by the available information given by $\mathscr{F}_t$. This can be interpret that X is adapted if and only if, for every t, X_t is known at time t. A standard Brownian motion on $[0,\infty)$ for a filtration $\{\mathscr{F}, t \in [0,\infty)\}$ is an example of adapted process.

Definition 1.10 ***(Portfolio Strategy)***
Portfolio strategy can be defined as a stochastic process that is given by

$$\pi = \{\pi_t = \left(\Pi_{b,t}, \Pi_{s,t}\right), t \in [0,T]\}, \tag{1.1}$$

that obeys the following conditions:

(i) π must be measurable progressively.

(ii) π must be an adapted process, that is $\forall_t, \pi_t$ is $\mathscr{F}_t$ measurable.

In equation 1.1 this portfolio strategy can be interpreted in a sense that $\Pi_{b,t}$ represents the number of units in risk free assets (bonds) that were held by an insurance company at a given time t also $\Pi_{s,t}$ can be interpreted as the number of units in risky assets (stocks) that were held by an insurance company at the same time.

1.2.1 Brownian Motion

If the stochastic process W_t is the Brownian motion then;

(i) For each t , W_t is a Gaussian random variable with mean 0 and variance t.

(ii) The stochastic process W_t has increments which are independent , that is for $0 \leq t_1 < t_2 < \ldots. < t_n$, the random variables $W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent.

(iii) Almost all sample paths of W_t are continuous.

(iv) It starts at 0 with probability of 1.

A Brownian motion is said to be n-dimensional if it is $\mathbb{R}^n$ valued stochastic process given by $W = \left(W_t^1, W_t^2, W_t^3, \ldots, W_t^n\right)$ where the components W_t^i are all independent one dimension Brownian motion (Löffler & Kruschwitz, 2019).

1.2.2 Itô Process

Let $W = (W_t)_{t\in[0,\infty)}$ be a 1-dimensional Brownian motion on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t\in[0,\infty)}, \mathbb{P})$. Then a stochastic integral (*Itô* process) is a process $Y = (Y_t)_{t\in[0,\infty)}$ valued in $\mathbb{R}$ such that almost surely

$$Y_t = Y_0 + \int_0^t \sigma(s, Y_s)dW_s + \int_0^t b(s, Y_s)ds, \quad t \in [0, \infty), \tag{1.2}$$

where Y_0 is $\mathscr{F}_0$ -measurable, $b : \mathscr{T} \times \mathbb{R} \to \mathbb{R}$ are progressively measurable processes valued in $\mathbb{R}$ such that

$$\int_0^t |\sigma_s|^2 ds + \int_0^t |\mu_s| ds < \infty, \quad a.s. \quad \forall t \in [0, \infty). \tag{1.3}$$

Itô process in differential form can be written as

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dW_t, \tag{1.4}$$

where σ and μ are the diffusion (dispersion) and drift coefficients respectively (Shreve, 2004; Steele, 2012).

Theorem 1.1 ***(1-Dimensional Itô's Formula)***
Let Y_t be the Itô process and let $f(t,y) \in C^2([0,\infty) \times \mathbb{R})$. Then $Z_t = f(t, Y_t)$ is also an *Itô* process , and

$$dZ_t = \frac{\partial f}{\partial t}(t, Y_t)dt + \frac{\partial f}{\partial y}(t, Y_t)dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(t, Y_t)(dY_t)^2, \tag{1.5}$$

where $(dY_t).(dY_t) = (dY_t)^2$ is obtained by using the rule $dtdt = dtdW_t = dW_tdt = 0, dW_tdW_t = dt$ (Shreve, 2004; Steele, 2012).

Proof:
Suppose Y_t is an Itô process given by equation 1.4. Suppose $Z_t = f(t, Y_t)$ from 1.5 then,

$$dZ_t = df(t, Y_t) = \frac{\partial f}{\partial t}(t, Y_t)dt + \frac{\partial f}{\partial y}(t, Y_t)dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(t, Y_t)(dY_t)^2.$$

Using 1.4;-

$$dZ_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial y}(\mu_t dt + \sigma_t dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dY_t)^2,$$

$$dZ_t = \frac{\partial f}{\partial t}dt + \mu_t\frac{\partial f}{\partial y}dt + \sigma_t\frac{\partial f}{\partial y}dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dY_t)^2. \tag{1.6}$$

Also
$(dY_t)^2 = (\mu_t dt + \sigma_t dW_t)^2 = \mu_t^2(dt)^2 + 2\mu_t\sigma_t dtdW_t + \sigma_t^2(dW_t)^2.$

Now, by using the rule $dtdt = dtdW_t = dW_tdt = 0, dW_tdW_t = dt$, thus

$$(dY_t)^2 = \sigma_t^2 dt. \tag{1.7}$$

Equation 1.6 become

$$dZ_t = \frac{\partial f}{\partial t}dt + \mu_t \frac{\partial f}{\partial y}dt + \sigma_t \frac{\partial f}{\partial y}dW_t + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial y^2}dt. \tag{1.8}$$

Equation 1.8 can be written as

$$dZ_t = \left[\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial y} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial y^2}\right]dt + \sigma_t \frac{\partial f}{\partial y}dW_t. \tag{1.9}$$

From 1.9 let the drift part $\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial y} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial y^2} = \hat{\mu}_t$ and the diffusion part $\sigma_t \frac{\partial f}{\partial y} = \hat{\sigma}_t$, therefore $dZ_t = \hat{\mu}_t dt + \hat{\sigma}_t dW_t$ which is also an Itô process.
End of the proof.

Theorem 1.2 ***(Generalized Itô's Formula)***

Let $dY_t = \mu(t,Y_t)dt + \sigma(t,Y_t)dW_t$ be an n-dimensional *Itô* process with $Y_t = \begin{pmatrix} Y_{1,t} \\ Y_{2,t} \\ \vdots \\ Y_{n,t} \end{pmatrix}$,
$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}$, $\sigma = \begin{pmatrix} \sigma_{11} & \dots & \sigma_{1m} \\ \vdots & & \vdots \\ \sigma_{n1} & \dots & \sigma_{nm} \end{pmatrix}$ and $dW_t = \begin{pmatrix} dW_{1,t} \\ dW_{2,t} \\ \vdots \\ dW_{n,t} \end{pmatrix}$. Let $f(t,Y_t) = (f_1(t,Y_t), f_2(t,Y_t), ..., f_p(t,Y_t))$ be C^2 map from $[0,\infty) \times \mathbb{R}^n$ into $\mathbb{R}^p$. Then the process $Z_{t,\omega} = f(t,Y_t)$ is also an *Itô* process whose component number k, Z_k is given by

$$dZ_k = \frac{\partial f_k}{\partial t}(t,Y)dt + \sum_i \frac{\partial f_k}{\partial y_i}(t,Y)dY_i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 f_k}{\partial y_i \partial y_j}(t,Y)(dY_i)(dY_j), \tag{1.10}$$

where dY_idY_j is obtained by using the rule $dW_idW_j = \delta_{ij}dt, dW_idt = dtdW_i = 0$ and δ_{ij} is the Kronecker delta function given by

$$\delta_{ij} = \begin{cases} 1 & when \quad i = j \\ 0 & when \quad i \neq j \end{cases}$$

(Shreve, 2004; Steele, 2012).

1.2.3 White Noise and Perturbation Procedure

The White Noise process $\xi(t)$ is formally defined as the derivative of the Brownian motion $B(t)$,

$$\xi(t) = \frac{dB(t)}{dt} = B'(t). \tag{1.11}$$

It does not exist as a function of t in the usual sense, since a Brownian motion $B(t)$ is nowhere differentiable. If $\sigma(x,t)$ is the intensity of the noise at point x at time t, then it is agreed that,

$$\int_0^T \sigma(X(t),t)\xi(t)dt = \int_0^T \sigma(X(t),t)B'(t)dt = \int_0^T \sigma(X(t),t)dB(t), \tag{1.12}$$

where the integral is Itô integral. Stochastic Differential Equations arise, for example, when the coefficients of ordinary equations are perturbed by white noise (Klebaner, 2012).

To introduce the stochastic term into non-stochastic model, that is to perturb the non-stochastic model, then the procedure is to perturb the constant and the perturbation of the constant is done by using the white noise, and the white noise goes together with the intensity of the noise and intensity of the noise is the one that introduces the volatility sigma to be used together with the stochastic term in the perturbed model (Klebaner, 2012).

Definition 1.10 ***(Martingales)***
Consider a given real-valued $\mathscr{F}_t$-adapted stochastic process $Y = \{Y_t\}_{t\in[0,\infty)}$ that satisfy $\mathrm{E}[Y_t] < \infty$ for all $t \geq 0$. Then, for $0 \leq s < t$,

(i) The process Y_t is referred as a sub-martingale if $E[Y_t|\mathscr{F}_s] \geq Y_s$.

(ii) The process Y_t is referred as a super-martingale if $E[Y_t|\mathscr{F}_s] \leq Y_s$.

(iii) The process Y_t is referred as a martingale if $E[Y_t|\mathscr{F}_s] = Y_s$.

Note: A process $Y = \{Y_t\}_{t\in[0,\infty)}$ is a martingale if and only if it is both a sub-martingale and a super-martingale (Serfozo, 2009; Liao, 2013).

Theorem 1.3 A one-dimensional Brownian motion $B = \{B_t\}_{t\in[0,\infty)}$ is a martingale.

Proof:
Let $0 \leq s < t$ then it follows that

$$\begin{aligned}
E[B_t|\mathscr{F}_s] &= E[B_t - B_s + B_s|\mathscr{F}_s], \\
&= E[B_t - B_s|\mathscr{F}_s] + E[B_s|\mathscr{F}_s] \quad (\text{by} \quad \text{independence}), \\
&= B_s \quad (B_s + E[B_t - B_s] \quad \text{is} \quad \mathscr{F}_s - \text{measurable}), \\
&= B_s + 0, \\
&= B_s.
\end{aligned}$$

End of the proof.

Definition 1.11 ***(Leibniz Integral Rule)***
This rule illustrates how to take derivatives under the integration sign. It is useful when it comes to the conversion of integral equations into differential equations.

(i) For $a(x) < b(x)$ functions of x, the Leibniz Integral rule is given by

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} F(x,y)dy\right) = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}(F(x,y))dy + F(x,b(x))\frac{db(x)}{dx} - F(x,a(x))\frac{da(x)}{dx}. \tag{1.13}$$

(ii) For $a < b$ some constants, the Leibniz Integral rule is given by

$$\frac{d}{dx}\left(\int_{a}^{b} F(x,y)dy\right) = \int_{a}^{b} \frac{\partial}{\partial x}(F(x,y))dy. \tag{1.14}$$

1.2.4 Ruin Time and Ruin Probability

Ruin theory was introduced by the well known model called the Cramér–Lundberg model (or Poisson risk process, classical compound-Poisson risk model or classical risk process). It was introduced back in 1903 by the Swedish Filip Lundberg. The work was republished in the 1930s by Harald Cramér. The model describes an insurance company which experiences two opposing cash flows: incoming cash premiums and outgoing claims. For instance if an insurer's surplus is modelled by $X(t)$ for $t \geq 0$ and the company started with initial surplus of $X(0) = x$ then the ruin time can be defined as the time when this surplus goes below zero for the first time, mathematically it can be expressed as

$$\tau = \inf\{t > 0 : X(t) < 0\}. \tag{1.15}$$

The Fig. 1 illustrate more where $X(\tau-)$ surplus prior to ruin and $X(\tau)$ the deficit at ruin.

The ruin probability is the probability that the portfolio becomes negative in finite time. This is to say that ruin occurs when the surplus process, modelled as a stochastic process, becomes negative for the first time. Mathematically ruin probability can also be presented as

$$\begin{aligned}\psi(x) &= E\left[I(\tau < \infty)|X(0) = x\right],\\ &= \mathbb{P}\left[\tau < \infty|X(0) = x\right].\end{aligned} \tag{1.16}$$

Where in equation 1.16, the I is the indicator function, that is, $I(\tau < \infty) = 1$ if $\tau < \infty$, and

$I(\tau < \infty) = 0$ if $\tau = \infty$. Note that the probability of ruin before or at time t is denoted by

$$\begin{aligned} \psi(x,t) &= E\left[I(\tau \le t)|X(0) = x\right], \\ &= \mathbb{P}\left[\tau \le t|X(0) = x\right]. \end{aligned} \tag{1.17}$$

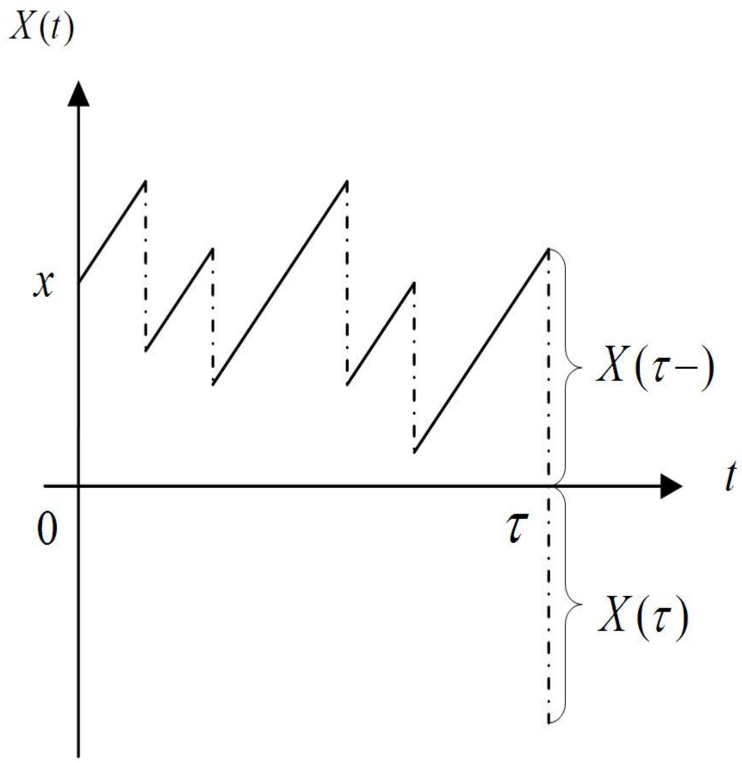


Figure 1: Ruin Time

1.2.5 Utility Functions

In this section a brief survey on the utility functions is given. In the investment process the insurer is required to make decisions while minimizing the risk at the same time. Thus it is necessary to find a way of associating decisions with respect to preferences, that is the measure of a degree of insurer's satisfactions. Expected utility criterion is one of the criteria the insurer can use. Among other criteria it also includes the probability of ruin. Morgenstern and Von Neumann suggested that the preferences can be represented by the expectation of some functions called utility functions (Fishburn, 1989). A utility function is a function of the form $U : [0,\infty) \to [0,\infty)$ that satisfies;

(i) $U(x) \in C^2(\mathbb{R})$.

(ii) $U(x)$ is a strictly increasing function, this means $U'(x) > 0$; non-satiation.

(iii) $U(x)$ is a strictly concave function, this means $U''(x) < 0$; risk-averse.

(iv) $U(x) > 0 \quad \forall x \in \mathbb{R}_+$ and $U(x) = -\infty \quad \forall x \in \mathbb{R}_-$.

where x represents the amount of the insurer's wealth.

Pratt and Zeckhauser (1987) also Crainich and Eeckhoudt (2008) suggested that there are two measures of degree of risk-aversion obtained from the utility functions, namely ;

(i) Absolute risk aversion (ARA) function the ratio $A(x) = -\frac{U''(x)}{U'(x)}$.

(ii) Relative risk aversion (RRA) function the ratio $R(x) = -x\frac{U''(x)}{U'(x)}$.

Insurer may use the above measures to make decisions in the process of portfolio optimisation while taking into account the risk associated with each asset. Actually, the utility functions gives an opportunity to visualize the preferences relation between various levels in wealth and for various portfolio strategies.

The following are some examples of various utility functions which are commonly used;

Exponential Utility Function $U(x) = 1 - \mathrm{e}^{-ax}, \quad a > 0$.
The ARA in the case is constant given by $A(x) = a$, this means that the behaviour of the insurer towards the risk doesn't depend on the initial wealth. These utility functions are among the most widely used functions to represent the attitude of the investors towards risk in optimisation of the portfolio (Çanakoğlu & Özekici, 2009).

Logarithmic Utility Function $U(x) = \log x$.
The ARA is $A(x) = \frac{1}{x}$, this decreases as the wealth increases. This mean that somebody with higher capital is less afraid of taking risk than somebody with lower wealth which actually makes sense from the economics point of view.

Power Utility Function $U(x) = \frac{x^{\alpha}}{\alpha}, \quad 0 < \alpha < 1$.
The ARA is $A(x) = \frac{1-\alpha}{x}$, which is a decreasing function of wealth. This can be interpreted in similar way as logarithmic utility functions.

Generally, insurance companies are very vital as a mechanism for recovery from loss and they must exist as going concerns. Scholars have the responsibility to research the performance of insurance companies and give suggestions/recommendations to the insurance industry where necessary. Many insurance companies became bankrupt due to failure to manage their portfolios; as a result clients suffer by losing their rights to be served by the companies in keeping with their contracts. The goal of the insurer is to find the process that enables him/her to make as much money as possible. Therefore in that regard utility functions are assigned to wealth so that the expected utility of the wealth can be maximised at some future time.

1.3 Statement of the Research Problem

Shareholders of insurance businesses are interested in optimizing the returns from the insurance portfolio as well as ensuring that the business remains afloat over a long-time horizon. To achieve this, the managers of the company have to optimally run the business with objectives of maximizing returns and reducing ruin probability. Even in extreme care, many times ruin is inevitable. Most studies in the literature for example Schmidli (2002), Kasozi *et al.* (2013) and Kasumo *et al.* (2018) do not consider recovery from ruin, once it hits. This study seeks to develop and analyse an insurance portfolio optimisation model and determine the optimal investments and refinancing strategies when there are possibilities of recovery after ruin. This is the gap that this research comes to fill.

1.4 Research Justification

In investments there is a trade-off between risks and returns. In turn, to increase the expected returns from investment, investors must be willing to tolerate greater risks (Kolm *et al.*, 2014). Portfolio management theory helps in studying how to model the trade-off for the given collections of several possible investments (Taillard, 2012).

Investigating companies that have suffered from ruin is one of the very important area one can choose to research. Some research studies have been done to investigate portfolio optimisation, most of them applied reinsurance and refinancing approaches (Liu & Hu, 2014; Kasumo, 2019). However more research is needed on these insurance companies that have suffered from ruin because little has been done to investigate how these companies with the possibility of recovery after ruin can be managed financially to become profitable again.

This research will open the way to other researchers to conduct more research on how to manage the portfolio of insurance companies that have suffered from ruin for them to recover. Refinancing and investment will be considered as approaches to be used in making these insurance companies get back to profitable operations. Also, the study employs an approach of converting the Volterra integro-differential equations into an ordinary differential equations for solving this kind of problem. This approach has not been used before to solve these kinds of insurance modelling problems.

1.5 Research Objectives

1.5.1 Main Objective

To formulate and analyse an insurance portfolio optimisation model and determine the optimal control strategies with possibilities of recovery after ruin.

1.5.2 Specific Objectives

The specific objectives of the research are:

(i) To formulate and analyse the insurance portfolio optimisation model.

(ii) To determine the optimal control strategies that maximises the insurance company's wealth at some future time.

(iii) To determine the optimal investments and refinancing strategies under ruin probability constraints.

1.6 Research Questions

(i) How can the insurance portfolio optimisation model be formulated and analysed?

(ii) What is the optimal control strategy that maximises the insurance company's wealth at some future time?

(iii) What are the optimal investments and refinancing strategies under ruin probability constraints?

1.7 Significance of the Study

(i) The study will help the managers (decision-makers) in insurance industry to understand how they should control their portfolio so that their companies can perform better upon recovery after ruin.

(ii) The study will give the vast knowledge on how to invest and refinance under ruin probability constrains. This knowledge will be very useful to insurance mathematicians and managers (decision-makers) of insurance companies.

(iii) The study will increase knowledge of the current body of knowledge on insurance mathematics and insurance industry at large.

(iv) The study will be submitted as a partial fulfilments of the requirements for the award of the degree of Doctor of Philosophy of the NM-AIST.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Much research has been conducted highlighting various approaches to portfolio optimisation. The intention of this chapter is to review relevant and important literature related to this study, the review has been subdivided into four parts namely investing, refinancing, ruin probability and portfolio optimisation.

2.2 Investing

Frajtova-Michalikova *et al.* (2015) focused on investments fund in insurance and banking managed by a company originated in Luxembourg. A company invested in several portfolios of equities and other securities up to ten percent of the actual assets of the given fund. Their portfolio included companies operating in various sectors such as consumer finance, banks, brokerage, investment banks, insurance, and asset management. Dynamic investors were the specific target for the fund. To achieve their objectives of investment, the fund was allowed to be utilized in other financial derivatives such as index options and futures, interest rates, swaps, and foreign currencies forward transactions. The intention was to make sure the value of the funds is increased by selecting profitable stocks upon a thorough analysis of companies. They selected section for each month to quantify the recovery, for calculation of the best, logarithmic quantification was chosen because it is more accurate in determining its height. In their study they didn't address insurance companies that have suffered from ruin but their aim of optimising the investment portfolio was achieved, their results offered an alternative to better compositions and optimisation had been improving to the investor and eventually achieve their intended benefits.

Badaoui *et al.* (2018) considered the problems insurance companies where the wealth of insurers were given by Cramér-Lundberg process. An insurer was free to make the investment in risky assets having stochastic volatility upon looking at the influences of economic factors and remaining surplus in the bank account. A system of stochastic differential equation was used to model a price in the risky assets and economic factors. They then assumed the market was incomplete and studied a problem in maximization of utility of the expected terminal wealth. With exponential preferences to the insurer they proved the uniqueness and existence theorem for a non-linear equation of Hamilton–Jacobi–Bellman (HJB). They managed to produce the optimal strategies and value function in a closed-form. They presented two numerical methods namely a method of Monte-Carlo which was based on stochastic representations of solutions of the insurer's problem using Feynman-Kac's formula and the method of mixed finite differences

Monte-Carlo, in here they aimed at showing the connection between the correlation coefficient and decision of the insurer. They finalized by presenting the Scott model results although their study only addressed profitable companies, they didn't look for insurance companies with possibilities of recovery after suffering from ruin.

Meng *et al.* (2016) investigated optimal reinsurance-investments strategies of an insurer who faces model uncertainties. Insurer acquires new business then invested in financial markets that consisted of one asset that has no risk and one risky asset in which the price process follows a geometric Brownian motion. The expected quadratics distances of the terminal wealth of a given benchmark in the "worst-case" scenario were minimized to obtain closed-form solutions of the optimal strategy and their value function was determined after solving the HJB equation. An example was presented numerically showing the impact of the parameters of the model in the optimal strategy. Other researchers like Zhao *et al.* (2018) decided to introduce a spectral risk measure (SRM) into an optimisation problem of insurance investments. They realized that spectral risks measure could best describe the degree of risk aversion, so the underlying strategies might take the investor's risks attitude into accounts. They observed that when aversive levels were increased to some extent, the impacts on investment strategy disappeared because of marginal effects of risk aversion, although in these studies they didn't explain how to handle insurance companies that have suffered from ruin which have possibilities of recovery.

In their study, Brachetta and Schmidli (2019) considered a diffusion approximation model to insurance risk models where an external driver models the stochastic environments. They considered a type of SAHARA utility function to maximise the expected terminal utility of wealth. Finally, they managed to obtain explicit results. In other paper titled optimal proportional re-insurances and investments for stochastic factors models, Brachetta and Ceci (2019) investigated optimal proportional insurances and investment strategies of insurance companies that wished to have the maximization of its exponential expected utility of the terminal wealth in finite time horizons. With this paper, their goal was the extension of the classical Cramér-Lundberg models by introducing stochastic factors that affect the intensity of the claims arrival process that was described using the Cox process, also the reinsurance and insurance premia. They used classical stochastic controls method based upon the HJB equation they characterized the optimal strategies and provided a verification result of the value functions using a classical solution of the backward partial differential equation. They also presented and discussed the uniqueness and existence of those solutions. Results in various premium calculations principles were illustrated and the new premium calculations rule was proposed so that to obtain realistic strategy and to better fit the stochastic factors model. To finalize their work numerical simulation was done. These studies also didn't look at insurance companies that have suffered from ruin but have possibilities of recovery.

Kasozi *et al.* (2013) used a model which was obtained after perturbation of the basic insurance model, the new model was then used to stand for the dynamics of the wealth of a given insurance company. Later on, the model was compounded by return on investment process using Black-Scholes. The two models from the risky process were used in managing the risk levels, the company entered with the re-insurer into a quota-share kind of reinsurance. Second-order Volterra integro-differential equation was derived and transformed it into a linear volterra integral equations of a second kind and these equations were then solved numerically by the block-by-block method. Their results indicated that the quota-share reinsurance performed better in improving the insurer's survival. In their study they didn't consider an approach of converting the Volterra integro-differential equations into a system of differential equations, this approach will be used in this study.

2.3 Refinancing

Bulinskaya *et al.* (2015) in their paper titled discrete-time insurance model with capital injections and reinsurance, they used capital injection (refinancing) at the end of each period to maintain the company surplus above a chosen level, below which the ruin could occur. One period insurance claims formed a sequence of i.i.d non-negative random variables with finite mean. They applied a non-proportional reinsurance to minimize the total expected discounted injections during a given planning horizon of n periods. There after they calculated insurance and reinsurance premiums by using the expected value principle and managed to establish the optimal reinsurance strategy and the numerical results that illustrates the theoretical ones were provided for three claims distributions. In their study they didn't use the perturbation on the Cramér-Lundberg model, this study uses the perturbed Cramér-Lundberg model to encounter for uncertainty due to claim and premium variation.

The study on a jump-diffusion model was done by Yin and Yuen (2014) to investigate how optimal control on dividends can be achieved. Three different practical optimisation problems were studied one of which capital refinancing was used as an approach. In their first problem, they considered the classical dividend problem without capital injections, in the second problem the intention was to maximise the expected discounted dividend payments minus the expected discounted costs of capital injections over strategies with positive surplus at all times, and the last problem had the same aim as that of the second one, but without having the constraints on capital injections. Although in their study they didn't address how insurance companies that have already suffered from ruin with possibilities of recovery can be handled to came back in profitable operations. Finally under the assumption of the proportional transaction costs, they identified the value function and the optimal strategies for any given distribution of gains.

In another study, Eisenberg and Schmidli (2011) without taking into consideration insurance companies that have suffered from ruin with possibilities of recovery, they had aim of minimising the expected discounted capital injections over all admissible reinsurance strategies. For the diffusion approximation case, they used HJB approach to obtain the optimal strategy and obtaining expression for the value function explicitly. For the case of the classical risk model they managed to show that there is an existance of a weak solution and they used numerical approximations approach to calculate the value function.

Eisenberg (2010) considered optimal control of capital injections by reinsurance and investments. The objective of the study was to investigate how the insurer's capital injections for the future can be controlled by using investments and/or reinsurance, he used the Cramér-Lundberg model and in a diffusion approximation with preference rate greater or equal to zero, to achieve this aim. The Cramér-Lundberg model was also studied by Kasozi *et al.* (2011) where they used the Homotopy Analysis Method (HAM) as the numerical approach to calculate the value function of the dividend payments in a given basic insurance process. In these two studies companies that have already suffered from ruin that have possibilities of recovery were not considered.

Liu and Hu (2014) had the objective of finding the strategy that can maximise the expected present values of their dividends payout minus the equity issuance up to the time of ruin. Although this study also ignored companies that have already suffered from ruin that have possibilities of recovery but finally by using an approach of constructing two different suboptimal control problems they managed to solve the optimal problem and identify the optimal strategy.

He and Liang (2008) studied optimal financing and dividend control of the insurance company with proportional reinsurance policy. It was the first time that the financing process in an insurance model has been considered, which was more realistic. They managed to identify the value functions and the optimal strategies corresponding to the suboptimal models depending on the relationships between the coefficients. In another study, He and Liang (2009) considered optimal financing and dividend control of the insurance company with fixed and proportional transaction costs. It was the first time that the financing process in an insurance model was considered with two kinds of transaction costs which come from real financial market. They aimed at maximizing the expected present value of the dividends payout minus the equity issuance before bankruptcy although in both papers they didn't address companies that have suffered from ruin with possibilities of recovery.

2.4 Ruin Probability

Ruin probability has been studied widely in literature, scholars like Shiu (1989), Tang (2005), Asmussen and Albrecher (2010), Loeffen *et al.* (2013) and Kasumo *et al.* (2018) gives various opinions on how to deal with the ruin probability under various settings of the insurance mathematical models.

Schmidli (2002) considered a classical risk model and allowed investment to be done into a risky asset which was modelled using Black–Scholes model as well as the proportional reinsurance. In their study they didn't introduce perturbation in the Cramér-Lundberg model although by using the HJB approach they found a candidate for the optimal strategy and they managed to develop a procedure for solving HJB equation numerically. They wanted to show that for any increasing solution to the HJB equation it is bounded and solves the optimisation problem, thus they decided to prove the verification theorem. They also proved that the solution to the HJB equation which is always increasing function exists. Then they ended by discussing two numerical examples.

Paulsen *et al.* (2005) studied a numerical method for the aim of finding the probability of ultimate ruin for a given classical risks model with stochastic returns on investments. In their study survival probability function was given a sufficient conditions of being four times continuously differentiable, this actually implied that the survival probability function is a solution of a second order integro-differential equation. After using an approach of transforming this integro-differential (second order) equation into an another equation which was ordinary Volterra integral equation of second kind, they managed to analyze the properties of its numerical solutions upon using the block-by-block method together with the Simpson's rule. Despite the fact that they didn't use an approach of converting the Volterra integro-differential equations into a system of differential equations, by using numerical examples they realized that their methods also works very well.

Ma and Jiang (2018) investigated the discrete dynamical Pareto optimisation model in China upon looking at portfolio for natural disaster insurance. They proposed a model for managing risks that was based on the cooperative insurance among the insurance market, operating government and public. They divided their study areas into units whereby in each unit they analysed the risk stochastic process of the operating government and insurers. The operating government provided the subsidy and policy support. Their risk stochastic process took into consideration the premium income, the fixed initial risk value, the claim and the transaction costs. There after they introduced the ruin probability together with the stopping time of the ruin probability, the stable operation of insurers, and the recovery capability of the public. Finally, they used numerical simulation for verification of the model results.

Liu and Yang (2004) studied optimal investment strategies of an insurance company in order to minimize the probability of ruin to the company. They assumed that the rate of receiving premiums to the insurance company is constant and they used compound Poison process to model the total claims of the company. Insurance company was allowed to invest in risky asset such as stocks and in the money market. By including a risk-free asset their model became the generalization of the model used by (Hipp & Plum, 2000). Using various claim-size distributions they investigated numerically the investment behaviour. The associated HJB equation for the optimal policy was solved for each distribution. Their results provided insights for the managers in insurance companies on how to hove good investments for minimizing the ruin probability. These studies didn't address how to handle insurance companies that have suffered from ruin with possibilities of recovery.

Weibull distribution is one of the distributions that can be used to model data with a higher degree of positive skewness which is mostly seen in the claim amounts. Das and Nath (2019) used Weibull distribution to fit the set of insurance claim data and the probability of ultimate ruin was computed for those Weibull distributed claim data by using two different methods. The consistency has been found in the values obtained from both methods. The influence of the surplus process being subjected to the force of interest earnings and tax payments on the probability of ultimate ruin was also studied. On another side, Deshpande *et al.* (2019) decided to study risk discriminating portfolio optimisation where they described an investment portfolio optimisation method that used both non-linear and linear asymmetric dependence of assets. Due to its nature of return-seeking, it was realized that risk discriminating portfolio optimisation had the chance to outperform the simple mean-variance efficient portfolio. These studies also didn't address how to handle insurance companies that have suffered from ruin with possibilities of recovery.

Kasumo (2019) worked on diffusion and perturbed risks model consisting of an investment return and surplus generating processes. The investments return process obeyed standard Black–Scholes type modeled using geometric Brownian motion. A company was free to purchase non-cheap proportional reinsurance whose price was obtained using expected value principles. By the use of Hamilton–Jacobi–Bellman (HJB) equation a second-order Volterra integro-differential equation was derived and the equation was later transformed into a linear Volterra integral equation of the second kind. The formed equation was then solved by using the method of block-by-block to get the numerical solutions of the optimal reinsurance and retention level that minimized the ultimate ruin probability. Numerical solutions based on a light and a heavy-tailed individual claims amount distribution showed that the proportional investment and reinsurance play vital roles in the enhancement of the survival of insurance company but it was observed that ruin probabilities show sensitivity to the given volatility of stock prices. In his study he didn't address how to handle insurance companies that have suffered from ruin but have possibilities of recovery, also he didn't consider an approach of converting the Volterra integro-differential equations into a system of differential equations, which will be used in this study.

2.5 Portfolio Optimisation

Kümmerle and Rudolf (2016) studied portfolio optimisation under illiquid life insurance investment. By un-smoothing the data from German life insurers they were able to obtain the return or risk patterns of the given underlying financial assets. Mean-variance portfolios were analysed engaging a smoothed with a profit contract which was compared to an unsmoothed unit-linked. They found out that the life insurance was attractive for the case of conservative investors and can't be simply replicated. In addition, their results indicated that investor can't regain expected utility when allocated to the undesirable amounts in the illiquid with-profits contracts neither to the reallocation of his liquid investments nor on collateral lending. Although their study couldn't explain how to handle portfolios of the insurance companies that have suffered from ruin, they finally observed that with-profit contracts were similar in characteristics than some illiquid assets. The major difference is that by law allow the investors to play part in the smoothed returns contrary to other financial market products whereby the investor shall just participate for his holding period returns.

Oliynyk *et al.* (2015) observed how to manage effectively the insurance portfolio of the company. They carried out procedures of formulating mathematical models and they proposed scientific mathematical approaches in performing optimisation of an insurance portfolio of any company in developing countries. They later conducted implementation practically for those methodologies for some insurers of Ukraine. On the other hand, Zhu *et al.* (2015) analysed proportional investments and reinsurance for an insurer of defaultable markets. They assumed that the exponential premium principle was used to calculate the reinsurance premium. The insurer may distribute his wealth in the following securities: a risky stock asset, a corporate bond, and a bank account. The starting optimisation problem was divided into two small problems: a pre-default and a post-default. They also derived explicitly the optimal investments and reinsurance policies that can maximise the utility of wealth, finally, they gave simulations for numerical results and discussed relevant economic meaning obtained from their results. Both the two studies could not address how to handle insurance companies that have suffered from ruin with possibilities of recovery and none of them considered an approach of converting the volterra integro-differential equations into a system of differential equations.

Oyatoye and Arogundade (2011) intended to design a stochastic model capable of predicting the optimum portfolio of insurance business at an acceptable risk exposure level. Although they studied healthier insurance companies and their study ignored companies that have suffered from ruin, they thought this was important because it would guarantee the acceptable risk levels for a viable insurance company, evaluate the retention rate of insurance portfolio at a given risk rate, but also it would provide good knowledge on the importance of reinsurance on risk adjustment in times of larger claim, and finally, it would examine the unbearable risk level that would require co-insurance. They adopted the application of Markowitz's portfolio optimisation method to finance and insurance risks. Risk return analysis and catastrophe exposure analysis were performed and they observed that there is a need to revert to stochastic modelling, which canvases the use of risk, variances and expected values for mathematical computation.

Recently considerable attention on the part of insurance companies is given to the procedures of the formation of a given insurance portfolio because it serves as an indicator of the quality of insurance liabilities. Oliynyk (2015) studied the basic methodological principles of formation and management of insurance portfolio to achieve its equilibrium and to ensure that the financial stability of insurance companies is maintained. One of the stage in the company's insurance portfolio management is to deal with portfolio optimisation. This stage was discussed as it leads to the reduction of risks and an increase in profitability levels. The study finally observed that the proposed scientific and methodical approach to building and managing an insurance portfolio to achieve its equilibrium based on nonlinear programming has a differentiated character. For each company, this model chose an optimal structure of an insurance portfolio that ensures maximal profits and minimal risks.

Ma *et al.* (2018) were motivated to come up with an extension of the work of Zhu *et al.* (2015) to include defaultable securities. The insurer was given a chance of buying a proportional kind of reinsurance, and put his wealth in stock, a defaultable corporate bond, and a money account. The intention was maximizing their expected utility of wealth. In their paper, they chose the constant elasticity of variance (CEV) process for describing the behavior of the stocks. The reason for selecting CEV model was that it can also be used as an alternative model to describe the stochastic volatility behavior of the price of the stock and it had several empirical pieces of evidence to support it. By using theories of stochastic control they derived an equation for Hamilton–Jacobi–Bellman (HJB) and later divided the original problem into two parts a pre-default case and a post-default case. Value functions and expressions of the optimal strategy were derived, finally, they presented examples in numerical forms as illustrations to their results. In their study an approach of forming and later converting the volterra integro-differential equations into an ordinary differential equation was not considered.

2.6 Conclusion

This chapter has reviewed the relevant literature to this study, from the above review, it is clearly observed that researchers contributed a lot in various areas of insurance portfolio optimisation such as investment-reinsurance strategies, proportional reinsurance, portfolio management, and others. However, most of the studies are for profitable companies and not companies that have the possibility of recovery after ruin. To the best of my knowledge, there is no any study that investigated the insurance portfolio optimisation for the companies having possibility of recovery after ruin. This research seeks to fill this gap in the literature. The next chapter presents about the procedure of the model formulation and analysis of the model will be done.

CHAPTER THREE

MATERIALS AND METHODS

3.1 Introduction

The process of model formulation and analysis are achieved in this chapter. The study considers a perturbed Cramér-Lundberg model with investment and refinancing. Before looking at this model it is better to state the Cramér-Lundberg model which is also referred to as the classical risk process, since it forms the basis for the entire model to be formulated and analysed in this study.

3.2 Cramér-Lundberg Model

Ernest Filip Oskar Lundberg Swedish actuary and mathematician, is one of the founders of mathematical risk theory, one among the very important contribution was his introduction of the simple and good model that was capable of entailing the basic dynamics of a homogeneous insurance portfolio. The main concept of the Lundberg model and its extension was the determination of probability of ruin (Mikosch, 2009). In 1930's Harald Cramér a Swedish mathematician, statistician and actuary who specialized in mathematical statistics and probabilistic number theory, decided to extend the Lundberg model and due to that extension till today the model is know as the Cramér-Lundberg Model (CLM).

Cramér-Lundberg model is given by

$$X_t = p + ct - \sum_{i=1}^{N_t} Y_i \quad , \quad t \geq 0. \tag{3.1}$$

Where by

- X_t is the surplus of the insurance company at time t.
- $X_0 = p$ is the initial reserve or surplus.
- c is the premium rate, that is, the insurer's premium income per unit time assumed to be received continuously.
- $\{N_t\}$ is a homogeneous Poisson process with intensity λ , this is the counting process for the claims.
- $\{Y_i\}$ is a sequence of strictly positive independent and identically distributed (iid) random variables representing the claim sizes, with distribution function F having finite first moment μ and finite second moment σ^2.
- $\{N_t\}$ and $\{Y_i\}$ are independent and $\sum_{i=1}^{N_t} Y_i$ is a compound Poisson process with an average number of claims per time period of λ. This process (also called the total claim amount process) represents the aggregate losses to the insurance company.

The classical risk process in equation 3.1 is the surplus process and has dynamics given by

$$dX_t = cdt - d\left(\sum_{i=1}^{N_t} Y_i\right). \tag{3.2}$$

The Cramér-Lundberg model is among the most popular and useful model in insurance mathematics, simple and powerful enough to calculate probabilities of interest but it is too simple to be realistic. The main reason includes the following, it does not include interest earned on the surplus, nor long tail business with claims which are settled a long time after occurrence of claim, nor time dependence or even randomness of premium income. Due to these weaknesses, this study makes some modifications on the Cramér-Lundberg model by taking into consideration the uncertainty due to claim and premium variation and incorporate investment of the surplus into risk and risk-free assets but also incorporating the refinancing aspect.

3.3 Model Assumptions

(i) Trading in the insurance and financial markets in general is continuous.

(ii) The company has a fixed premium rate, only depending on the safety loadings of the insurer.

(iii) There is no transaction cost for the investment.

(iv) The moment when deficit occurs is just the time the company refinances (inject the capital).

(v) There are no costs for refinancing (capital injection) unless borrowed.

(vi) There is no restriction on the investment policy (that is, both borrowing money at the risk-free interest rate and short selling of the risky asset are allowed).

3.4 Model Variables and Parameters

We assumed the model variables and parameters to meet the need of the study. Some are newly used, while some are similar to those used in other studies. Descriptions of the model variables and parameters are given in Table 1 and Table 2, respectively.

3.5 Model Formulation

In this work we consider continuous time stochastic processes, and the time interval $[0,T]$, where $0 < T < \infty$. In actual sense a stochastic process is a family $X = (X_t)_{t\in[0,T]}$ of random variables defined on the probability space $(\Omega, \mathscr{F}, P)$ and valued in a measurable space $\mathbb{R}$ and indexed by time t. For each $\omega \in \Omega$, the mapping $X(\omega) : t \in [0,T] \to X(t;\omega)$ is called the path of the process for the event ω.

All stochastic quantities and random variables are defined on a large enough stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F})_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions, that is $(\mathscr{F})_{t\in[0,T]}$ is right continuous and $\mathbb{P}$-complete, $\mathbb{P}$ is the probability measure defined on $\mathscr{F}$ and $(\mathscr{F})_{t\in[0,T]}$ is an augmented filtration.

3.5.1 Basic Insurance Process without Financing and Investment

In actual sense the income of the insurer is not deterministic, there are exist fluctuations in the number of customers, claim arrival but also premium income. If both financing (capital injection) and investment are absent, to model all these additional uncertainties, upon following the procedure of introducing perturbation to non-perturbed model as explained in chapter one, lets introduce perturbation with intensity σ_X to model 3.1 to have risk process X_t defined by

$$X_t = p + ct + \sigma_X W_{X,t} - \sum_{i=1}^{N_{X,t}} Y_{X,i} \quad , \quad t \geq 0. \tag{3.3}$$

In this case c is the premium rate and is calculated by expected value principle, that is $c = (1+\theta)\lambda_X \mu_X$ where $\theta > 0$ is the relative safety loading of the insurer. Also $X_0 = p$ is the initial capital of the insurance company and W_X is a standard Brownian motion independent of the compound Poisson process $\sum_{i=1}^{N_{X,t}} Y_{X,i}$. Here λ_X is the intensity of the counting process $N_{X,t}$ for the claims and let F_X be the distribution function of the claims $Y_{X,i}$. It is assumed that F_X is continuous and concentrated on $(0,\infty)$.

Table 1: State variables for the model

Variable	Description
X_t	State variable representing the surplus in the absence of refinancing and investment.
X_t^M	State variable representing the surplus with refinancing.
R_t	State variable representing the return on investments at time t.
P_t^M	State variable representing the surplus in the presence of refinancing and investment.
$W_{X,t}$	1-dimensional standard Brownian motion or Wiener process.
$Y_{X,i}$	Amount of the i-th claim.
$N_{X,t}$	Number of claims received by the insurer at time t.

Equation 3.3 can be interpreted as follows, ct is the premium income received by an insurance company up to time t. The Brownian term $\sigma_X W_X$ is meant to take care of small perturbations in premium income and claim sizes. $N_{X,t}$ is the claim number process and $Y_{X,i}$ are claim sizes. It is assumed that $F_X(0) = 0$ and at least one of σ_X or λ_X is non-zero.

A vast number of researchers have studied this classical risk process perturbed by diffusion model in insurance industry, some of them includes; Cai and Xu (2006), Kasozi *et al.* (2013), Kasumo *et al.* (2018), Hu *et al.* (2018), Kasumo (2019) and many others.

Table 2: Parameters for the model

Parameter	Description
c	Fixed premium rate in the absence of refinancing and investment.
σ	Diffusion or volatility coefficient.
θ	Safety loading of the insurer.
λ	Number of claims received per unit time.
ρ	Correlation coefficient.
r_0	Risk-free interest rate for the bond.
r	Instantaneous rate of stock return.

3.5.2 Basic Insurance Process with Refinancing

Let M be an increasing process with $M_0 - = 0$. The process with refinancing (capital injections) is denoted by $X_t^M = X_t + M_t$ with X_t being the surplus process and $X_0 = p$. The injection process M has to be chosen such that $X_t^M \geq 0$ for all t (almost surely), it could then be optimal to inject capital already before the process reaches zero. Lets assume when the insurance company take risk Y_i and at the same time capital refinanced is M then the insurance company will retain $\max(M, Y_i) = y \wedge M$. In return the company has to pay cost of refinancing for example to the lender in case money were borrowed, suppose this cost is calculated as premium using expected value principle with safety loading η and let a rate be given by $(1+\eta)\lambda_X \mathbb{E}[(M-Y_i)^+]$. This means under refinancing process the premium rate left to the company is $c_M = c - (1+\eta)\lambda_X \mathbb{E}[(M-Y_i)^+]$ and assume further that $\eta > \theta$ for healthy refinancing. Therefore, by using equation 3.3 the following equation can be obtained;

$$X_t^M = p + c_M t + \sigma_X W_{X,t} - \sum_{i=1}^{N_{X,t}} Y_{X,i} + M_t \quad , \quad t \geq 0. \tag{3.4}$$

3.5.3 Investment Process

Suppose that in addition to refinancing an insurer is now allowed to invest some of the surplus into financial market consisting of a risk-free asset (for example bond) with a positive interest rate r and the rest to a risky asset (for example stock). An attempt of incorporating investment income in a Cramér-Lundberg Model was done for the first time in 1942 by Segerdahl, the assumption was that the capital earns interests with a fixed rate r (Yuan & Hu, 2008).

Other researchers has been increasingly attracted in this area and many of them extended the researches therein to make this area among very popular in insurance mathematics. Paulsen and Gjessing (1997) studied a risk process with stochastic return on investment, Kasumo (2011) modified Gjessing and Paulsen model by incorporating investment and a proportional reinsurance. This work extends the model by incorporating refinancing and investment.

Assume in the same way as in Meng *et al.* (2016) that the risk-free price process is given by

$$dB_t = r_0 B_t dt, \tag{3.5}$$

where $r_0 \geq 0$ is the interest rate for the risk-free asset, which is assumed to be constant. B_t is the price of the bond at time t.

Let us also describe the risky asset (stock) price process by the Geometric Brownian Motion (GBM) similar to that of Badaoui *et al.* (2018) given by

$$dS_t = rS_t dt + \sigma_S S_t dW_{S,t}, \tag{3.6}$$

where S_t is the stock price at time t, $r \geq 0$ is the expected instantaneous rate of stock return, $\sigma_S \geq 0$ is the diffusion of the stock price and $W_{S,t} : t \geq 0$ is a standard Brownian motion defined on the complete probability space $(\Omega, \mathscr{F}, (\mathscr{F})_{t\in[0,T]}, \mathbb{P})$.

Paulsen and Gjessing (1997) in their work gave the generalized return on investment process R_t by using the equation 3.7 ;-

$$R_t = rt + \sigma_R W_{R,t} + \sum_{i=1}^{N_{R,t}} S_{R,i} \quad , \quad t \geq 0, \quad R_0 = 0, \tag{3.7}$$

where $W_{R,t}$ is a Brownian motion independent of the surplus process R_t also $\sum_{i=1}^{N_{R,t}} S_{R,i}$ is a Poisson process with intensity λ_R which represents the sudden changes in income (jumps), the term $\sigma_R W_{R,t}$ represents the fluctuation in income of an insurance company and the rt is non risk part of the investment process. If we assume $\lambda_R = 0$ that is there is no jumps, the resulting model is the Black-Scholes model given by

$$R_t = rt + \sigma_R W_{R,t} \quad t \geq 0, \quad R_0 = 0. \tag{3.8}$$

Equation 3.8, is the return in investment model. r is the risk-free part, hence $R_t = rt$ means one unit that was invested at time zero will be worth e^{rt} at time t. The process R_t in equation 3.8 is known as Black and Scholes option pricing formula.

3.5.4 Risk Process with Refinancing and Investment

In this part the insurance process compounded with refinancing and return from investment is obtained. It is obtained by combining equations 3.4 and 3.7 above. For the case of reinsurance and investment the process has been extensively studied for ultimate ruin probability, this can be observed in studies such as Paulsen and Gjessing (1997), Paulsen and Rasmussen (2003), Paulsen *et al.* (2005), Kasozi *et al.* (2013) and many others. Following the similar approach as in Kasozi *et al.* (2013) the process $P^M = \{P_t^M\}_{t\in[0,\infty)}$ which represent the insurance portfolio is given by:

$$P_t^M = X_t^M + \int_0^t P^M(s^-)dR(s), \tag{3.9}$$

which is the solution of the SDE

$$dP_t^M = dX_t^M + P^M(t^-)dR(t), \tag{3.10}$$

where in this case X_t^M is the basic insurance process given in equation 3.4 and $R(t)$ is the return on investment process given in equation 3.8 and $P^M(t^-)$ stands for the insurer's surplus just prior to time t.

3.5.5 Stochastic Differential Equation for the Wealth

In this section the basic insurance process with investment, which is expressed by the stochastic differential equation for the wealth after refinancing, is formulated. Now let us consider the investment problem of an insurance company which wants to transfer current wealth into the bond and stock. The company prefers to choose the dynamic portfolio strategies in order to maximise its expected utility of wealth at some future time T. Therefore in order to describe the company's actions the portfolio strategy is formulated.

Assume that the joint distribution of the $W_{X,t}$ and $W_{S,t}$ is bi-variate normal and we denote their correlation coefficient by ρ, that is $E\left[W_{X,t}W_{S,t}\right] = \rho t$. The company needs to monitor its wealth, let the amount of money invested in risky asset (stock) at time t under investment policy π be denoted by π_t , where $\{\pi_t\}$ is a portfolio strategy suitable and admissible control process, that is to say π_t satisfies $\int_0^T \pi_t^2 dt < \infty$ a.s., for all $T < \infty$.

Let $\{Z_t, t \geq 0\}$ denote the corresponding wealth process, then the dynamic Z_t is given by

$$dZ_t = \pi_t \frac{dS_t}{S_t} + (X_t^M - \pi_t)\frac{dB_t}{B_t} + dX_t^M, \tag{3.11}$$

with $Z_0 = z > 0$ being the initial wealth of the company.

By using equations 3.4, 3.5 and 3.6 then the wealth process with investment and refinancing will follow the following stochastic differential equation

$$dZ_t = \big(\pi_t r + (X_t^M - \pi_t) r_0\big)\,dt + \sigma_S \pi_t dW_{S,t} + dX_t^M. \tag{3.12}$$

3.5.6 Optimal Control Problem for Maximizing the Expected Utility of Terminal Wealth

Ferguson (1965) studied the problem of expected utility of wealth in the discrete time for a given investor. In that study it was conjectured that minimizing the ruin probability is strictly related to maximizing exponential utility of terminal wealth of the investor, the assumption behind the conjecture was that the investor is allowed to borrow an unlimited amount of money and without risk-free interest rate.

Let a strategy α describe the stochastic process $\{\pi_t, M_t\}$, where π_t is the amount invested in the risky asset at time t and M_t is the capital refinanced/injected at time t and denote the set of all admissible strategies by α_s. Suppose now that the insurer is interested in maximizing the utility function of its terminal wealth, say at time T . The utility function $u(z)$ is typically concave and increasing $(u''(z) < 0)$. For a strategy α , lets define the utility attained by the insurer from state z at time t as follows;

$$V_\alpha(t,z) = \mathbb{E}\left[u\left(Z(T)\right)/Z(t) = z\right]. \tag{3.13}$$

Therefore the objective is to find the optimal value function

$$V(t,z) = \sup_{\alpha \in \alpha_s} V_\alpha(t,z), \tag{3.14}$$

and the optimal strategy $\alpha^*\{\pi_t^*, M_t^*\}$ such that $V_{\alpha^*}(t,z) = V(t,z)$.

3.6 Dynamic programming and Hamilton-Jacobi-Bellman equation

At this point the interest is to solve the stochastic optimal control problem 3.14 by looking for a maximum value of the performance function 3.13 subject to the state which is the wealth equation given by 3.12.

First the the necessary definitions and theorems are given then the Bellman's principle of optimality, which is commonly known as the Dynamic programming principle (DPP) is stated in the Theorem 3.2 .

Definition 3.1 ***(Infinitesimal generator)***
Let $(Z_t)_{t\geq 0}$ be an Itô diffusion in $\mathbb{R}^n$. Then the infinitesimal generator $\mathscr{L}$ of Z_t is defined by

$$\mathscr{L}g(s,z) = \lim_{t\to s} \frac{\mathbb{E}^{s,z}[g(t,Z_t)] - g(s,z)}{t-s} \qquad ,z \in \mathbb{R}^n, \tag{3.15}$$

and g is in the domain $\mathscr{D}_{\mathscr{L}}$ which is the class of functions $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ for which the limit exists for all s,z (Pham, 2009).

Proposition 3.1
Let $\mathscr{L}$ be the differential operator defined on $C^{1,2}$ by

$$\mathscr{L} = \frac{\partial}{\partial t} + \sum_{i}^{n} b_i \frac{\partial}{\partial z} + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial z_i \partial z_j}, \tag{3.16}$$

with $a_{ij} = (\sigma\sigma^T)_{ij}$ and let g be defined on $C^{1,2}$ such that for all $0 \leq s \leq t, z \in \mathbb{R}$. Then

$$\mathbb{E}^{s,z}\left[\int_s^t |\mathscr{L}g(u,Z_u)|du\right] < \infty \quad \text{and} \quad \mathbb{E}^{s,z}\left[\int_s^t [(D_z g(u,Z_u))^T \sigma(u,Z_u)]^2 du\right] < \infty, \tag{3.17}$$

thus $g \in \mathscr{D}_{\mathscr{L}}$ and $\mathscr{L}g = \mathscr{L}g$ (Ndounkeu, 2010)

Proof
The proof is available in (Ndounkeu, 2010).

Remark 3.1
It can be observed that $\mathscr{L}$ is also a generator of Itô diffusion. Therefore when it is applied to the function $g \in C^{1,2}$ it results in $\mathscr{L}g(t,z) = g_t(t,z) + (D_z g(t,z))^T b + \frac{1}{2} tr((D_{zz} g(t,z))^T a)$ where in this case $D_{zz}g(t,z)$ and $D_z g(t,z)$ are the Hessian matrix and gradient vector respectively of the function $g \in C^{1,2}$ while $tr(B) = \sum_1^n b_{ii}$ is the trace of any square matrix $B = (b_{ij}) \in \mathbb{R}^{n\times n}, 1 \leq i,j \leq n$.

Theorem 3.1 ***((Dynkin's formula)***
Suppose $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}$. Let τ be a stopping time in such a way that $\mathbb{E}^z[\tau] < \infty$. Then

$$\mathbb{E}^z[g(\tau, Z_\tau)] = g(z) + \mathbb{E}^z\left[\int_0^\tau \mathscr{L}g(s,Z_s)ds\right]. \tag{3.18}$$

Note: If τ is the first exit time from the bounded set $\mathscr{G} \in \mathbb{R}^n$, then Dynkin's formula holds for any $g \in C^{1,2}$.

Proof
The proof is available in (Øksendal, 2003).

Theorem 3.2 ***(Bellman's equation)***
For all $(t_0, y) \in [0,T] \times \mathbb{R}^n$ and $t_1 \in [t_0, T]$,

$$V(t_0,y) = \sup_{\alpha \in \alpha_s} \mathbb{E}^{t_0,y}\left[\int_{t_0}^{t_1} \phi(s,Z_s,\alpha_s)ds + V(t_1, Z_{t_1})\right] \quad ,\forall \quad 0 \leq t_0 \leq t_1 \leq T. \tag{3.19}$$

In other words, this is to say that, an optimal policy from t_0 to T that passes through t_1 is also optimal in $[t_1, T]$ (Yong & Zhou, 1999; Ndounkeu, 2010).

Proof

See Yong and Zhou (1999) also (Ndounkeu, 2010).

After imposing some assumption on the value function, Bellman's equation given in Theorem 3.2 is very useful in the derivation of the so called HJB equation upon applying Itô's formula 1.5 to the value function V if it is smooth enough, and with some reasoning leads to HJB equation.

According to Yong and Zhou (1999) upon considering a modified state equation with a control α given by

$$\begin{cases} dZ_t^{\alpha} = b(t, Z_t, \alpha_t) + \sigma(t, Z_t, \alpha_t) dW_t \quad , \quad t \in [0, T], \\ Z_{t_0} = z \quad , \quad z \in \mathbb{R}^n, \end{cases} \tag{3.20}$$

together with the performance function given by

$$J(t_0, Z_t, \alpha_t) = \mathbb{E}\left[\int_{t_0}^{T} f(t, Z_t, \alpha_t) dt + h(Z_T)\right]. \tag{3.21}$$

Then if the value function V is smooth enough it is the solution of a second order non-linear partial differential equation

$$\sup_{\alpha \in \alpha_s} \left\{ \phi(t, z, \alpha) + V_t(t, z) + (D_z(V(t, z))^T b(t, z, \alpha) + \frac{1}{2} tr(D_{zz}(V(t, z)) a(t, z, \alpha) \right\} = 0. \tag{3.22}$$

Equation 3.22 can also be written by using the differential operator $\mathscr{L}$ given by equation 3.16 to get

$$\sup_{\alpha \in \alpha_s} \{\phi(t, z, \alpha) + \mathscr{L}^{\alpha} V(t, z)\} = 0. \tag{3.23}$$

Here the supremum is taken over all the admissible control. So, for a fixed z, the quantity will be maximised only through $\alpha \in U$ where U is a given utility function, then the following equation can be formulated upon using 3.23 ;-

$$\begin{cases} \sup_{\alpha \in U} \{\mathscr{L}^{\alpha} V(t, z)\} = 0, \\ V(T, z) = U(T, z) = U(z). \end{cases} \tag{3.24}$$

Equation 3.24 is called HJB equation, and also known as the dynamic programming equation which is very vital in optimal control and U(z) is the utility function under consideration. This will give a non-linear partial differential equations (PDEs) which sometimes can be complicated and hard to solve.

From equation 3.12 the generator is

$$\mathscr{L}^{\pi} g(t,z) = g_t + \left[\pi_t r - \pi_t r_0 + X_t^M r_0\right] g_z + \frac{1}{2}\left[\sigma_s^2 \pi_t^2 + 2\rho\sigma_s\pi_t + 1\right] g_{zz}. \tag{3.25}$$

No constraints are put on the control π_t, it is allowed that $\pi_t < 0$ or also $\pi_t > Z_t$. This assumption was first applied by Ferguson (1965) when he considered a discrete time problem for an ordinary investor. Under $\pi_t < 0$ an insurance company is considered to short the stock while for the case of $\pi_t > Z_t$ an insurance company is borrowing money to invest long in the stock. This is realistic provided that the insurance company has positive net wealth, that is $Z_t > 0$, it can be allowed to borrow money for investment, but when the ruin occurs, that is an insurance company is bankrupt it will not be allowed to borrow money for investment. On account of this fact the probability of ruin and its possibility to occur is of real great concern.

3.6.1 Maximizing the Exponential Utility of Terminal Wealth

An ordinary investor under discrete time and space was studied by Ferguson (1965) where it was found that when the investor had an exponential utility function such as $u(z) = -e^{-\theta z}$ and aiming at maximizing the utility of terminal wealth at fixed terminal time then the optimal policy was investing a fixed constant. The conclusion given by a strategy was in general optimal for minimizing the probability of ruin or maximizing the probability of survival.

Using equations 3.13 and 3.14 , let π_t^* denote the optimal policy and suppose that the company is now having an exponential utility of the form 3.26 , where $\gamma > 0$ and $\theta > 0$.

$$u(z) = \lambda - \frac{\gamma}{\theta} e^{-\theta z}. \tag{3.26}$$

This kind of utility function have constant ARA since $-u''(z)/u'(z) = \theta$, it plays a very important function/role in actuarial and insurance mathematics at large.

Theorem 3.3
The optimal policy to maximise expected utility at a terminal time T is to invest at each time $t \leq T$ the constant amount given by

$$\pi_t^* = \frac{r}{\sigma_s^2 \theta} - \frac{\rho}{\sigma_s}. \tag{3.27}$$

Then optimal value function become

$$V(t,z) = \lambda - \frac{\gamma}{\theta} exp\left\{-\theta z + (T-t)Q(\theta)\right\}. \tag{3.28}$$

In here $Q(.)$ is the quadratic function which is defined by

$$Q(\theta)=\frac{1}{2}(1-\rho^2)\theta^2-\left(X_t^M r_0-\rho\left(\frac{r-r_0}{\sigma_s}\right)\right)\theta-\frac{1}{2}\left(\frac{r-r_0}{\sigma_s}\right)^2. \qquad (3.29)$$

Proof

For our problem of maximizing utility from terminal wealth at a fixed terminal time T. Then the HJB equations for $t<T$ can be obtained as follows

$$\begin{cases} \sup_{\pi_t}\{\mathscr{L}^{\pi_t}V(t,z)\}=0, \\ V(T,z)=u(z). \end{cases} \qquad (3.30)$$

where in here $V(t,z)=\sup_{\pi_t}\mathbb{E}^{t,z}[u(Z_T^{\pi_t})]$ this is the same as saying for each (t,z) we need to solve the non linear PDE of 3.30 and there after find a value of π_t that can maximise the function 3.31 ;-

$$V_t+\left[\pi_t r-\pi_t r_0+X_t^M r_0\right]V_z+\frac{1}{2}\left[\sigma_s^2\pi_t^2+2\rho\sigma_s\pi_t+1\right]V_{zz}. \qquad (3.31)$$

Suppose we assume that the HJB equation 3.30 consists of a classical solution V that satisfies $V_z>0$, $V_{zz}<0$ now differentiating with respect to π_t and equating to zero in 3.31 the following optimizer is obtained

$$\pi_t=-\frac{\rho}{\sigma_s}-\left(\frac{r-r_0}{\sigma_s^2}\right)\left(\frac{V_z}{V_{zz}}\right). \qquad (3.32)$$

Substituting equation 3.32 back into equation 3.31 then after some simplifications equation 3.30 become

$$\begin{cases} V_t+\left[X_t^M r_0-\rho\left(\frac{r-r_0}{\sigma_s}\right)\right]V_z-\frac{1}{2}\left(\frac{r-r_0}{\sigma_s}\right)^2\frac{V_z^2}{V_{zz}}+\frac{1}{2}\left(1-\rho^2\right)V_{zz}=0 \quad \text{for} \quad t<T, \\ V(T,z)=u(z). \end{cases} \qquad (3.33)$$

The PDE obtained in equation 3.33 are quit different from those obtained in other studies of utility maximization such as those in Browne (1995) and that of (Zou & Cadenillas, 2014). Since we want to solve the PDE under a given case when $u(z)=\lambda-\frac{\gamma}{\theta}e^{-\theta z}$. To solve the PDE in equation 3.33 under this case lets assume that it has the solution of the following form

$$V(t,z)=\lambda-\frac{\gamma}{\theta}e^{-\theta z+g(T-t)}, \qquad (3.34)$$

where by $g(.)$ is a given suitable function, with this assumption, then

$$\begin{cases} V_t(t,z)=[V(t,z)-\lambda][-g'(T-t)], \\ V_z(t,z)=[V(t,z)-\lambda][-\theta], \\ V_{zz}(t,z)=[V(t,z)-\lambda]\left[\theta^2\right]. \end{cases} \qquad (3.35)$$

Since the boundary condition is $V(T,z)=\lambda-\frac{\gamma}{\theta}e^{-\theta z}$ this mean that $g(0)=0$ now let us insert

3.35 into 3.33 and simplify to get

$$-g'(T-t)+\frac{1}{2}(1-\rho^2)\theta^2-\left(X_t^M r_0-\rho\left(\frac{r-r_0}{\sigma_s}\right)\right)\theta-\frac{1}{2}\left(\frac{r-r_0}{\sigma_s}\right)^2=0. \tag{3.36}$$

Now letting $Q(\theta)=\frac{1}{2}(1-\rho^2)\theta^2-\left(X_t^M r_0-\rho\left(\frac{r-r_0}{\sigma_s}\right)\right)\theta-\frac{1}{2}\left(\frac{r-r_0}{\sigma_s}\right)^2$ gives

$$g'(T-t)=Q(\theta). \tag{3.37}$$

Integrating equation 3.37 and using $g(0)=0$ gives the value function 3.28.

Since the value function is known, we can now obtain the control 3.27 by substituting the values of V_z and V_{zz} from equation 3.35 into equation 3.32.

Finally we need to show that the value function and the control obtained above are optimal. This is revealed upon checking the value function 3.28 since it is twice continuously differentiable thus it is clear that it satisfies the conditions of the classical verification theorems as stated by Ankirchner *et al.* (2019), therefore these are the optimal value function and controls. This brings us to the end of the proof.

3.7 Minimizing Ruin or Maximizing Survival Function

Lets consider equation 3.9 for the purpose of minimizing ruin or maximizing survival function for the insurance company. Since both X and R have stationary independent increments, then P is a homogeneous strong Markov process. By using Itô's formula the infinitesimal generator for P can be given by

$$\mathscr{L}g(p)=\frac{1}{2}\left(\sigma_R^2p^2+\sigma_X^2\right)g''(p)+(rp+c_M)g'(p)+\lambda_X\int_0^\infty\left(g(p-(y\wedge M))-g(p)\right)dF_X(y). \tag{3.38}$$

The integro-differential operator presented in equation 3.38 is quite complicated and explicit analytical computations involving it are hard to perform. However Paulsen and Gjessing (1997) have presented and proved the following very useful results.

Theorem 3.4

Let $\tau_p=\inf\{t:P_t<0\}$ be the time of ruin where $\tau_p=\infty$ means ruin never occurs and then let $\psi(p)=P(\tau_p<\infty)$ be the probability of eventual ruin to occur. For all t then using the above notations we have the following:

(i) Assume that $\Psi(p)$ is twice continuous differentiable and bounded on $p\geq 0$ with a bounded first derivative there, where at $p=0$ is meant the right hand derivative .

If Ψ solves $\mathscr{L}\Psi(p)=0$ on $p>0$ together with the boundary conditions

$$\begin{aligned} \Psi(p) &= 1 \quad \text{on} \quad p < 0, \\ \Psi(0) &= 1 \quad \text{if} \quad \sigma_X^2 > 0, \\ \lim_{p\to\infty} \Psi(p) &= 0, \end{aligned} \tag{3.39}$$

then

$$\Psi(p) = P(\tau_p < \infty).$$

(ii) Assume that $q_\alpha(p)$ is twice continuous differentiable and bounded on $p \geq 0$ with a bounded first derivative there, where at $p = 0$ is meant the right hand derivative .

If q_α solves $\mathscr{L} q_\alpha(p) = 0$ on $p > 0$ together with the boundary conditions

$$\begin{aligned} q_\alpha(p) &= 1 \quad \text{on} \quad p < 0, \\ q_\alpha(0) &= 1 \quad \text{if} \quad \sigma_X^2 > 0, \\ \lim_{p\to\infty} q_\alpha(p) &= 0, \end{aligned} \tag{3.40}$$

then

$$q_\alpha(p) = \mathbb{E}\left[e^{\alpha \tau_p}\right].$$

Proof

The proof is available in (Paulsen & Gjessing, 1997).

Now, according to Paulsen *et al.* (2005) replace the first part of the theorem with the survival $\phi(p) = 1 - \Psi(p)$ with boundary conditions given by equation 3.41

$$\begin{aligned} \phi(p) &= 0 \quad \text{on} \quad p < 0, \\ \phi(0) &= 0 \quad \text{if} \quad \sigma_X^2 > 0, \\ \lim_{p\to\infty} \phi(p) &= 1. \end{aligned} \tag{3.41}$$

Because minimizing ruin is the same as maximizing survival for the insurance company the goal is now to maximise survival $\phi(p)$. Therefore the value function for this intention is defined as

$$V(p) = \sup_{M \geq 0} \phi^M(p), \tag{3.42}$$

and if it exists, we determine the corresponding refinancing strategy $M_t \in [0, \infty)$ that will satisfy the objective function. Therefore in this section we are interested to find the optimal refinancing strategy in presence of investments in the risk process that is risky and risk free assets. We refer to this strategy as optimal in a sense that it maximises ultimate survival function which is the same as minimizing the probability of ultimate ruin. In other words the survival function is the objective function and the refinancing strategy M_t is the control variable to be adjusted such that the objective function is maximised.

3.7.1 Hamilton-Jacobi-Bellman Equation and Integro-Differential Equation

Under this subsection the HJB equation for the value function given by 3.42 is derived and solved then later on the integro-differential equations for the survival function is formulated and solved too. The solution of the HJB equation is the value function which gives the optimal cost to go for a given dynamical system. In literature several HJB equations of similar kind have been used for example the reader may refer Schmidli (2002), Paulsen *et al.* (2005), Kasozi *et al.* (2013) and Kasumo *et al.* (2018) for more details.

(i) Hamilton-Jacobi-Bellman Equation

To derive the HJB equation for the value function given by 3.42, let $(0,h]$ be a small interval and suppose that for each surplus $p(h) > 0$ at time h we have refinancing strategy M^{ε} such that $\delta M^{\varepsilon}(p(h)) > \delta(p(h)) - \varepsilon$. Let also that $M_t = M \in [0,\infty)$ for $t \leq h$. Then by Markov property one has the following

$$\begin{aligned}\phi(p) \geq \phi^M(p) &= \mathbb{E}\left[(\phi^{M^{\varepsilon}}(P^M(h)); \tau_p > h\right],\\ &= \mathbb{E}\left[(\phi^{M^{\varepsilon}}(P^M(\tau_p \wedge h))\right],\\ &\geq \mathbb{E}\left[(\phi^{M^{\varepsilon}}(P^M(\tau_p \wedge h))\right] - \varepsilon.\end{aligned}$$

But ε is arbitrary one can choose $\varepsilon = 0$ to get

$$\phi(p) \geq \mathbb{E}\left[(\phi^{M^{\varepsilon}}(P^M(\tau_p \wedge h))\right]. \tag{3.43}$$

Let us assume that $\phi(p)$ is twice continuously differentiable, by using Itô's formula we obtain

$$\begin{aligned}\phi(P^M(\tau_p \wedge h)) = \phi(p) + \int_0^{\tau_p \wedge h} &\Big\{(rp + c_M)\phi'(P^M(s)) + \frac{1}{2}(\sigma_R^2 p^2 + \sigma_X^2)\phi''(P^M(s))\\ &+ \lambda_X\left[\int_0^p \phi(P^M(s) - (y \wedge M))dF_X(y) - \phi(P^M(s))\right]\Big\}ds,\end{aligned} \tag{3.44}$$

where $y \wedge M = \max(M, Y_i)$ denote the retained amount to the insurance company.

Now, put 3.44 into 3.43 to get

$$\begin{aligned}\mathbb{E}\Big[\int_0^{\tau_p \wedge h} &\Big\{(rp + c_M)\phi'(P^M(s)) + \frac{1}{2}(\sigma_R^2 p^2 + \sigma_X^2)\phi''(P^M(s))\\ &+ \lambda_X\left[\int_0^p \phi(P^M(s) - \max(M, Y_i))dF_X(y) - \phi(P^M(s))\right]\Big\}ds\Big] \leq 0.\end{aligned} \tag{3.45}$$

Provided the limit and expectation can be interchanged then dividing the later equation by h

and letting $h \to 0$ gives the following

$$(rp+c_M)\phi'(p)+\frac{1}{2}(\sigma_R^2p^2+\sigma_X^2)\phi''(p)+\lambda_X\left[\int_0^p\phi(p-\max(M,Y_i))dF_X(y)-\phi(p)\right]\leq 0. \tag{3.46}$$

This equation 3.46 must hold for all $M>0$, that is to write

$$\sup_{M>0}\left[(rp+c_M)\phi'(p)+\frac{1}{2}(\sigma_R^2p^2+\sigma_X^2)\phi''(p)+\lambda_X\left[\int_0^p\phi(p-\max(M,Y_i))dF_X(y)-\phi(p)\right]\right]\leq 0. \tag{3.47}$$

Suppose that there is an optimal strategy $M\in[0,\infty)$ such that $\lim_{t\downarrow 0}M(t)=M(0)$. Then using similar approach we have

$$(rp+c_M)\phi'(p)+\frac{1}{2}(\sigma_R^2p^2+\sigma_X^2)\phi''(p)+\lambda_X\left[\int_0^p\phi(p-\max(M,Y_i))dF_X(y)-\phi(p)\right]=0. \tag{3.48}$$

Finally this gives us the HJB equation

$$\sup_{M>0}\left[(rp+c_M)\phi'(p)+\frac{1}{2}(\sigma_R^2p^2+\sigma_X^2)\phi''(p)+\lambda_X\left[\int_0^p\phi(p-\max(M,Y_i))dF_X(y)-\phi(p)\right]\right]=0, \tag{3.49}$$

whose boundary conditions are $\phi(p)=0$ on $p<0$ and $\lim_{p\to\infty}\phi(p)=1$.

An optimal strategy is obtained from the solution set $(\phi(p),M^*(p))$ of the equation 3.49 in which $M^*(p)$ is a point at which the supremum in 3.49 is obtained. The insurance company has a non negative net premium income if $c>(1+\eta)\lambda_X\mathbb{E}[(M-Y_i)^+]$.

Let $\underline{M}$ be the value where the equality holds that is $c=(1+\eta)\lambda_X\mathbb{E}[(M-Y_i)^+]$ but the aim is to find a non decreasing solution of 3.49, thus lets write it as follows;

$$\sup_{M>\underline{M}}\left[(rp+c_M)\phi'(p)+\frac{1}{2}(\sigma_R^2p^2+\sigma_X^2)\phi''(p)+\lambda_X\left[\int_0^p\phi(p-\max(M,Y_i))dF_X(y)-\phi(p)\right]\right]=0, \tag{3.50}$$

whose boundary conditions are $\phi(p)=0$ on $p<0$ and $\lim_{p\to\infty}\phi(p)=1$ as explained in Theorem 3.4 .

According to Hipp and Plum (2000) the function $\phi(p)$ will satisfy equation 3.50, only if $\phi(p)$ is strictly concave, strictly increasing, twice continuously differentiable and it satisfies the second condition, that is $\lim_{p\to\infty}\phi(p)=1$.

Now, let us assume that $\phi(p)$ is concave so that to insure smoothness and concavity, the claim density function must be locally bounded. The following results provide verification of the existence and property of the solution of the HJB equation 3.49.

Proposition 3.2 ***(Existence of Solution)***

Let the claim size distribution have a locally bounded density. Then the HJB equation has a bounded twice continuously differentiable solution $\phi \in C^2(0,\infty) \cap C^1[0,\infty)$.

Proof

A proof is similar to that of (Hipp & Plum, 2003).

Proposition 3.3 ***(Property of the Solution)***

If $\phi(p)$ is twice continuously differentiable and solves the HJB equation 3.49, then it is strictly concave.

Proof

See (Schmidli, 2002).

Remark 3.2

According to Hipp and Vogt (2003) if $\phi(p)$ is a smooth solution of the HJB equation 3.49 with properties of Theorem 3.4, then the supremum over $M > \underline{M}$ is either attained at $M = 0$ when there is no refinancing for small claims or at $M = p$ or $\underline{M} < M < p$.

(ii) Integro-Differential Equation

From the HJB equation 3.49 then the integro-differential equation for the survival function $\phi(p)$ takes the following form

$$\mathscr{L}\phi(p) = 0, \quad p \geq 0, \tag{3.51}$$

where $\mathscr{L}$ is the infinitesimal generator defined by the equation 3.38 for the underlying risk process with refinancing and investment given by equation 3.9. Thus from the HJB equation 3.49, the integro-differential equation for the survival function is given by

$$(rp + c_M)\phi'(p) + \frac{1}{2}(\sigma_R^2 p^2 + \sigma_X^2)\phi''(p) + \lambda_X \int_0^p \phi(p - \max(M, Y_i))dF_X(y) - \lambda_X \phi(p) = 0, \tag{3.52}$$

for $0 < p \leq \infty$.

Equation 3.52 is a second order integro-differential equation of Volterra type (VIDE). In this study VIDE equation 3.52 is converted into an ordinary differential equation (ODE) which can be solved numerically in order to determine the optimal strategies.

(iii) Converting VIDE into an ODE

In this section, we begin the process of solving VIDE given by equation 3.52. The equation is first converted into an ODE and later it will be solved numerically. The equation will be solved on the assumption that the claims are exponentially distributed. If $\sigma_R = 0$ and $r = 0$ then there is no investment, for this case the analytical solution to the similar problem is given by Belhaj (2010) and if $\lambda_X = 0$ similar case was solved analytically in Paulsen and Gjessing (1997) however when $\lambda_X \neq 0$, $\sigma_R \neq 0$ and $r \neq 0$ equation 3.52 has no analytical solution.

Consider exponential distribution given by

$$\begin{aligned} f_X(y) &= \beta e^{-\beta y}, \\ F_X(y) &= 1 - e^{-\beta y}, \\ dF_X(y) &= \beta e^{-\beta y}. \end{aligned} \tag{3.53}$$

Then equation 3.52 become

$$(rp + c_M)\phi'(p) + \frac{1}{2}(\sigma_R^2 p^2 + \sigma_X^2)\phi''(p) + \lambda_X \int_0^p \phi(p - \max(M, Y_i))\beta e^{-\beta y} dy - \lambda_X \phi(p) = 0, \tag{3.54}$$

for $0 < p \leq \infty$.

Differentiating with respect to p and simplifications gives

$$\frac{1}{2}(\sigma_R^2 p^2 + \sigma_X^2)\phi'''(p) + (rp + c_M + \sigma_R^2 p)\phi''(p) + (r - \lambda_X)\phi'(p) - \lambda_X \beta e^{-\beta p}\phi(p - \max(M, Y_i)) = 0 \tag{3.55}$$

for $0 < p \leq \infty$.

Equation 3.55 is an ODE that can later be solved numerically.

3.7.2 Treating Possibility of Recovery After Ruin for Insurance Companies

In this section an approach on how to handle the possibility of recovery after ruin for insurance companies is suggested and developed for the first time in insurance mathematics.

Assume that an insurance company had a wealth $X_{t_{\tau^-}}$ before the time of ruin suppose an insurance company has a possibility $\gamma \in [0, 1]$ of recovery after ruin, where $\gamma = 0$ means the company has no possibility of recovery at all and $\gamma = 1$ means the company has a possibility of recovering a full wealth after ruin.

Now, this study suggest that the new wealth or capital of the company for running the insurance business be given by the following formula

$$X_t^N = \gamma X_{t_{\tau^-}}. \tag{3.56}$$

Therefore, this approach will be used in the next chapter to perform simulation on various cases of possibilities γ under various situations of wealth or capital X_t when compounded with investment and/or refinancing.

3.8 Summary of Materials and Methods

In previous sections, the perturbed model was formulated upon using the Cramér-Lundberg model. Later on the model was compounded with return on investment and refinancing, with the help of stochastic control the optimisation problem was formed and later solved using utility function. Upon using a model that includes the investment and refinancing then the stochastic control theory was used again to derive the HJB equation from which the volterra integro-differential equation (VIDE) was derived and later transformed into an ODE ready for numerical analysis.

3.8.1 Numerical Method

In this section let us present the numerical method called Runge–Kutta to be applied in solving the formed ODE to obtain the survival $\phi(p)$ given by equation 3.52. The method will not be derived rather it will just be stated, for details of the derivation a reader can refer a book by (Richard & Burden, 2011).

Runge–Kutta methods are used in numerical analysis to solve initial value ODE and system of ODEs. These methods were developed by the German mathematicians Carl Runge and Wilhelm Kutta around 1900. There are six Runge–Kutta methods namely first order, second order, third order, fourth order, fifth order and sixth order Runge–Kutta methods, out of these six the fourth order is the most used. The main computational effort in using Runge-Kutta Methods is based on the evaluation of the function f. In second order method, the local truncation error is $O(h^2)$, and the cost is two function evaluations per each step while the fourth order Runge-Kutta method requires four functions evaluations per each step with local trunction error of $O(h^4)$. The advantage of the Runge–Kutta methods is that they eliminate the need to compute and evaluate the derivatives of $f(x_i, y_i)$ (Richard & Burden, 2011).

In this study a fourth order Runge–Kutta method will be used, the method is presented below;

$$\begin{cases} y_0 = \alpha, \\ k_1 = hf(x_i, y_i), \\ k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right), \\ k_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right), \\ k_4 = hf(x_{i+1}, y_i + k_3), \\ y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4). \end{cases} \tag{3.57}$$

Where α and h are the initial condition and step size respectively.

Next we transform an ODE given by equation 3.55 into a system of ODEs that will be solved numerically by using a fourth order Runge–Kutta method given by equation 3.57.

Letting $Z_1(x) = \phi(p)$, $Z_2(x) = \phi'(p) = Z_1'(x)$ and $Z_3(x) = \phi''(p) = Z_2'(x)$ then the following system of first order ODEs is obtained

$$\begin{cases} Z_1' = Z_2, \\ Z_2' = Z_3, \\ Z_3' = \frac{2}{(\sigma_R^2 p^2 + \sigma_X^2)} \left[\lambda_X \beta e^{-\beta p} Z_1(p - \max(M, Y_i)) - (r - \lambda_X) Z_2 - (rp + c_M + \sigma_R^2 p) Z_3 \right]. \end{cases} \tag{3.58}$$

In the next chapter the numerical solution using a fourth order Runge–Kutta method for this system 3.58 will be discussed.

3.8.2 Materials

(i) All the data simulations in this dissertation will be performed using a HP ENVY 17 with an Intel(R) Core(TM) i7-8550U CPU processor at 1.80GHz to 1.99GHz and 16.0GB of RAM.

(ii) The numerical method described in subsection 3.8.1 will be implemented in the next chapter by using MATLAB R2020a.

(iii) All the figures in the next chapter will be constructed using MATLAB R2020a.

3.9 Conclusion

In this chapter the perturbed model incorporating refinancing and investment was formulated. By using the Hamilton-Jacobi-Bellman (HJB) approach, the Volterra Integro-Differential equation (VIDE) corresponding to the optimisation problem was derived and solved using exponential utility function. The stochastic control theory was also used to derive the HJB equation from which the volterra integro-differential equation (VIDE) was derived and later transformed into an ODE ready for numerical analysis. The chapter also introduced an approach for handling the possibility of recovery after ruin for insurance companies and it finally ends with the subsection addressing numerical methods.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, we present this study's numerical results, simulations, and their discussions of the basic models such as the Cramér-Lundberg Model using the fourth order Runge-Kutta method. Various scenarios are presented and discussed thereafter. For the purpose of the simulations, some parameter values were obtained from the literature. Other values were not available in the literature thus there was a need for a trial, out of several trials the values that performed better were assumed for this study. The parameters values are presented in Table 3. Numerical results and their graphical representations were performed using MATLAB R2020a.

Table 3: Model parameters and their values.

Symbol	Definition	Value(s)	Source
λ	Intensity of the counting process	2, 1.75, 1, 5, 10, 20, 200	(Kasumo *et al.*, 2018).
r_0	Risk-free interest rate for the bond	0.02, 0.04	(Liu & Yang, 2004).
σ	The volatility or diffusion coefficient	0.25, 0.1, 0.2, 1, 1.3, 1.5, 2	(Mtunya *et al.*, 2017).
c	Insurer's premium income per unit time	2	(Kasumo, 2011).
δ	Preference rate to refinancing	0.04, 0.08, 0.12	(Eisenberg, 2010).
β	Mean of the exponential distribution	0.5	(Kasumo, 2011).
r	Instantaneous rate of stock return	0.05, 0.5	(Kasozi *et al.*, 2013).
ρ	Correlation coefficient	0.03	(Hu *et al.*, 2018).
r_i	The value of the investment rate	0.05, 0.1, 0.2	(Kasumo, 2011).
X_t^M	Refinanced surplus process	50, 70, 100, 1000	(Liu & Yang, 2004).
θ	Safety loading of the insurer	0.8, 2, 3, 5	(Kasumo, 2019).
γ	Possibility of recovery	2%, 25%, 80%	Assumed.
M	Capital refinanced	600	Assumed.
p	Initial capital	10000	Assumed.
Y	Capital risked	500	Assumed.

4.2 Survival in the Cramér-Lundberg Model

Consider a Cramér-Lundberg model given by equation 3.1, this model is the same as model 3.4 when there is no perturbation and refinancing.

By Itô's formula we can obtain the infinitesimal generator for X_t given by an equation 4.1;-

$$\mathscr{L}g(x) = cg'(x) + \lambda \int_0^x \Big(g(x-y) - g(x) \Big) dF(y). \tag{4.1}$$

This gives the VIDE given by equation 4.2 ;-

$$c\phi'(x) + \lambda \int_0^x \Big(\phi(x-y) - \phi(x) \Big) dF(y) = 0. \tag{4.2}$$

Integrating by parts on $[0,x]$ transforms 4.2 into a VIE of the second kind that can easily be converted into an ODE.

$$c\big(\phi(x) - \phi(0)\big) + \lambda \int_0^x F(x-y)\phi(y)dy - \lambda \int_0^x \phi(y)dy = 0. \tag{4.3}$$

$$\begin{aligned} c\phi(x) &= c\phi(0) + \lambda \int_0^x \phi(y)dy - \lambda \int_0^x F(x-y)\phi(y)dy, \\ &= c\phi(0) + \lambda \int_0^x \Big(1 - F(x-y) \Big) \phi(y)dy, \\ &= c\phi(0) + \lambda \int_0^x \overline{F}(x-y)\phi(y)dy. \end{aligned} \tag{4.4}$$

Therefore we get,

$$\phi(x) = \phi(0) + \frac{\lambda}{c} \int_0^x \overline{F}(x-y)\phi(y)dy. \tag{4.5}$$

This means that 4.5 can be expressed as a linear VIE of the second kind

$$\phi(x) = g(x) + \int_0^x K(x-y)\phi(y)dy, \tag{4.6}$$

whereby $g(x) = \phi(0)$ is the forcing function and $K(x-y) = \frac{\lambda \overline{F}(x-y)}{c}$ is the kernel with $\overline{F}(x-y) = 1 - F(x-y)$.

This study focuses on solving an equation of the form 4.5, by transforming it into an ODE, and later into a system of ODEs that can be solved numerically by using the fourth order Runge-Kutta Method given by equation 3.57.

Now, by using the exponential distribution given by equation 3.53 and with the help of Leibniz Integral Rule we transform this VIE into an ODE of third order.

Consider exponential distribution given by

$$\begin{aligned} f(y) &= \beta e^{-\beta y}, \\ F(y) &= 1 - e^{-\beta y}, \\ dF(y) &= \beta e^{-\beta y} dy. \end{aligned} \tag{4.7}$$

Then $\overline{F}(x-y) = 1 - F(x-y) = e^{-\beta(x-y)}$.

Equation 4.5 becomes

$$\phi(x) = \phi(0) + \frac{\lambda}{c}\int_0^x e^{-\beta(x-y)}\phi(y)dy. \tag{4.8}$$

Since $\phi(0) \in \mathbb{R}$ then differentiate throughout with respect to x gives

$$\phi'(x) = \frac{d}{dx}\left(\frac{\lambda}{c}\int_0^x e^{-\beta(x-y)}\phi(y)dy\right) = \frac{\lambda}{c}\frac{d}{dx}\left(\int_0^x e^{-\beta(x-y)}\phi(y)dy\right). \tag{4.9}$$

But we know from Leibniz Integral Rule given by equation 1.13 ;-

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} F(x,y)dy\right) = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}(F(x,y))dy + F(x,b(x))\frac{db(x)}{dx} - F(x,a(x))\frac{da(x)}{dx}. \tag{4.10}$$

Then in equation 4.8 letting $a(x) = 0, b(x) = x$ and $F(x,y) = e^{-\beta(x-y)}\phi(y)$ we have;

$\frac{\partial}{\partial x}F(x,y) = -\beta e^{-\beta(x-y)}\phi(y)$, $\frac{da(x)}{dx} = 0$ and $\frac{db(x)}{dx} = 1$.

Then upon using equation 1.13 we have

$$\frac{d}{dx}\left[\int_0^x e^{-\beta(x-y)}\phi(y)dy\right] = -\beta\int_0^x e^{-\beta(x-y)}\phi(y)dy + \phi(x). \tag{4.11}$$

Thus, equation 4.9 becomes

$$\phi'(x) = \frac{d}{dx}\left(\frac{\lambda}{c}\int_0^x e^{-\beta(x-y)}\phi(y)dy\right) = \frac{\lambda}{c}\left[-\beta\int_0^x e^{-\beta(x-y)}\phi(y)dy + \phi(x)\right]. \tag{4.12}$$

Therefore

$$\phi'(x) = \frac{-\lambda\beta}{c}\left[\int_0^x e^{-\beta(x-y)}\phi(y)dy\right] + \frac{\lambda}{c}\phi(x). \tag{4.13}$$

Differentiating once again throughout with respect to x we get

$$\phi''(x) = \frac{-\lambda\beta}{c}\frac{d}{dx}\left[\int_0^x e^{-\beta(x-y)}\phi(y)dy\right] + \frac{\lambda}{c}\phi'(x). \tag{4.14}$$

Then using the same procedure with the help of equation 4.9 and simplifying we have;

$$\phi''(x) = \frac{\lambda}{c}\phi'(x) - \frac{\lambda\beta}{c}\phi(x) + \frac{\lambda\beta^2}{c}\left[\int_0^x e^{-\beta(x-y)}\phi(y)dy\right]. \tag{4.15}$$

Now, we differentiate 4.15 for the last time to get a third order ODE

$$\phi'''(x) = \frac{\lambda}{c}\phi''(x) - \frac{\lambda\beta}{c}\phi'(x) + \frac{\lambda\beta^2}{c}\frac{d}{dx}\left[\int_0^x e^{-\beta(x-y)}\phi(y)dy\right]. \tag{4.16}$$

Then using the same procedure with the help of equations 3.41 and 4.8 with simplification we have;

$$\phi'''(x) = \frac{\lambda}{c}\phi''(x) - \frac{\lambda\beta}{c}\phi'(x) + \frac{\lambda\beta^2}{c}\phi(x) - \frac{\lambda\beta^3}{c}(\beta - \beta e^{-\beta x}). \tag{4.17}$$

Now, we transform a third order ODE given by equation 4.17 into a system of first order ODEs to be solved by the fourth order Runge-Kutta Method given by equation 3.57.

Letting $Z_1 = \phi(x)$, $Z_2 = \phi'(x) = Z_1'$ and $Z_3 = \phi''(x) = Z_2'$ then the following system of first order ODEs is obtained

$$\begin{cases} Z_1' = Z_2, \\ Z_2' = Z_3, \\ Z_3' = \frac{\lambda}{c}Z_3 - \frac{\lambda\beta}{c}Z_2 + \frac{\lambda\beta^2}{c}Z_1 - \frac{\lambda\beta^3}{c}(\beta - \beta e^{-\beta x}). \end{cases} \tag{4.18}$$

Now, this system 4.18 of first order ODEs is solved numerically using the fourth order Runge–Kutta method given by equation 3.57, implemented using MATLAB codes and results are discussed. Values of the parameters used for simulations are presented in Table 3.

We observe clearly in Fig. 2(a) that, as initial surplus increases the survival function also increases this in turn will increase the survival in the Cramér-Lundberg model. It is further observed in Fig. 2(b) that the increase in survival function tend to reduce with respect to intensity of the counting process this suggest that the counting process should be high for faster survival increase in the Cramér-Lundberg model.

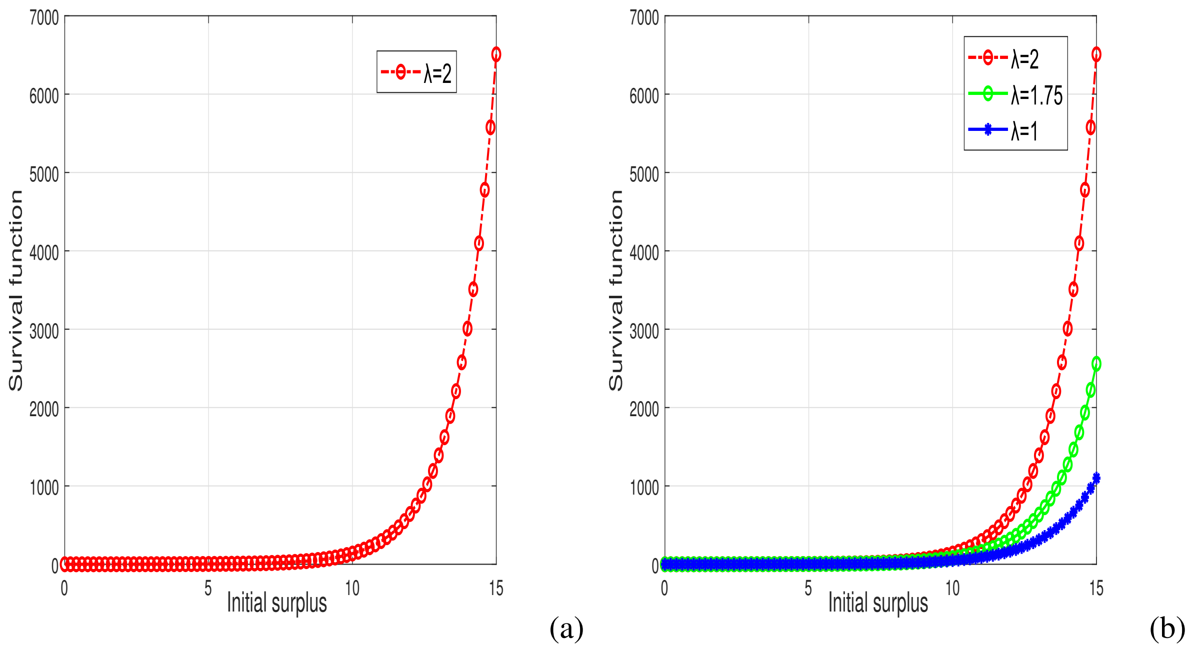


Figure 2: Behaviour of the survival function against the initial surplus

4.2.1 Survival in the Cramér-Lundberg Model Compounded by Investments

In this section we are interested to study the effect of investments on the increase of the survival in the Cramér-Lundberg model. We consider a Cramér-Lundberg model given by equation 3.1 then we shall compound this model by a constant force of interest so that we can study how the value of the investment rate affects the survival in the Cramér-Lundberg model.

Compounding this model 3.1 by a constant force of interest we have

$$X_t = p + ct - \sum_{i=1}^{N_t} Y_i + r_i \int_0^t X(y)dy \quad , \quad t \geq 0, \tag{4.19}$$

where r_i is the investment rate.

Then according to Kasozi and Paulsen (2005) the $\phi(x)$ for equation 4.19 satisfies the VIDE given by equation 4.20 ;-

$$(r_i x + c)\phi'(x) + \lambda \int_0^x \Big(\phi(x-y) - \phi(x)\Big) dF(y) = 0. \tag{4.20}$$

Now, Integrating by parts equation 4.20 on $[0;x]$ will transform it into a linear VIE of the second kind that can easily be converted into an ODE. That is,

$$\big[(r_i z + c)\phi(z)\big]_0^x - r_i \int_0^x \phi(z)dz + \lambda \int_0^x \Big(F(x-y)\phi(y) - \phi(y)\Big) dy = 0. \tag{4.21}$$

Then

$$(r_i x + c)\phi(x) - c\phi(0) = r_i \int_0^x \phi(y)dy - \lambda \int_0^x \Big([-1 + F(x-y)]\phi(y)\Big) dy. \tag{4.22}$$

Then we have

$$\begin{aligned} \phi(x) &= \frac{c}{r_i x + c}\phi(0) + \frac{r_i}{r_i x + c}\int_0^x \phi(y)dy + \frac{\lambda}{r_i x + c}\int_0^x \Big([1 - F(x-y)]\phi(y)\Big) dy, \\ &= \frac{c}{r_i x + c}\phi(0) + \frac{r_i}{r_i x + c}\int_0^x \phi(y)dy + \frac{\lambda}{r_i x + c}\int_0^x \overline{F}(x-y)\phi(y)dy, \\ &= \frac{c}{r_i x + c}\phi(0) + \int_0^x \frac{r_i + \lambda \overline{F}(x-y)}{r_i x + c}\phi(y)dy. \end{aligned} \tag{4.23}$$

This means that equation 4.23 can be expressed as a linear VIE of the second kind

$$\phi(x) = g(x) + \int_0^x K(x-y)\phi(y)dy, \tag{4.24}$$

whereby $g(x) = \frac{c}{r_i x + c}\phi(0)$ is the forcing function and $K(x-y) = \frac{r_i + \lambda \overline{F}(x-y)}{r_i x + c}$ is the kernel with

$\overline{F}(x-y) = 1 - F(x-y)$. Now, using the same procedures this VIE is transformed into an ODE of third order.

By using equation 4.5 then 4.23 become

$$\phi(x) = \frac{c}{r_i x + c}\phi(0) + \int_0^x \frac{r_i + \lambda e^{-\beta(x-y)}}{r_i x + c}\phi(y)dy. \tag{4.25}$$

Differentiate throughout with respect to x we get

$$\begin{aligned} \phi'(x) &= \frac{d}{dx}\left(\frac{c}{r_i x + c}\phi(0) + \int_0^x \frac{r_i + \lambda e^{-\beta(x-y)}}{r_i x + c}\phi(y)dy\right), \\ &= \frac{d}{dx}\left(\frac{c}{r_i x + c}\phi(0)\right) + \frac{d}{dx}\left(\int_0^x \frac{r_i + \lambda e^{-\beta(x-y)}}{r_i x + c}\phi(y)dy\right). \end{aligned} \tag{4.26}$$

Since $\phi(0) \in \mathbb{R}$ then

$$\frac{d}{dx}\left(\frac{c}{r_i x + c}\phi(0)\right) = \frac{-r_i c}{(r_i x + c)^2}\phi(0) + \frac{c}{r_i x + c}\phi'(0) = \frac{-r_i c}{(r_i x + c)^2}\phi(0). \tag{4.27}$$

Also by using Leibniz integral rule we get

$$\begin{aligned} \frac{d}{dx}\left(\int_0^x \frac{r_i + \lambda e^{-\beta(x-y)}}{r_i x + c}\phi(y)dy\right) &= \int_0^x \frac{(r_i x + c)(-\lambda\beta e^{-\beta(x-y)}) + ((r_i)^2 + r_i\lambda e^{-\beta(x-y)})}{(r_i x + c)^2}\phi(y)dy \\ &\quad + \frac{r_i + \lambda}{r_i x + c}\phi(x). \end{aligned} \tag{4.28}$$

Then equation 4.26 can be written as

$$\phi'(x) = \frac{r_i + \lambda}{r_i x + c}\phi(x) - \frac{r_i c}{(r_i x + c)^2}\phi(0) + \int_0^x \frac{(r_i x + c)(-\lambda\beta e^{-\beta(x-y)}) + ((r_i)^2 + r_i\lambda e^{-\beta(x-y)})}{(r_i x + c)^2}\phi(y)dy. \tag{4.29}$$

Differentiate once again throughout with respect to x we get

$$\phi''(x) = \frac{d}{dx}\left[\frac{r_i + \lambda}{r_i x + c}\phi(x) - \frac{r_i c}{(r_i x + c)^2}\phi(0) + \int_0^x \frac{(r_i x + c)(-\lambda\beta e^{-\beta(x-y)}) + ((r_i)^2 + r_i\lambda e^{-\beta(x-y)})}{(r_i x + c)^2}\phi(y)dy\right]. \tag{4.30}$$

Then

$$\begin{aligned} \phi''(x) &= \frac{d}{dx}\left[\frac{r_i + \lambda}{r_i x + c}\phi(x)\right] - \frac{d}{dx}\left[\frac{r_i c}{(r_i x + c)^2}\phi(0)\right] \\ &\quad + \frac{d}{dx}\left[\int_0^x \frac{(r_i x + c)(-\lambda\beta e^{-\beta(x-y)}) + ((r_i)^2 + r_i\lambda e^{-\beta(x-y)})}{(r_i x + c)^2}\phi(y)dy\right]. \end{aligned} \tag{4.31}$$

Differentiating the first two terms we have

$$\begin{aligned}\phi''(x) &= \frac{r_i+c}{r_ix+c}\phi'(x) - \frac{(r_i)^2+\lambda r_i}{(r_ix+c)^2}\phi(x) - \frac{2(r_i)^3cx+2(r_i)^2c}{(r_ix+c)^4}\phi(0)\\ &+ \frac{d}{dx}\left[\int_0^x \frac{(r_ix+c)(-\lambda\beta e^{-\beta(x-y)})+((r_i)^2+r_i\lambda e^{-\beta(x-y)})}{(r_ix+c)^2}\phi(y)dy\right].\end{aligned} \tag{4.32}$$

Then with the help of Leibniz integral rule we get

$$\begin{aligned}\phi''(x) &= \frac{r_i+c}{r_ix+c}\phi'(x) - \frac{(r_i)^2+\lambda r_i}{(r_ix+c)^2}\phi(x) - \frac{2(r_i)^3cx+2(r_i)^2c}{(r_ix+c)^4}\phi(0)\\ &+ \int_0^x \frac{\beta\lambda(r_i+\beta)(r_ix+c)-r_i\lambda(\beta+2r_i)-2(r_i)^3}{r_ix+c}e^{-\beta(x-y)}\phi(y)dy\\ &+ \frac{(r_i)^2+\lambda r_i-\lambda\beta(r_ix+c)}{(r_ix+c)^2}\phi(x).\end{aligned} \tag{4.33}$$

Simplifications gives

$$\begin{aligned}\phi''(x) &= \frac{r_i+c}{r_ix+c}\phi'(x) - \frac{\lambda\beta}{r_ix+c}\phi(x) - \frac{2(r_i)^3cx+2(r_i)^2c}{(r_ix+c)^4}\phi(0)\\ &+ \int_0^x \frac{\beta\lambda(r_i+\beta)(r_ix+c)-r_i\lambda(\beta+2r_i)-2(r_i)^3}{r_ix+c}e^{-\beta(x-y)}\phi(y)dy.\end{aligned} \tag{4.34}$$

Now, we differentiate equation 4.34 for the last time to get a third order ODE that can be converted into a system of ODEs that can be solved numerically by using the fourth order Runge–Kutta method.

$$\begin{aligned}\phi'''(x) &= \frac{d}{dx}\left(\frac{r_i+c}{r_ix+c}\phi'(x)\right) - \frac{d}{dx}\left(\frac{\lambda\beta}{r_ix+c}\phi(x)\right) - \frac{d}{dx}\left(\frac{2(r_i)^3cx+2(r_i)^2c}{(r_ix+c)^4}\phi(0)\right)\\ &+ \frac{d}{dx}\left(\int_0^x \frac{\beta\lambda(r_i+\beta)(r_ix+c)-r_i\lambda(\beta+2r_i)-2(r_i)^3}{r_ix+c}e^{-\beta(x-y)}\phi(y)dy\right).\end{aligned} \tag{4.35}$$

Differentiation of the first three terms gives

$$\begin{aligned}\phi'''(x) &= \frac{r_i+c}{r_ix+c}\phi''(x) - \frac{r_i(r_i+c)}{(r_ix+c)^2}\phi'(x) - \frac{\lambda\beta}{r_ix+c}\phi'(x)\\ &+ \frac{\lambda\beta r_i}{(r_ix+c)^2}\phi(x) + \frac{(r_i)^2c((r_i)^2x+1)-2(r_i)^3c(r_ix+c)}{(r_ix+c)^5}\phi(0)\\ &+ \frac{d}{dx}\left(\int_0^x \frac{\beta\lambda(r_i+\beta)(r_ix+c)-r_i\lambda(\beta+2r_i)-2(r_i)^3}{r_ix+c}e^{-\beta(x-y)}\phi(y)dy\right).\end{aligned} \tag{4.36}$$

Then using the same procedure with the help of Leibniz integral rule and equation 3.41 we have;

$$\begin{aligned}\phi'''(x) &= \left(\frac{r_i+c}{r_ix+c}\right)\phi''(x) - \left(\frac{r_i(r_i+c)}{(r_ix+c)^2} + \frac{\lambda\beta}{r_ix+c}\right)\phi'(x) \\ &\quad + \left(\frac{\beta\lambda(r_i+\beta)(r_ix+c) - r_i\lambda(\beta+2r_i) - 2(r_i)^3}{r_ix+c} + \frac{\lambda\beta r_i}{(r_ix+c)^2}\right)\phi(x).\end{aligned} \tag{4.37}$$

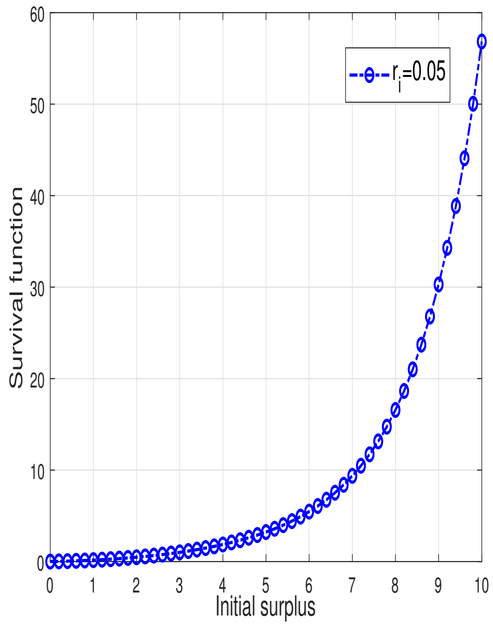
Now, we transform a third order ODE given by equation 4.37 into a system of first order ODEs to be solved by the fourth order Runge-Kutta Method given by equation 3.57.

Letting $Z_1 = \phi(x)$, $Z_2 = \phi'(x) = Z_1'$ and $Z_3 = \phi''(x) = Z_2'$ then the following system of first order ODEs is obtained

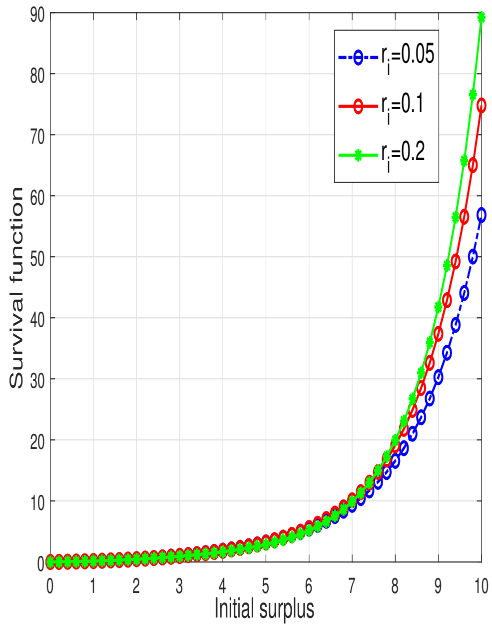
$$\begin{cases} Z_1' = Z_2, \\ Z_2' = Z_3, \\ Z_3' = \left(\frac{r_i+c}{r_ix+c}\right)Z_3 - \left(\frac{r_i(r_i+c)}{(r_ix+c)^2} + \frac{\lambda\beta}{r_ix+c}\right)Z_2 + \left(\frac{\beta\lambda(r_i+\beta)(r_ix+c)-r_i\lambda(\beta+2r_i)-2(r_i)^3}{r_ix+c} + \frac{\lambda\beta r_i}{(r_ix+c)^2}\right)Z_1. \end{cases} \tag{4.38}$$

Now, using similar approach this system 4.38 of first order ODEs is solved numerically using the fourth order Runge–Kutta method given by equation 3.57, implemented using MATLAB codes and results are discussed. Values of the parameters used for simulations are presented in Table 3.

We observe in Fig. 3(a) that, as initial surplus increases the survival function also increases because the company become more liquid as surplus increases, as a result it is expected that survival in the Cramér-Lundberg model will be increased. In addition to that it is also observed in Fig. 3(b) that the rate of increase in survival function tend to increase with respect to the value of the investment rate. For example, we see that using $r_i = 0.05$ (that is surplus is invested at 5%) results in lower survival function than the one obtained when the surplus is invested at 10% (that is, when $r_i = 0.1$) and 20%(that is, when $r_i = 0.2$).



(a)



(b)

Figure 3: Behaviour of the survival function against the investment rate

4.2.2 Survival in the Cramér-Lundberg Model Compounded by Refinancing

Now we are interested to study the effect of refinancing on the survival in the Cramér-Lundberg model. We consider the Cramér-Lundberg model 3.1 and compound it by a refinancing process.

Compounding this model 3.1 by a refinancing process we have

$$X_t = p + ct - \sum_{i=1}^{N_t} Y_i + M_t \quad , \quad t \geq 0, \tag{4.39}$$

where M_t is the capital injected at time t. Let $\delta \geq 0$ be a preference rate to refinancing such that $\delta > 0$ means that the investor prefers refinancing tomorrow to refinancing today while $\delta = 0$ means investor prefers not to refinance. Then, according to Eisenberg (2010) the $\phi(x)$ for equation 4.39 satisfies the VIDE given by equation 4.40 ;-

$$c\phi'(x) + \lambda \int_0^x \Big(\phi(x-y) - \phi(x)\Big) dF(y) - (\delta + \lambda)\phi(x) = 0. \tag{4.40}$$

Now, Integrating equation 4.40 by part on $[0,x]$ will transform it into a linear VIE of the second kind that can easily be converted into an ODE. That is,

$$c\big(\phi(x) - \phi(0)\big) + \lambda \int_0^x F(x-y)\phi(y)dy - \lambda \int_0^x \phi(y)dy - (\delta + \lambda) \int_0^x \phi(y)dy = 0. \tag{4.41}$$

$$
\begin{aligned}
c\phi(x) &= c\phi(0) + \lambda \int_0^x \phi(y)dy - \lambda \int_0^x F(x-y)\phi(y)dy + (\delta+\lambda)\int_0^x \phi(y)dy, \\
&= c\phi(0) + \lambda \int_0^x \Big(1 - F(x-y)\Big)\phi(y)dy + (\delta+\lambda)\int_0^x \phi(y)dy, \qquad . \\
&= c\phi(0) + \lambda \int_0^x \overline{F}(x-y)\phi(y)dy + (\delta+\lambda)\int_0^x \phi(y)dy
\end{aligned} \tag{4.42}
$$

Therefore we get,

$$
\phi(x) = \phi(0) + \frac{\lambda}{c}\int_0^x \overline{F}(x-y)\phi(y)dy + \frac{\delta+\lambda}{c}\int_0^x \phi(y)dy. \tag{4.43}
$$

This can be written as

$$
\phi(x) = \phi(0) + \int_0^x \frac{\delta+\lambda+\lambda\overline{F}(x-y)}{c}\phi(y)dy. \tag{4.44}
$$

This means that 4.44 can be expressed as a linear VIE of the second kind

$$
\phi(x) = g(x) + \int_0^x K(x-y)\phi(y)dy. \tag{4.45}
$$

whereby $g(x) = \phi(0)$ is the forcing function and $K(x-y) = \frac{\delta+\lambda+\lambda\overline{F}(x-y)}{c}$ is the kernel with $\overline{F}(x-y) = 1 - F(x-y)$. Now, using the same procedure this VIE is transformed into an ODE of third order.

By substituting $\overline{F}(x-y)$ from equation 4.5 in 4.44 results in 4.46 as follows;-

$$
\phi(x) = \phi(0) + \int_0^x \frac{\delta+\lambda+\lambda e^{-\beta(x-y)}}{c}\phi(y)dy. \tag{4.46}
$$

Since $\phi(0)$ is a constant differentiating throughout with respect to x gives

$$
\begin{aligned}
\phi'(x) &= \frac{d}{dx}\left(\phi(0) + \int_0^x \frac{\delta+\lambda+\lambda e^{-\beta(x-y)}}{c}\phi(y)dy\right), \\
&= \frac{d}{dx}\Big(\phi(0)\Big) + \frac{d}{dx}\left(\int_0^x \frac{\delta+\lambda+\lambda e^{-\beta(x-y)}}{c}\phi(y)dy\right), \\
&= \frac{d}{dx}\left(\int_0^x \frac{\delta+\lambda+\lambda e^{-\beta(x-y)}}{c}\phi(y)dy\right).
\end{aligned} \tag{4.47}
$$

Now, Leibniz integral rule gives

$$
\phi'(x) = \frac{\delta+2\lambda}{c}\phi(x) + \int_0^x \frac{\delta+\lambda-\lambda\beta e^{-\beta(x-y)}}{c}\phi(y)dy. \tag{4.48}
$$

Differentiate once again throughout with respect to x we get

$$\phi''(x) = \frac{\delta + 2\lambda}{c}\phi'(x) + \frac{d}{dx}\left(\int_0^x \frac{\delta + \lambda - \lambda\beta e^{-\beta(x-y)}}{c}\phi(y)dy\right). \tag{4.49}$$

Then by Leibniz integral rule we get

$$\phi''(x) = \frac{\delta + 2\lambda}{c}\phi'(x) + \frac{\delta + \lambda - \lambda\beta}{c}\phi(x) + \int_0^x \frac{\delta + \lambda + \lambda\beta^2 e^{-\beta(x-y)}}{c}\phi(y)dy. \tag{4.50}$$

Now, we finally differentiate 4.50 to get a third order ODE that can be converted into a system of ODEs to be solved numerically by using the fourth order Runge–Kutta method.

$$\phi'''(x) = \frac{\delta + 2\lambda}{c}\phi''(x) + \frac{\delta + \lambda - \lambda\beta}{c}\phi'(x) + \frac{d}{dx}\left(\int_0^x \frac{\delta + \lambda + \lambda\beta^2 e^{-\beta(x-y)}}{c}\phi(y)dy\right). \tag{4.51}$$

Then using the same procedure with the help of Leibniz integral rule and equation 3.41 we have;

$$\phi'''(x) = \frac{\delta + 2\lambda}{c}\phi''(x) + \frac{\delta + \lambda - \lambda\beta}{c}\phi'(x) + \frac{\delta + \lambda + \lambda\beta^2}{c}\phi(x). \tag{4.52}$$

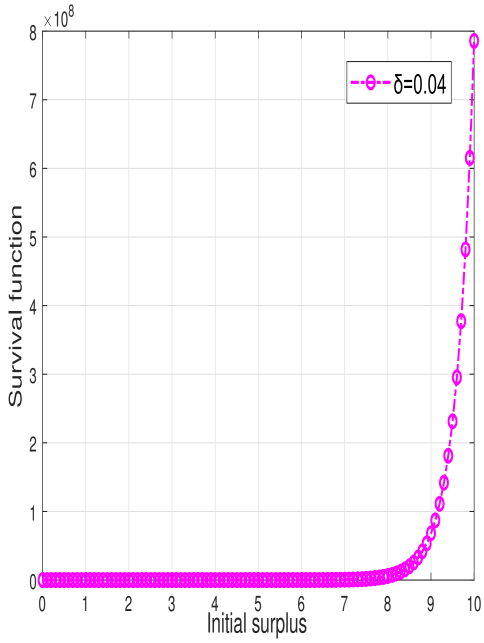
Now, we transform the third order ODE given by equation 4.52 into a system of first order ODEs to be solved by the fourth order Runge-Kutta Method given by equation 3.57.

Letting $Z_1 = \phi(x)$, $Z_2 = \phi'(x) = Z_1'$ and $Z_3 = \phi''(x) = Z_2'$ leads to the following system of first order ODEs

$$\begin{cases} Z_1' = Z_2, \\ Z_2' = Z_3, \\ Z_3' = \frac{\delta+2\lambda}{c}Z_3 + \frac{\delta+\lambda-\lambda\beta}{c}Z_2 + \frac{\delta+\lambda+\lambda\beta^2}{c}Z_1. \end{cases} \tag{4.53}$$

By using similar approach, this system of first order ODEs is solved numerically using the fourth order Runge–Kutta method given by equation 3.57, implemented using MATLAB and the results are discussed. Values of the parameters used for simulations are presented in Table 3.

We observe in Fig. 4(a) that, with the same reason of increase in liquidity, the survival function increases with an increase in initial surplus, it is further observed in Fig. 4(b) that as preference rate to refinancing tomorrow is increased the survival function also increases this indicate that when the insurance company is about to reduce its survival (get ruin) then the decision to refinance is much better to overcome the situation.



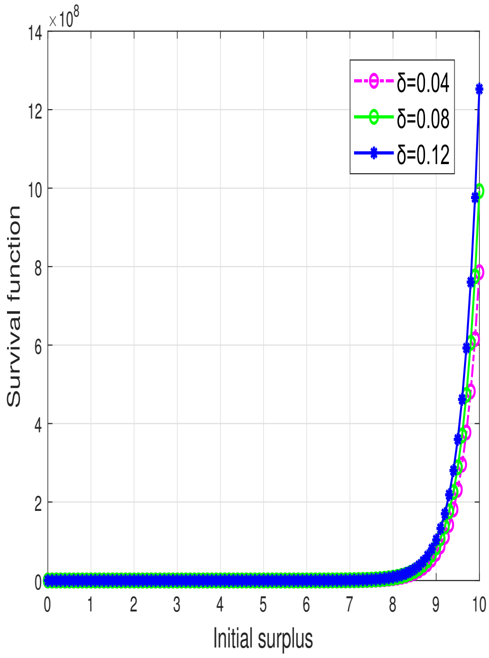


Figure 4: Behaviour of the survival function against the preference rate to refinancing

4.3 Survival in the Diffusion-Perturbed Model

We consider the survival in a Cramér-Lundberg model when the model is perturbed. Let us consider a perturbed Cramér-Lundberg model given by equation 3.3. This model is the same as model 3.4 when there is no refinancing.

By using Itô's formula we can obtain the infinitesimal generator for X_t in 3.3 given by an equation 4.54;-

$$\mathscr{L}g(x) = \frac{1}{2}\sigma^2 g''(x) + cg'(x) - \lambda g(x) + \lambda \int_0^x g(x-y)dF(y). \tag{4.54}$$

This gives the VIDE given by equation 4.55;-

$$\frac{1}{2}\sigma^2\phi''(x) + c\phi'(x) - \lambda\phi(x) + \lambda \int_0^x \phi(x-y)dF(y) = 0. \tag{4.55}$$

Using 4.6 we get

$$\frac{1}{2}\sigma^2\phi''(x) + c\phi'(x) - \lambda\phi(x) + \lambda \int_0^x \phi(x-y)\beta e^{-\beta y}dy = 0. \tag{4.56}$$

This can be written as

$$\phi''(x) = -\frac{2c}{\sigma^2}\phi'(x) + \frac{2\lambda}{\sigma^2}\phi(x) - \frac{2\lambda}{\sigma^2}\int_0^x \phi(x-y)\beta e^{-\beta y}dy. \tag{4.57}$$

Then differentiating throughout yields.

$$\phi'''(x) = -\frac{2c}{\sigma^2}\phi''(x) + \frac{2\lambda}{\sigma^2}\phi'(x) - \frac{2\lambda}{\sigma^2}\frac{d}{dx}\left(\int_0^x \phi(x-y)\beta e^{-\beta y}dy\right). \tag{4.58}$$

Using Leibniz integral rule and equation 3.41 we get;

$$\phi'''(x) = -\frac{2c}{\sigma^2}\phi''(x) + \frac{2\lambda}{\sigma^2}\phi'(x) - \frac{2\lambda}{\sigma^2}e^{-\beta x}\phi(0). \tag{4.59}$$

For $\sigma^2 > 0$ we have

$$\phi'''(x) = -\frac{2c}{\sigma^2}\phi''(x) + \frac{2\lambda}{\sigma^2}\phi'(x). \tag{4.60}$$

Now, we transform the third order ODE 4.60 into a system of first order ODEs to be solved by the fourth order Runge-Kutta Method given by equation 3.57.

Letting $Z_1 = \phi(x)$, $Z_2 = \phi'(x) = Z_1'$ and $Z_3 = \phi''(x) = Z_2'$ leads to the following system of first order ODEs

$$\begin{cases} Z_1' = Z_2, \\ Z_2' = Z_3, \\ Z_3' = -\frac{2c}{\sigma^2}Z_3 + \frac{2\lambda}{\sigma^2}Z_2. \end{cases} \tag{4.61}$$

By a similar approach, this system of first order ODEs is solved numerically using the fourth order Runge–Kutta method given by equation 3.57, implemented using MATLAB and the results are discussed. Values of the parameters used for simulations are presented in Table 3.

We observe in Fig. 5(a) that, with the same reason of increase in liquidity, the survival function increases with the increase in initial surplus, it is further seen that when volatility coefficient is increased the survival function tends to decrease indicating that it is much risk and very likely to have more survival when the perturbation is low.

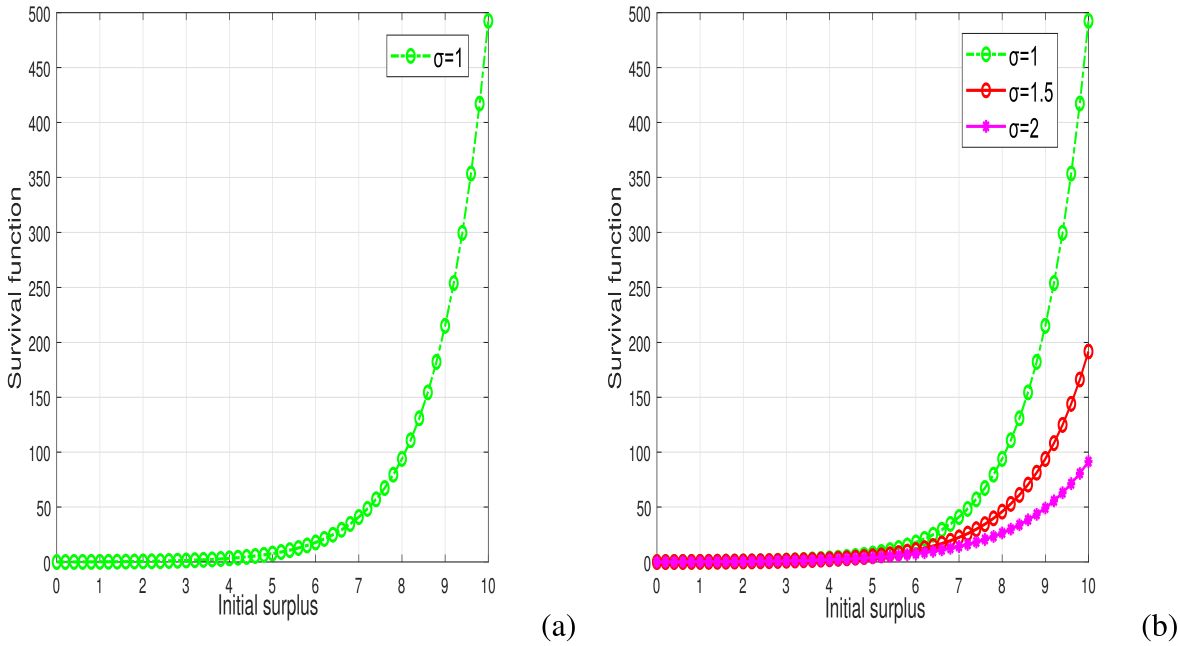


Figure 5: Behaviour of the survival function against diffusion coefficient in perturbed model

4.3.1 Survival in the Diffusion-Perturbed Model Compounded by Investment

We now consider the effect of investments on the increase of survival in the Cramér-Lundberg model when there is perturbation in the model. Let us consider a perturbed Cramér-Lundberg model given by equation 3.3 and then compound this model by a constant force of interest so that we can study how the value of the investment rate affects the survival in the perturbed Cramér-Lundberg model.

Compounding this model 3.3 by a constant force of interest we have

$$X_t = p + ct + \sigma W_t - \sum_{i=1}^{N_t} Y_i + r_i \int_0^t X(y)dy \quad , \quad t \geq 0, \tag{4.62}$$

where r_i is the investment rate.

By using Itô's formula we can obtain the infinitesimal generator for X_t given by an equation 4.63 ;-

$$\mathscr{L}g(x) = \frac{1}{2}\sigma^2 g''(x) + (r_i x + c)g'(x) - \lambda g(x) + \lambda \int_0^x g(x-y)dF(y). \tag{4.63}$$

This gives the VIDE given by equation 4.64

$$\frac{1}{2}\sigma^2 \phi''(x) + (r_i x + c)\phi'(x) - \lambda \phi(x) + \lambda \int_0^x \phi(x-y)dF(y) = 0. \tag{4.64}$$

Using 4.6 we get

$$\frac{1}{2}\sigma^2\phi''(x)+(r_ix+c)\phi'(x)-\lambda\phi(x)+\lambda\int_0^x\phi(x-y)\beta e^{-\beta y}dy=0. \tag{4.65}$$

This can be written as

$$\phi''(x)=-\frac{2(r_ix+c)}{\sigma^2}\phi'(x)+\frac{2\lambda}{\sigma^2}\phi(x)-\frac{2\lambda}{\sigma^2}\int_0^x\phi(x-y)\beta e^{-\beta y}dy. \tag{4.66}$$

Then differentiating throughout we have.

$$\phi'''(x)=-\frac{2(r_ix+c)}{\sigma^2}\phi''(x)+\frac{2(\lambda-r_i)}{\sigma^2}\phi'(x)-\frac{2\lambda}{\sigma^2}\frac{d}{dx}\left(\int_0^x\phi(x-y)\beta e^{-\beta y}dy\right). \tag{4.67}$$

Applying Leibniz integral rule and equation 3.41 we get;

$$\phi'''(x)=-\frac{2c}{\sigma^2}\phi''(x)+\frac{2(\lambda-r_i)}{\sigma^2}\phi'(x)-\frac{2\lambda}{\sigma^2}e^{-\beta x}\phi(0). \tag{4.68}$$

For $\sigma^2>0$ we have

$$\phi'''(x)=-\frac{2c}{\sigma^2}\phi''(x)+\frac{2(\lambda-r_i)}{\sigma^2}\phi'(x). \tag{4.69}$$

Now, we transform the third order ODE given by equation 4.60 into a system of first order ODEs to be solved by the fourth order Runge-Kutta Method given by equation 3.57.

Letting $Z_1=\phi(x)$, $Z_2=\phi'(x)=Z_1'$ and $Z_3=\phi''(x)=Z_2'$ leads to the following system of first order ODEs

$$\begin{cases} Z_1'=Z_2, \\ Z_2'=Z_3, \\ Z_3'=-\frac{2c}{\sigma^2}Z_3+\frac{2(\lambda-r_i)}{\sigma^2}Z_2. \end{cases} \tag{4.70}$$

By a similar approach, this system of first order ODEs is solved numerically using the fourth order Runge–Kutta method given by equation 3.57, implemented using MATLAB and the results are discussed. Values of the parameters used for simulations are presented in Table 3.

We observe in Fig. 6(a) that, with the same reason of increase in liquidity, the survival function increases with the increase in initial surplus, it is further observed in Fig. 6(b) that as the value of the investment rate was increased the survival function was decreasing indicating that in the perturbed model investment should be made with careful observation because increasing investment rate without considering which kind of investment should be made, that is risk or risk-free, may lead to lower survival to the insurance company.

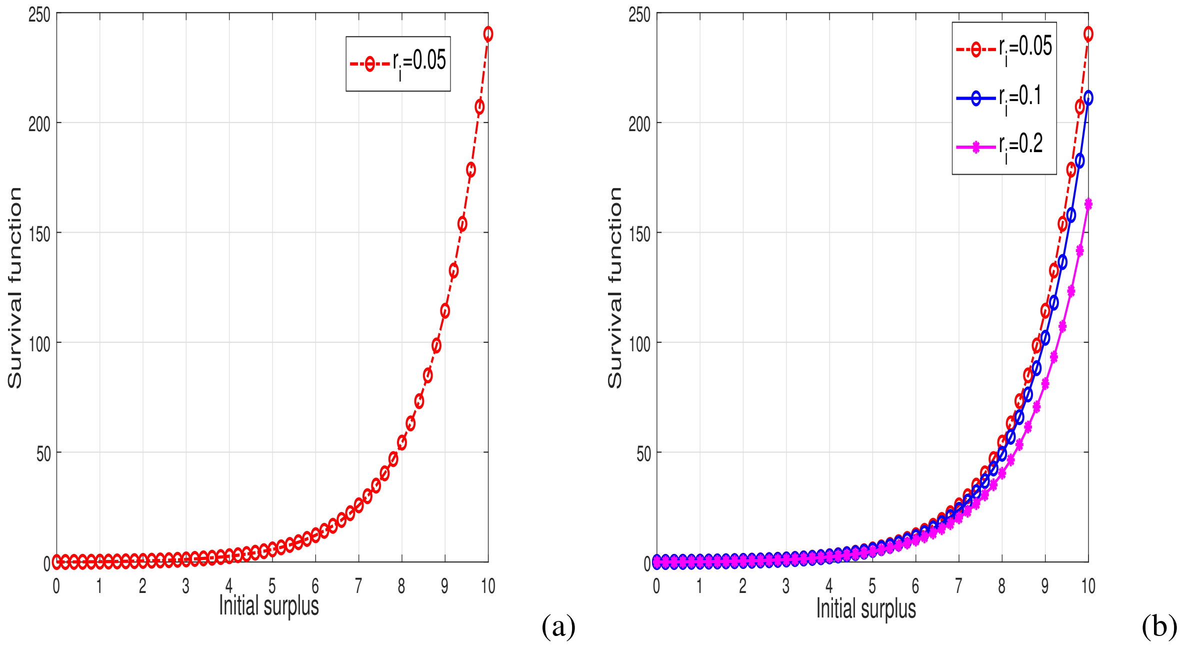


Figure 6: Behaviour of the survival function against the investment rate in perturbed model

4.3.2 Survival in the Diffusion-Perturbed Model Compounded by Refinancing

In this section we study the effect of refinancing on the increase of survival in the perturbed Cramér-Lundberg model. Let us consider a perturbed Cramér-Lundberg model given by equation 3.3 and then we shall compound this model by refinancing to get new model for this investigation purpose.

Compounded this model 3.3 by a refinancing process we have

$$X_t = p + ct + \sigma W_t - \sum_{i=1}^{N_t} Y_i + M_t \quad , \quad t \geq 0, \tag{4.71}$$

where M_t is the capital injected/refinanced at time t. Once again let $\delta \geq 0$ be a preference rate to refinancing such that $\delta > 0$ means that the investor prefers refinancing tomorrow to refinancing today while $\delta = 0$ means investor prefers not to refinance. Then, according to Eisenberg (2010) the $\phi(x)$ for equation 4.71 satisfies the VIDE given by equation 4.72 ;-

$$\frac{1}{2}\sigma^2\phi''(x) + c\phi'(x) - (\lambda+\delta)\phi(x) + \lambda \int_0^x \phi(x-y)dF(y) = 0. \tag{4.72}$$

Using 4.6 we get

$$\frac{1}{2}\sigma^2\phi''(x) + c\phi'(x) - (\lambda+\delta)\phi(x) + \lambda \int_0^x \phi(x-y)\beta e^{-\beta y}dy = 0. \tag{4.73}$$

This can be written as

$$\phi''(x) = -\frac{2c}{\sigma^2}\phi'(x) + \frac{2(\lambda+\delta)}{\sigma^2}\phi(x) - \frac{2\lambda}{\sigma^2}\int_0^x \phi(x-y)\beta e^{-\beta y}dy. \tag{4.74}$$

Then differentiating throughout we have.

$$\phi'''(x) = -\frac{2c}{\sigma^2}\phi''(x) + \frac{2(\lambda+\delta)}{\sigma^2}\phi'(x) - \frac{2\lambda}{\sigma^2}\frac{d}{dx}\left(\int_0^x \phi(x-y)\beta e^{-\beta y}dy\right). \tag{4.75}$$

Using Leibniz integral rule and equation 3.41 we get;

$$\phi'''(x) = -\frac{2c}{\sigma^2}\phi''(x) + \frac{2(\lambda+\delta)}{\sigma^2}\phi'(x) - \frac{2\lambda}{\sigma^2}e^{-\beta x}\phi(0). \tag{4.76}$$

For $\sigma^2 > 0$ we have

$$\phi'''(x) = -\frac{2c}{\sigma^2}\phi''(x) + \frac{2(\lambda+\delta)}{\sigma^2}\phi'(x). \tag{4.77}$$

Now, we transform a third order ODE given by equation 4.77 into a system of first order ODEs to be solved by the fourth order Runge-Kutta Method given by equation 3.57.

Letting $Z_1 = \phi(x)$, $Z_2 = \phi'(x) = Z_1'$ and $Z_3 = \phi''(x) = Z_2'$ leads to the following system of first order ODEs

$$\begin{cases} Z_1' = Z_2, \\ Z_2' = Z_3, \\ Z_3' = -\frac{2c}{\sigma^2}Z_3 + \frac{2(\lambda+\delta)}{\sigma^2}Z_2. \end{cases} \tag{4.78}$$

By using similar approach, this system 4.78 of first order ODEs is solved numerically using the fourth order Runge–Kutta method given by equation 3.57, implemented using MATLAB codes and results are discussed. Values of the parameters used for simulations are presented in Table 3.

We observe in Fig. 7(a) that the survival function increases as initial surplus increases due to increase in company's liquidity, we further observe in Fig. 7(b) that as the preference rate to refinancing increases the survival function also increases this indicate the refinancing behave in the same way as in non-perturbed model, thus when the insurance company is about to reduce its survival (get ruin) the decision to refinance is better to be taken early for the sake of securing the company.

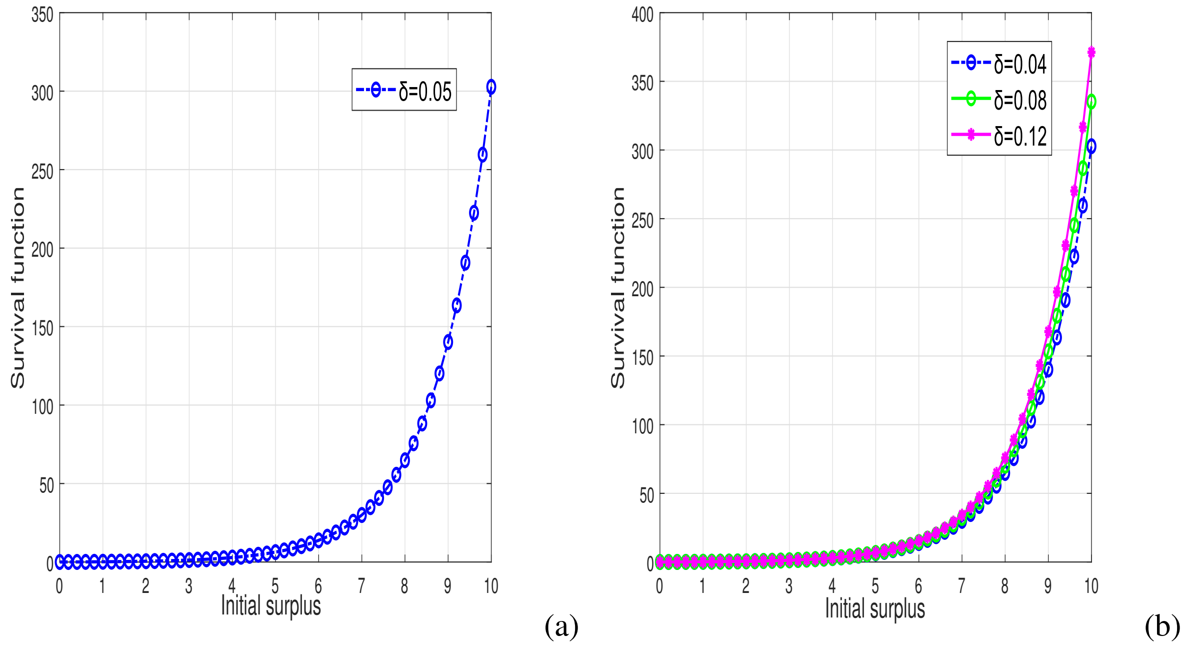


Figure 7: Behaviour of the survival function against preference rate to refinancing in perturbed model

4.4 Numerical Results for the Value Function

We proceed by analysing numerically the optimal value function given by equation 3.28. First we present a 3D plot shown in Fig. 8 that shows how the optimal value function behaves generally over initial wealth and time. Then, upon using a plot of value function against time shown in Fig. 9, the behaviours of the optimal value function is further studied against absolute risk aversion of the given utility function. Finally, the behaviour of the optimal value function is also studied against the volatility of the stock price, this can be observed in Fig. 10.

4.4.1 Numerical Experiments and Discussion of Results of the Value Function

Now, we analytically study the mathematical characteristics of the optimal value function given by equation 3.28. Mainly, the target in here was to maximise the exponential utility of the terminal wealth. Values of the parameters used for simulations are presented in Table 3 some were assumed and some were obtained from other studies.

In Fig. 8 we observe an increase in optimal value function at terminal time, this increase is observed to be irrespective of the initial wealth. Finally the value function attain its maximum value and this value is maintained for all time and initial wealth.

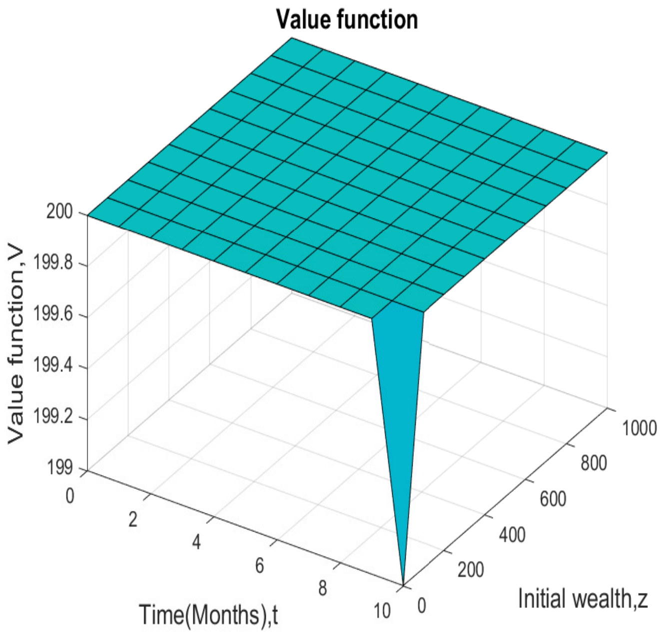


Figure 8: Dependence of the value function over the initial wealth and time

In Fig. 9 we observe that an insurance company has increasing absolute risk aversion (IARA) but also as wealth increases the value function also increases very rapidly at initial time. According to Johnson (2017) this implies that as wealth increases an insurance company will hold fewer investments in risky assets.

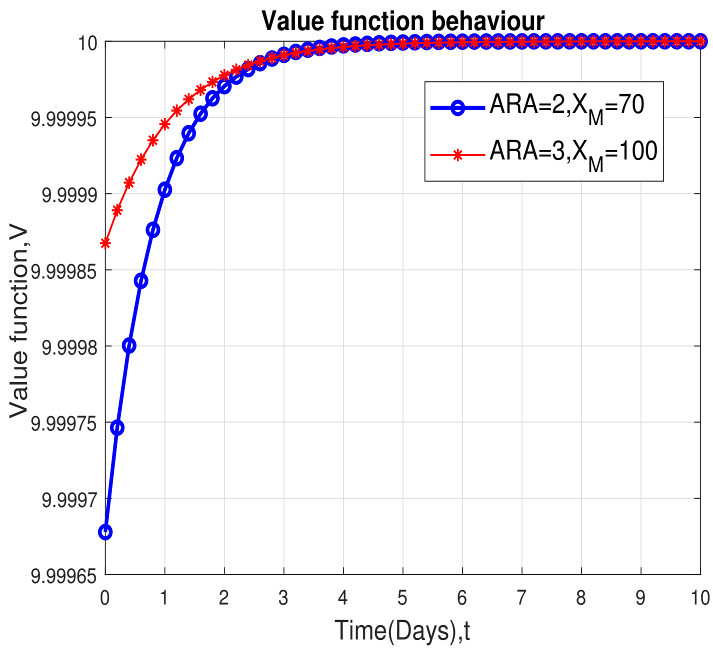


Figure 9: Behaviour of the value function against the absolute risk aversion and capital

In Fig. 10(a) we observe like two graphs coincide but a zoomed graph given in Fig. 10(b) we clearly observe that the value function increase with the increase on the volatility of the stock price which in turns it indicate that taking relatively higher risk on risk assets will give the insurance company a much better expected wealth utility. Actually this results resemble with the results of Hu *et al.* (2018) since insurance company's utility maximazation can really be realized for a large volatility of the stock price.

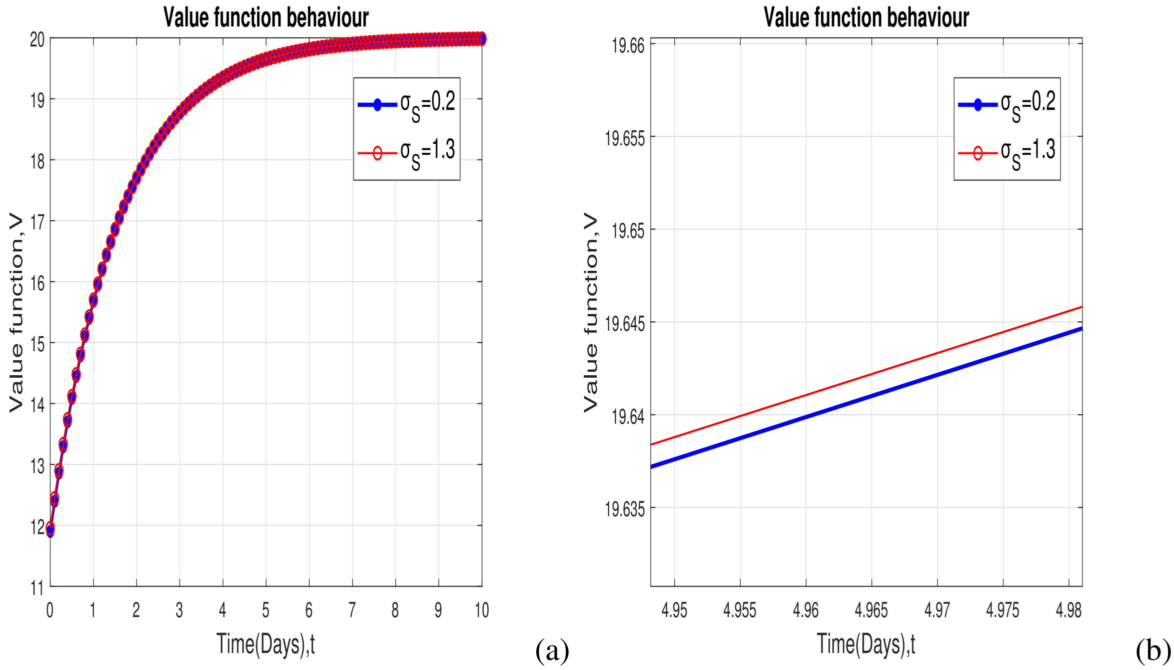


Figure 10: Behaviour of the value function against the volatility of the stock price

Also in Fig. 11(a) we observe like two graphs coincides but a zoomed graph given in Fig. 11(b) clearly shows that the value function increases with the increase in the possibility of recovery of the insurance company since we see that with the possibility of recovery of 25% the value function has small value as compared to 80% of possibility of recovery. In turn, this indicates that when an insurance company has a higher possibility of recovery it will have a better expected wealth utility since its value function is somehow larger.

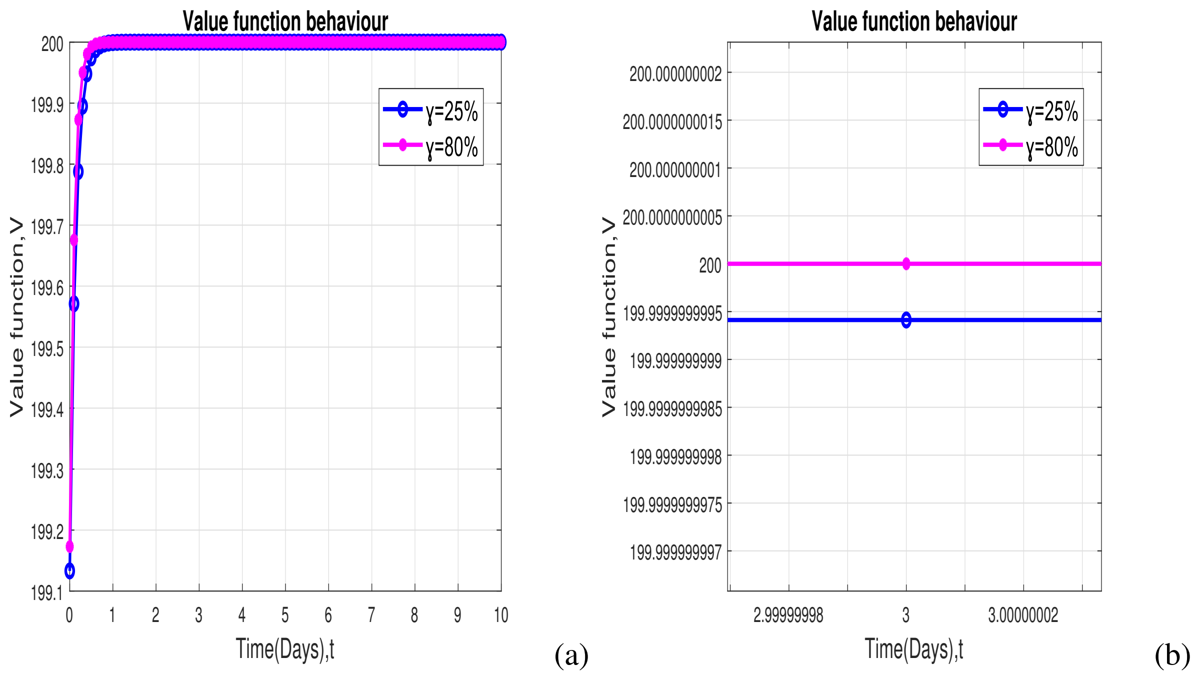


Figure 11: Behaviour of the value function against possibility of recovery of insurance company

4.5 Numerical Results for the Survival Function

In this section, the system 3.58 of first order ODEs is solved numerically using the fourth order Runge–Kutta method given by equation 3.57, implemented using MATLAB codes and results are discussed. Values of the parameters used for simulations are presented in Table 3. Generally, we observe similar numerical behavior in the fourth year, where the survival function increases because of liquidity improvement in the insurance company due to capital injection and investments.

In Fig. 12 we observe an increase in survival function at the fourth year due to liquidity in the insurance company. The survival function is observed to increase as the intensity of the counting process increases. This is due to the fact that as the counting process increases the insurance company services its clients much faster. As a result it becomes healthier and hence its probability of survival is increased.

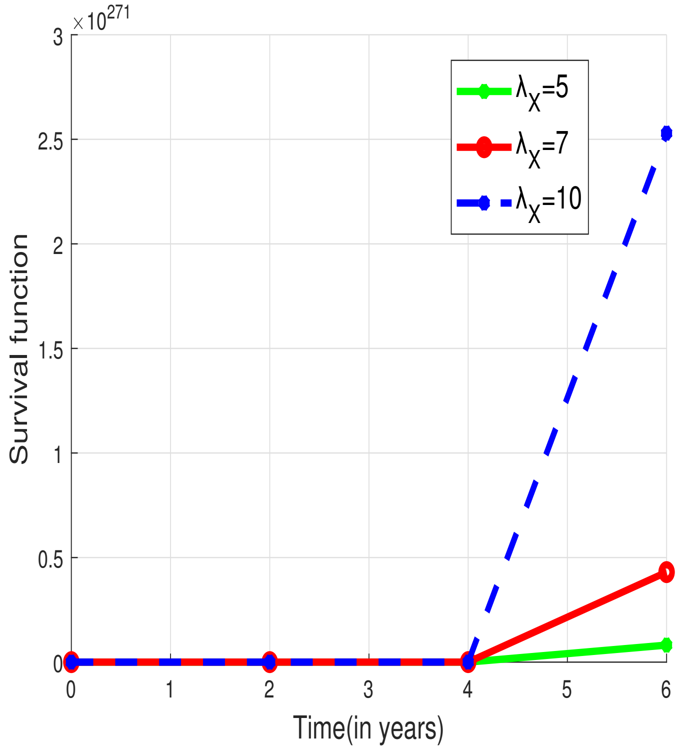


Figure 12: Dependence of the survival function on the intensity of the counting process

In Fig. 13 we observe that an insurance company has increasing survival function due to capital injection that leads to more liquidity but we further note that when instantaneous rate of stock return is increased the survival function is decreasing due to the fact that the risk is much higher.

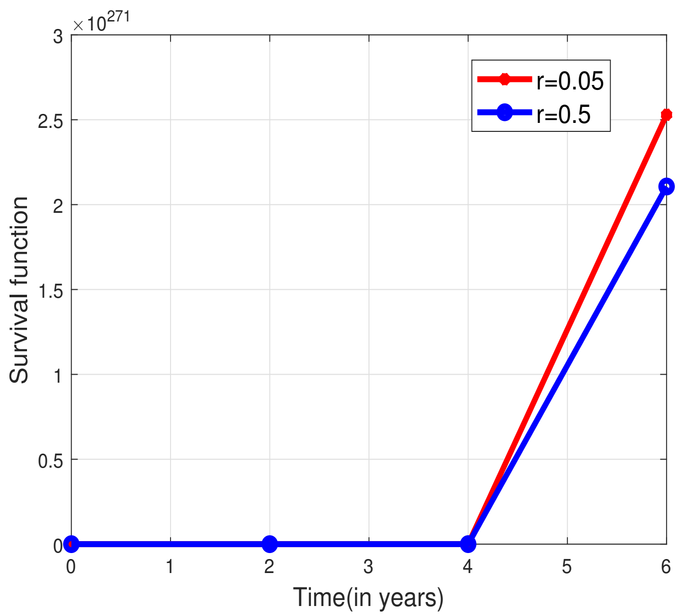


Figure 13: Dependence of the survival function on the instantaneous rate of stock return

Since return volatility is a measure of the dispersion of returns for a given security, we observe in Fig. 14, that an insurance company has inverse relationship with the return volatility, we note that when return volatility is increased just by 0.02 the survival function is decreasing very quickly due to the fact that the higher the volatility, the riskier the security. These results show that the survival function is extremely sensitive to the return volatility and support the earlier observations on the instantaneous rate of stock return. As a result, this study suggests that more investments should be made in risk-free assets rather than in risky assets when this situation occurs.

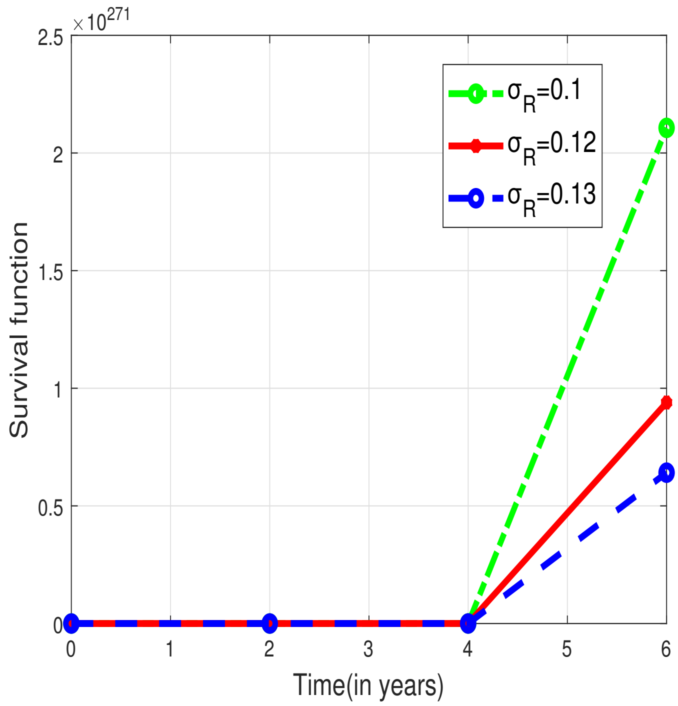


Figure 14: Dependence of the survival function on the return volatility

4.6 Conclusion

In this chapter, we presented and discussed the study findings. We analysed the survival function in various scenarios and established that it increases with the surplus process and investment rate while it decreases with the increase in the diffusion coefficient. Additionally, we analysed the value function in various scenarios and found out that it increases with the volatility of the stock price, the possibility of recovery, and the intensity of the counting process. However it decreases with the increase in the instantaneous rate of stock return. In the next chapter we present the general conclusion and recommendations basing on the research findings and we suggest some possible problems for future research.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

In this study, a way of maximizing the exponential utility of terminal wealth has been proposed. In addition, a risk process compounded by refinancing and investment was formulated. Thereafter a stochastic differential equation for the wealth was derived and an optimal control problem for maximizing the expected utility of terminal wealth was formulated and solved. An approach to treating the possibility of recovery after ruin for insurance companies was developed and proposed for the first time in insurance mathematics.

The study investigated the behaviour of the value function numerically and the results indicated that the value function increases irrespective of the initial wealth and time. The results on studying the behaviour of the value function with respect to the volatility of the stock price were similar to those of Hu *et al.* (2018) since it was observed that insurance company's utility maximization can be realized for large volatility of the stock price.

Additionally, this study observed that in case ruin occurs and a company performs refinancing, the value function increases very rapidly with the increase in refinancing amount. This observation was also supported by the behaviour of value function with respect to the possibility of recovery after ruin. It was also observed that as the possibility of recovery after ruin increases the value function also increases.

On the other hand, this study used the basic Cramér-Lundberg model to derive the Volterra Integro Differential Equation (VIDE) for the survival function of the insurance company. After some conversions, the VIDE was solved numerically by the Runge-Kutta method of order four. The results indicated that it is possible to maximise the survival function for the insurance company's portfolio and this in turns helps the company to reduce the possibility of ruin. The study further established that increasing the claim intensity has a positive effect in terms of increasing the survival function and hence reducing probability of ruin.

5.2 Recommendations

This study didn't consider the risks associated with refinancing or investment caused by unforeseen events such as war, credit crunch like what happened in 2008, and other unpredictable events. Based on the research findings, this study recommends all insurance companies to have well-trained staff in risk management who can study the company's portfolio and give suggestions to managers on how to avoid or minimize ruin and how to recover in case ruin occurs.

Also this study recommends that insurance companies should take steps to increase their claim intensities since this has the positive impact in improving the companies' services to customers. The study also recommends that insurance companies should invest more in risk-free assets when the instantaneous rate of stock return and return volatility are much higher or increased as this decision will reduce the company's risk.

The following are the recommendations for the possible future research areas as an extension of this research:

(i) Incorporation of the use of stochastic interest rates instead of the fixed one for both stocks and bonds to realize the actual fluctuation of the interest rates at any given time.

(ii) Due to some constraints and the wideness of the research, this study did not incorporate data collection, and thus model validation was not part of the study. Other researchers can check the possibility of model validation and the inclusion of reinsurance in the model.

(iii) To solve the obtained systems of ODEs by using MAPLE SOFTWARE. It will lead to analytical solutions ready for further analysis.

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RESEARCH OUTPUTS

The following two research papers based on this study findings were published in two different journals. The first paper has been published in the Journal of Mathematics and Informatics and the second one has been published in the Journal of Mathematics.

(1) Komunte, M., Kasumo, C., & Verdiana, V. G. (2021). Insurance Companies Portfolio Optimization with Possibilities of Recovery after Ruin: A Case of Exponential Utility Function. *Journal of Informatics and Mathematics.*

(2) Komunte, M., Kasumo, C., & Verdiana, V. G. (2022). Reducing the Possibility of Ruin by Maximizing the Survival Function for the Insurance Company's Portfolio. *Journal of Mathematics.*

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Insurance Companies Portfolio Optimization with Possibilities of Recovery after Ruin: A Case of Exponential Utility Function

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Abstract. In this paper, we propose and analyze the perturbed mathematical model for modeling the portfolio of insurance companies with possibilities of recovery after ruin. Return on investment and refinancing are used as approaches for overcoming ruin. The model is analyzed for different cases of possibilities of recovery after ruin within [0, 1]. The results indicate that the return on investment plays an important role in reducing the ultimate ruin and that as the possibility of recovery for insurance companies increases the return on investment reduces the ruin at a fast rate. Finally, the study recommends that all insurance companies should have well trained staff in risk management who can study the company's portfolio and gives suggestions to managers on how to avoid or minimize ruin and how to recover in case ruin occurs.

Keywords: Ruin probability, Value function, Refinancing, Jump-diffusion, Possibilities of recovery

AMS Mathematics Subject Classification (2010): 37N40

1. Introduction

Risks affect many aspects of human life and in some cases may even result in financial loss. Therefore it is important to secure expensive property and insurance provides that security. In [1], Kasumo observed that the provision of insurance requires competent management as poor management may lead to the eventual ruin of the insurance company, resulting in its failure to fulfill its obligations. This happens when the insurer's surplus level falls below zero, thus making the company bankrupt.

According to [2] insurance refers to a contract that is represented by a policy where individuals or entities receive financial protections from a given insurance company against losses. In [3], Kozmenko and Oliynyk have observed and suggested that in fulfilling its obligations, an insurance company will have a collection of investments that generate income to cover clients' claims and this collection of investments for an insurance company is known as an insurance portfolio. In addition, an insurance company holding a portfolio with many liquid investments reduces the investor's risk since these investments enables the company to fulfill claims whenever they arise.

One of the best measures that an insurance company should take is risk management on its portfolio. Several measures are available for managing risk in an insurance company. Refinancing and investment are some of the measures to overcome the risks of the insurance company and give optimal returns to the shareholders. [4] reveals that using investment, the insurer distributes part of those risks to the paying investments which in turn can save a company to cover clients' benefits during ruin. [5] worked under the martingale invariance hypothesis and assumed the existence of conditional density to study the optimal reinsurance and investment of the insurer where the surplus process of the insurer was assumed to satisfy a jump-diffusion process and the dynamics of the risky stock price followed a Heston model. They considered proportional reinsurance and also investment optimization problems for insurers existing in financial markets depending on risky stock assets, a savings account, and corporate bonds, while [6] studied ruin probability based on a dual risk model having risk-free investments.

Through investment and refinancing strategies, insurers may protect themselves against any potentially big losses or at least ensure that their earnings will remain relatively stable when there is a possibility of recovery after ruin. In the literature (see, for example, [7,8,9]) many optimization problems have arisen as part of the risk management process to study how insurance companies can control ultimate ruin. In [10], studied how to optimize the control in investments and reinsurance problems for an insurer using a jump-diffusion risk process but with the independence of the Brownian motions while [4] studied the optimal investment and reinsurance problem for an insurer and a reinsurer using jump-diffusion processes. This paper seeks to establish the ways of minimizing ruin through portfolio management by maximizing insurance portfolio for the case of exponential utility function when there are possibilities of recovery after ruin.

This paper is organized as follows: In Section 2, we propose a model for maximizing insurance portfolio for a case of exponential utility function when there are possibilities of recovery after ruin. We investigated the behaviour of the value function for the proposed model. In Section 3 the numerical simulations were carried out and their results are presented. In the last section, we present the conclusion recommendation and possible extension of the paper.

2. Model formulation and analysis

We now present the model formulation and its analysis. In this work, we consider continuous time stochastic processes in the time interval $[0,T]$ where $0<T<\infty$. A stochastic process is a family of random variables $X=(X_t)_{t\in[0,T]}$ defined on the probability space (Ω, F, P) and valued in a measurable space (Ω, F) and indexed by time t. For each $\omega \in \Omega$, the mapping $X(\omega): t \in [0,T] \to X(t;\omega)$ is called the path of the process for the event ω. All stochastic quantities and random variables are defined on a large enough stochastic basis $(\Omega, F, (F)_{t\in[0,T]}, P)$ satisfying the usual conditions, that is to say, $(F)_{t\in[0,T]}$ is right continuous and P-complete, P is the probability measure defined on F and $(F)_{t\in[0,T]}$ is an augmented filtration.

In reality, the income of the insurer is not deterministic, there exist fluctuations in the number of customers, claim arrivals, and premium income [11]. If both refinancing (capital injection) and investment are absent, then the basic model for the insurance process can be given by a perturbed risk process X_t defined by

$$X_t = p + ct + \sigma_X W_{X,t} - \sum_{i=1}^{N_{X,t}} Y_{X,i} \quad , t \geq 0 \tag{1}$$

In this case, c is the premium rate, that is the insurer's premium income per unit time assumed to be received continuously and is calculated by the expected value principle, that is $c = (1+\theta)\lambda_X \mu_X$ where $\theta > 0$ is the relative safety loading of the insurer. Also $X_0 = p$ is the initial capital of the insurance company and W_X is a standard Brownian motion independent of the compound Poisson process $\sum_{i=1}^{N_{X,t}} Y_{X,i}$. Here λ_X is the intensity of the counting process $N_{X,t}$ for the claims and let F_X be the distribution function of the claims $Y_{X,i}$. It is assumed that F_X is continuous and concentrated on $[0,\infty)$. We interpret equation (1) above as follows: ct is the premium income received by an insurance company up to time t. The Brownian term $\sigma_X W_X$ is meant to take care of small perturbations in premium income and claim sizes, $N_{X,t}$ is the claim number process and $Y_{X,i}$ are claim sizes. It is assumed that $F_X(0) = 0$ and at least one of σ_X or λ_X is non-zero. A vast number of researchers have studied this classical risk process perturbed by diffusion in the insurance industry, some of them being [1,4,11,12,13,14].

For refinancing process we let M be an increasing process with $M_{0^-} = 0$. The process with refinancing (capital injections) is denoted by $X_t^M = X_t + M_t$ with X_t being the surplus process and $X_0 = p$. The capital injection process M has to be chosen such that $X_t^M \geq 0$ for all t (almost surely); it could then be optimal to inject capital already before the process reaches zero. Therefore, by using equation (1) the model with refinancing will be given by equation (2) below;

$$X_t^M = p + ct + \sigma_X W_{X,t} - \sum_{i=1}^{N_{X,t}} Y_{X,i} + M_t \ , t \geq 0 \tag{2}$$

For the investment process, we assume the risk-free (bond) price process is given by

$$dB_t = r_0 B_t dt \tag{3}$$

where $r_0 \geq 0$ is the risk-free interest rate, which is assumed to be constant. B_t is the price of the risk-free bond at a time t.

According to [15] stocks are an important way for the company to raise funds. Let us also describe the risky asset (stock) price process by the Geometric Brownian Motion (GBM) given by

$$dS_t = rS_t dt + \sigma_S S_t dW_{S,t} \tag{4}$$

where S_t is the price of the stock at time t, $r \geq 0$ is the expected instantaneous rate of stock return, $\sigma_S \geq 0$ is the volatility of the stock price and $W_{S,t} : t \geq 0$ is a standard Brownian motion defined on the complete probability space $(\Omega, F, (F)_{t\in[0,T]}, P)$. [16] gave the generalized return on investment process R_t as

$$R_t = rt + \sigma_R W_{R,t} + \sum_{i=1}^{N_{R,t}} S_{R,i} \quad , t \geq 0,\ R_0 = 0 \tag{5}$$

where $W_{R,t}$ is a Brownian motion independent of the surplus process R_t, also $\sum_{i=1}^{N_{R,t}} S_{R,i}$ is a compound Poisson process with intensity λ_R which represents the sudden changes in income (jumps), the term $\sigma_R W_{R,t}$ represents the fluctuation in income of an insurance company and the rt is the non-risky part of the investment process. If we assume $\lambda_R = 0$, that is, there are no jumps, the resulting model which was also discussed by [17] is the Black-Scholes option pricing formula given by

$$R_t = rt + \sigma_R W_{R,t} \quad , t \geq 0,\ R_0 = 0 \tag{6}$$

Equation (6) above is the return on investment model, where r is the risk-free part, hence $R_t = rt$ means that one unit invested at time zero will be worth e^{rt} at time t.

2.1. Stochastic differential equation for the wealth

In this section, a basic insurance process with investment, which is expressed by the stochastic differential equation for the wealth after refinancing, is formulated. Now let us consider the investment problem of an insurance company that seeks to transfer current wealth into the bond and stock. The company's preference is to choose a dynamic portfolio strategy in order to maximize the expected utility of wealth at some future time *T*. Therefore in order to describe the company's actions the portfolio strategy is formulated.

Assume that the joint distribution of the $W_{X,t}$ and $W_{S,t}$ that are used is bivariate normal and we denote their correlation coefficient by ρ, that is $E[W_{X,t} W_{S,t}] = \rho t$. The company needs to monitor its wealth, let the amount of money invested in risky asset (stock) at time t under investment policy π be denoted by π_t, where $\{\pi_t\}$ is a portfolio strategy suitable and admissible control process, that is to say π_t satisfies $\int_0^T \pi_t^2 dt < \infty$ a.s., for all $T < \infty$. Let $\{Z_t, t \geq 0\}$ denote the corresponding wealth process, then the dynamic of Z_t is given by

$$dZ_t = \pi_t \frac{dS_t}{S_t} + \left(X_t^M - \pi_t\right)\frac{dB_t}{B_t} + dX_t^M \tag{7}$$

with $Z_0 = z > 0$ being the initial wealth of the company.

By substituting equations (3) and (4) above into equation (7) then the wealth process with investment and refinancing will follow the stochastic differential equation

$$dZ_t = \left(\pi_t r + \left(X_t^M - \pi_t\right) r_0\right)dt + \sigma_S \pi_t dW_{S,t} + dX_t^M \tag{8}$$

where X_t^M is given by equation (2).

2.2. Optimal control problem for maximizing the expected utility of terminal wealth

In [18], the authors studied the problem of the expected utility of wealth in the discrete time for a given investor. In the study it was conjectured that minimizing the ruin probability is strictly related to maximizing the exponential utility of terminal wealth of the investor, the assumption behind the conjecture was that the investor is allowed to borrow an unlimited amount of money and without risk-free interest rate.

Let a strategy α describe the stochastic process $\{\pi_t, M_t\}$, where π_t the amount invested in the risky asset at time t and M_t is the capital refinanced/injected at time t and denote the set of all admissible strategies by α_S . Suppose now that the insurer is interested in maximizing the utility function of its terminal wealth, say at time T . The utility function $u(z)$ is typically increasing and concave $(u''(z) < 0)$. For a strategy α let's define the utility attained by the insurer from state z at time t as follows;

$$V_\alpha(t,z) = \mathrm{E}\left[u(Z(T))/Z(t) = z\right] \tag{9}$$

Therefore the objective is to find the optimal value function

$$V(t,z) = \sup_{\alpha \in \alpha_S} V_\alpha(t,z) \tag{10}$$

and the optimal strategy $\alpha^*\{\pi_t^*, M_t^*\}$ such that $V_{\alpha^*}(t,z) = V(t,z)$.

2.3. Maximizing the exponential utility of terminal wealth

An ordinary investor under discrete time and space was studied by [18] where it was found that when the investor had an exponential utility function such as $u(z) = -e^{-\theta z}$ and aiming at maximizing the utility of terminal wealth at fixed terminal time, then the optimal policy was an investment of a fixed constant amount. The conclusion given by a strategy was in general optimal for minimizing the probability of ruin or maximizing the probability of survival.

Using equations (9) and (10) above, let π_t^* denote the optimal policy and suppose that the company is now having an exponential utility of the form (11) below, where $\gamma > 0$ and $\theta > 0$.

$$u(z) = \lambda - \frac{\gamma}{\theta} e^{-\theta z} \tag{11}$$

This kind of utility function has constant absolute risk aversion (ARA) since $-u''(z)/u'(z)=\theta$, it plays a very important role in actuarial and insurance mathematics at large.

Theorem 2.1. The optimal policy of maximizing expected utility at a terminal time T is investing at each time $t \le T$ a constant amount given by

$$\pi_t^* = \frac{r}{\sigma_S^2\theta} - \frac{\rho}{\sigma_S} \tag{12}$$

Then the optimal value function becomes

$$V(t,z) = \lambda - \frac{\gamma}{\theta}\exp\{-\theta z + (T-t)Q(\theta)\} \tag{13}$$

where $Q(.)$ is the quadratic function defined by

$$Q(\theta) = \frac{1}{2}(1-\rho^2)\theta^2 - \left(X_t^M r_0 - \rho\left(\frac{r-r_0}{\sigma_S}\right)\right)\theta - \frac{1}{2}\left(\frac{r-r_0}{\sigma_S}\right)^2 \tag{14}$$

Proof: For our problem of maximizing utility of terminal wealth at a fixed terminal time T. Then the HJB equations for $t < T$ can be obtained as follows

$$\begin{cases} \sup_{\pi_t}\{\ell^{\pi_t}V(t,z)\}=0 \\ V(T,z)=u(z) \end{cases} \tag{15}$$

where $V(t,z)=\sup_{\pi_t}\mathrm{E}^{t,z}\left[u\left(Z_T^{\pi_t}\right)\right]$ this is the same as saying for each (t, z) we need to solve the nonlinear PDE of (15) and there after find the value of π_t that will maximize the function (16) below

$$f(\pi_t) = V_t + \left[\pi_t r - \pi_t r_0 + X_t^M r_0\right]V_z + \frac{1}{2}\left[\sigma_S^2\pi_t^2 + 2\rho\sigma_S\pi_t + 1\right]V_{zz} \tag{16}$$

Suppose we assume that the HJB equation (15) consists of a classical solution V that satisfies $V_z > 0$ and $V_{zz} < 0$ now differentiating with respect to π_t and equating to zero in (16) the following optimizer is obtained

$$\pi_t = -\frac{\rho}{\sigma_S} - \left(\frac{r-r_0}{\sigma_S^2}\right)\left(\frac{V_z}{V_{zz}}\right) \tag{17}$$

Substituting equation (17) back into equation (16) then after some simplifications equation (15) become

$$\begin{cases} V_t + \left[X_t^M r_0 - \rho\left(\frac{r-r_0}{\sigma_S}\right)\right]V_z - \frac{1}{2}\left(\frac{r-r_0}{\sigma_S}\right)^2\frac{V_z^2}{V_{zz}} + \frac{1}{2}(1-\rho^2)V_{zz} = 0 \quad \text{for } t<T \\ V(T,z)=u(z) \end{cases} \tag{18}$$

The PDEs obtained in equation (18) above are quite different from those obtained in other studies of utility maximization such as those by [19, 20]. Since we want to solve the

PDE under a given case when $u(z)=\lambda-\frac{\gamma}{\theta}e^{-\theta z}$. To solve the PDE in equation (18) above under this case let's assume that it has the solution of the following form

$$u(z)=\lambda-\frac{\gamma}{\theta}e^{-\theta z+g(T-t)} \tag{19}$$

where $g(.)$ is a given suitable function, with this assumption, then

$$\begin{cases} V_t(t,z)=[V(t,z)-\lambda][-g'(T-t)] \\ V_z(t,z)=[V(t,z)-\lambda][-\theta] \\ V_{zz}(t,z)=[V(t,z)-\lambda][\theta^2] \end{cases} \tag{20}$$

Since the boundary condition is $V(T,z)=\lambda-\frac{\gamma}{\theta}e^{-\theta z}$ this mean that $g(0)=0$ now let us insert (20) into (18) and simplify to get

$$-g'(T-t)+\frac{1}{2}(1-\rho^2)\theta^2-\left(X_t^M r_0-\rho\left(\frac{r-r_0}{\sigma_S}\right)\right)\theta-\frac{1}{2}\left(\frac{r-r_0}{\sigma_S}\right)^2=0 \tag{21}$$

Now letting $Q(\theta)=\frac{1}{2}(1-\rho^2)\theta^2-\left(X_t^M r_0-\rho\left(\frac{r-r_0}{\sigma_S}\right)\right)\theta-\frac{1}{2}\left(\frac{r-r_0}{\sigma_S}\right)^2$ gives

$$g'(T-t)=Q(\theta) \tag{22}$$

Integrating equation (22) and using $g(0)=0$ gives the value function (13). Since the value function is known, we can now obtain the control (12) by substituting the values of V_z and V_{zz} from equation (20) into equation (17).

Finally we need to show that the value function and the control obtained above are optimal. This is revealed upon checking the value function (13) since it is seen to be twice continuously differentiable thus we conclude that it satisfies the conditions of classical verification theorems as stated by [21], therefore these are the optimal value function and controls.

2.4. Treating possibility of recovery after ruin for insurance companies

In this section, an approach on how to handle the possibility of recovery after ruin for insurance companies is suggested and developed for the first time in insurance mathematics. We assume that an insurance company had wealth $X_{t_{\tau^-}}$ before the time of ruin suppose an insurance company has a possibility $\gamma\in[0,1]$ of recovery after ruin, where $\gamma=0$ means that the company has no possibility of recovery at all and $\gamma=1$ means the company has a possibility of recovering a full wealth after ruin. Now, this study suggests that the new wealth or capital of the company for running the insurance business be given by the following formula

$$X_t^N=\gamma X_{t_{\tau^-}} \tag{23}$$

Therefore, this approach is used in performing the simulation on different cases of possibilities γ under different situations of wealth or capital X_t when compounded with investment and/or refinancing.

3. Numerical experiments and discussion of results

We numerically observe the mathematical characteristics of the optimal value function given by equation (13) above. Mainly, the target here was to maximize the exponential utility of the terminal wealth. All the model simulations in this paper were performed in an HP ENVY 17 with an Intel(R) Core(TM) i7-8550U CPU processor at 1.80GHz to 1.99GHz and 16.0GB of RAM and the figures were constructed by using MATLAB R2020a. Values of the parameters are presented in Table 1 below, some were estimated and some were obtained from other studies.

Table 1: Model parameters and their values

Symbol	Definition	Value(s)	Source
σ_S	The volatility of the stock price	0.2, 1.3	[22]
λ	Number of claims received per unit time	10, 20, 200	Estimated
θ	Safety loading of the insurer	0.8, 2, 3, 5	[1]
X_t^M	Refinanced surplus process	50, 70, 100, 1000	Estimated
r	Instantaneous rate of stock return	0.05	[13]
r_0	Risk-free interest rate for the bond	0.02, 0.04	[23]
ρ	Correlation coefficient	0.03	[4]
γ	Possibility of recovery	2%, 25%, 80%	Estimated

In Figure 1 we observe an increase in optimal value function at the terminal time, this increase is observed to be irrespective of the initial wealth. Finally the value function attains its maximum value and this value is maintained for all time and initial wealth.

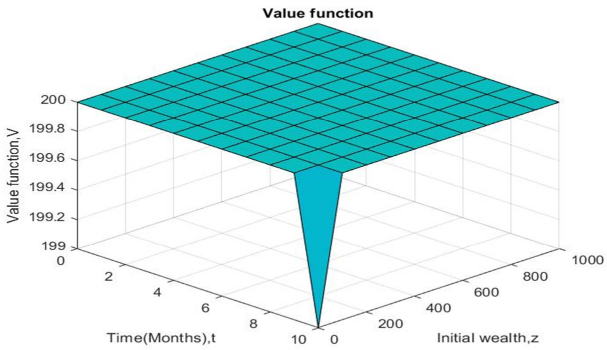
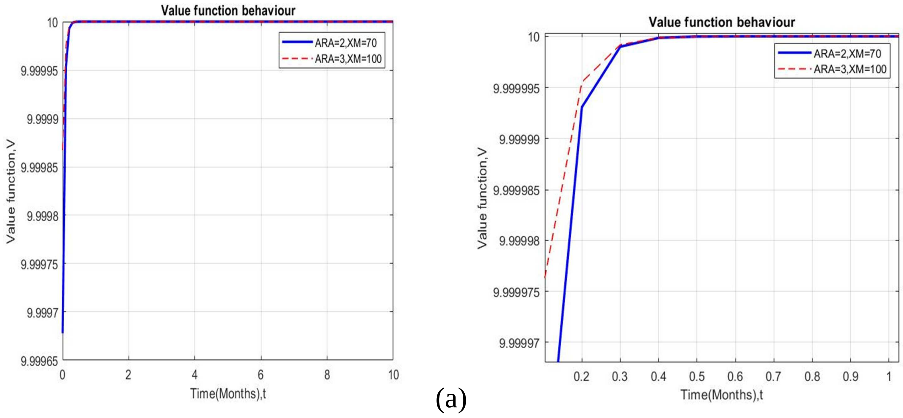


Figure 1: Dependence of the value function over the initial wealth and time.

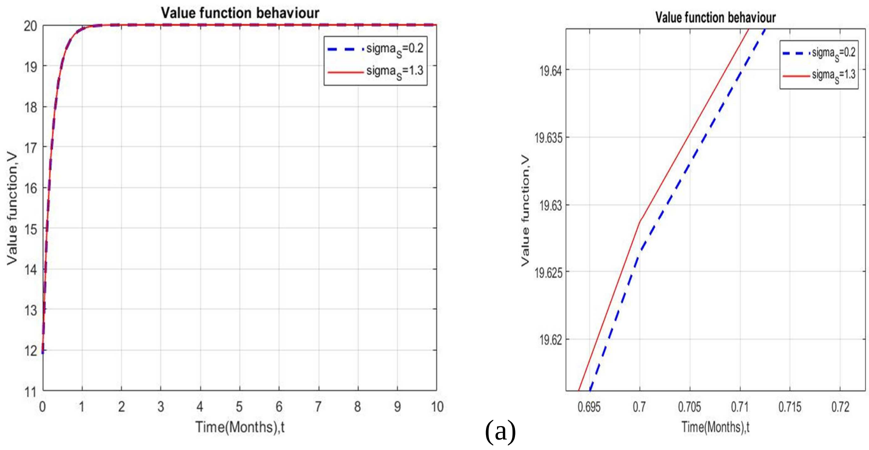
In Figure 2 we observe that an insurance company has increasing absolute risk aversion (IARA) but also as wealth increases the value function also increases very rapidly at the initial time. According to [24] this implies that as wealth increases an insurance company is advised to hold fewer investments in risky assets likewise this paper recommends that an insurance company should hold as few investments in risky assets as possible.



(a) (b)

Figure 2: Behaviour of the value function with respect to absolute risk aversion and capital

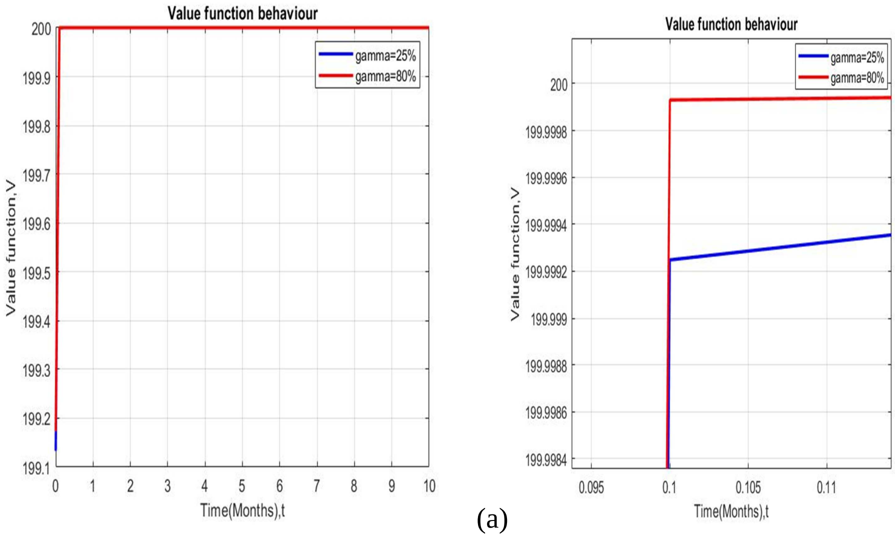
In Figure 3(a) it appears as if the two graphs coincide but a zoomed graph given in figure 3(b) shows that the value function increase with the increase in the volatility of the stock price. It is clearly observed from Figure 3(b) that when volatility was increased from 0.2 to 1.3 the value function also increased its value, this in turn indicates that taking a relatively higher risk on risky assets will give the insurance company a much better expected wealth utility. These results confirm the results of [4] since the insurance company's utility maximization can be realized for large volatility of the stock price.



(a) (b)

Figure 3: Behaviour of the value function with respect to the volatility of the stock price

Also in Figure 4(a) we observe like two graphs coincides but a zoomed graph given in Figure 4(b) clearly shows that the value function increases with the increase in the possibility of recovery of the insurance company since we see that with the possibility of recovery of 25% the value function has small value as compared to 80% of possibility of recovery. In turn, this indicates that when an insurance company has a higher possibility of recovery it will have a much better expected wealth utility since its value function is much larger.



(a) (b)

Figure 4: Behaviour of the value function with respect to the possibility of recovery of the insurance company.

4. Conclusion

In this study, we have proposed a way of maximizing the exponential utility of terminal wealth. We also formulated a risk process compounded by refinancing and investment thereafter a stochastic differential equation for the wealth was derived and an optimal control problem for maximizing the expected utility of terminal wealth was formulated and solved. An approach to treating the possibility of recovery after ruin for insurance companies was developed and suggested for the first time in insurance mathematics.

We were able to investigate the behavior of the value function numerically and the results indicated that the value function increases irrespective of the initial wealth and time. The results on studying the behavior of the value function with respect to the volatility of the stock price were similar to those of [4] since we observed that insurance company's utility maximization can be realized for large volatility of the stock price. Also, this study observed that in case ruin occurs and a company performs refinancing the value function increases very rapidly with the increase in refinancing amount. This observation was also supported by the behavior of value function with respect to the possibility of recovery after ruin, we also observed that as the possibility of recovery after ruin increases the value function also increases.

This study recommends that all insurance companies should have well-trained staff in risk management who can study the company's portfolio and give suggestions to managers on how to avoid or minimize ruin and how to recover in case ruin occurs.

Further study can be done as an extension where one can incorporate the use of stochastic interest rates instead of the fixed one for both stocks and bonds to realize the actual fluctuation of the interest rates at any given time.

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Research Article

Reducing the Possibility of Ruin by Maximizing the Survival Function for the Insurance Company's Portfolio

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In this paper, the intention was to reduce the possibility of ruin in the insurance company by maximizing its survival function. This paper uses a perturbed classical risk process as the basic model. The basic model was later compounded by refinancing and return on investment. The Hamilton–Jacobi–Bellman equation and integro-differential equation of Volterra type were obtained. The Volterra integro-differential equation for the survival function of the insurance company was converted to a third-order ordinary differential equation which was later converted into a system of first-order ordinary differential equations. This system was then solved numerically using the fourth-order Runge-Kutta method. The results show that the survival function increases with the increase in the intensity of the counting process but decreases with an increase in the instantaneous rate of stock return and return volatility. This is due to the fact that the insurance company faces more risk. Thus, this paper suggests that in this situation, more investments should be made in risk-free assets.

1. Introduction

The insurance industry is currently undergoing a fundamental transformation in terms of operations and competitiveness. Several disruptive factors in business have given rise to new players in the market with disruptive business models to outperform their competitors. Investment and refinancing can be used as survival approaches when insurance players consider how they should react to this major shift. With investing and refinancing, an insurance company can manage to operate much better even if it had suffered from ruin provided the investments are done properly and refinancing is done adequately and timely [1].

According to Kolm et al. [2] in investments, there is a trade-off between risks and returns. In turn, to increase the expected returns from investment, investors must be willing to tolerate greater risks. Portfolio management theory helps to model the trade-off for the given collections of several possible investments [3]. Investigation into companies that have suffered from ruin is one of the very important areas of actuarial research. Some research studies have been done to investigate portfolio optimization, most of them applying reinsurance and refinancing approaches (see, e.g., Kasumo [4], Liu, and Yang [5]). However, more research is needed on those companies that have suffered from ruin because little has been done to investigate how these companies can be managed financially to become profitable again.

Oyatoye and Arogundade [6] applied a stochastic model for predicting the optimum portfolio of insurance businesses at an acceptable risk exposure level, excluding ruin effects. They underscored the importance of this because it would guarantee the acceptable risk levels for a viable insurance company and evaluate the retention rate of the insurance portfolio at a given risk rate. But, it would also provide good knowledge on the importance of reinsurance in risk adjustment in times of larger claims. Finally, it would examine the unbearable risk level that would require coinsurance. Risk return analysis and catastrophe exposure analysis were performed. It was observed that there is a need to revert to stochastic modelling, which canvases the use of risk, variances, and expected values for mathematical computation.

Recently, considerable attention on the part of insurance companies has been given to the procedures of forming an assigned insurance portfolio because it serves as an indicator of the quality of insurance liabilities. Oliynyk [7] studied the basic methodological principles of formation and management of the insurance portfolio to achieve equilibrium and ensure that the financial stability of insurance companies is maintained. One of the stages in the company's insurance portfolio management process is to deal with portfolio optimization. This stage was discussed by Oliynyk [7] as it leads to the reduction of risks and an increase in profitability levels. The study finally observed that the proposed scientific and methodical approach to building and managing an insurance portfolio to achieve its equilibrium based on nonlinear programming has a differentiated character. For each company, this model chooses an optimal structure of an insurance portfolio that ensures maximal profits and minimal risks.

Ma et al. [8] extended the work of Zhu et al. [9] to include defaultable securities. The insurer was given a chance of buying proportional reinsurance and put his wealth in stock, a defaultable corporate bond, and a money account. The intention was to maximize their expected utility of wealth. In their work, Ma et al. [8] chose the constant elasticity of variance (CEV) process to describe the stocks' behaviour. The reason for selecting a CEV model was that it could also be used as an alternative model for describing the stochastic volatility behaviour of the stocks' price. It had several empirical pieces of evidence to support it. Using stochastic control theory, they derived a Hamilton–Jacobi–Bellman (HJB) equation and later divided the original problem into two parts a predefault case and a postdefault case. Value functions and expressions of the optimal strategy were derived. Finally, they presented numerical examples as illustrations of their results. Their study did not consider converting the Volterra integro-differential equations into an ordinary differential equation.

Shareholders of insurance businesses are interested in optimizing the returns from the insurance portfolio as well as ensuring that the business remains afloat over a long-time horizon. To achieve this, the company managers have to optimally run the business to maximize returns and reduce ruin probability. Even with extra care, many times, ruin is inevitable. Most studies in the literature, for example, Schmidli [10], Kasozi et al. [11], and Kasumo et al. [12], do not consider refinancing as a measure to overcome ruin once it hits. This paper seeks to develop and analyse an insurance portfolio optimization model incorporating investments and refinancing strategies and then find the best way to maximize the insurance company's survival function.

The rest of the paper is organized as follows: in Section 2, we derive a Volterra integro-differential equation (VIDE) corresponding to the model for maximizing the survival function for insurance companies. In Section 3, the numerical simulations were carried out using the fourth-order Runge-Kutta method after converting VIDE into a third-order ordinary differential equation that was later converted into a system of ODEs of the first order. The results are presented and discussed. In the last section, we present a conclusion for this paper.

2. Materials and Methods

2.1. Model Formulation and Analysis. For a mathematical formulation of the problem, we assume all the stochastic quantities and random variables are defined on a complete filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F})_{t\in[0,T]}, \mathbb{P})$, satisfying the usual conditions. The filtration $(\mathscr{F})_{t\in[0,T]}$, which represents the information available at time t and forms the basis for the decision making, is right continuous and $\mathbb{P}$-complete.

Due to the fact that there exist fluctuations in the real market, for example, the amount of premium income, claim arrivals, and a number of customers are not static or uniform, the model must capture these phenomena by considering the perturbed Cramér-Lundberg process X_t as in Kasumo et al. [12], given by

$$X_t = p + ct + \sigma_X W_{X,t} - \sum_{i=1}^{N_{X,t}} Y_{X,i}, \quad t \ge 0, \tag{1}$$

where $p > 0$ is the initial capital, W_X is a standard Brownian motion independent of the compound Poisson process $\sum_{i=1}^{N_{X,t}} Y_{X,i}$, and c is the premium rate per unit time which is calculated by the expected value principle; that is, $c = (1+\theta)\lambda_X\mu_X$ where $\theta > 0$ is the relative safety loading of the insurer and λ_X is the intensity of the counting process $N_{X,t}$ for the claims.

We proceed as in Schmidli [1] by assuming that the insurance company managers engage in the process of refinancing or capital injections, given by $X_t^M = X_t + M_t$, where X_t is the surplus process and M_t is the capital injected. Finally, using equation (1), the insurance model with capital injections is given by

$$X_t^M = p + c_M t + \sigma_X W_{X,t} - \sum_{i=1}^{N_{X,t}} Y_{X,i} + M_t, \quad t \ge 0, \tag{2}$$

where c_M is the premium rate during the capital injection.

To manage its portfolio, we assume the insurance company can undertake investment in either risk-free or risky assets. We assume the risk-free price process was modelled as in Meng et al. [13], given by

$$dB_t = r_0 B_t \mathrm{d}t, \tag{3}$$

where $r_0 \ge 0$ is the constant risk-free interest rate and B_t is the price of the risk-free bond at time t. Let us further assume as in Badaoui et al. [14] that the risky asset, such as the stock price process, is given by the geometric Brownian motion (GBM)

$$dS_t = \sigma_S S_t dW_{S,t} + rS_t \mathrm{d}t, \tag{4}$$

where S_t is the price of a stock at time t, $r \geq 0$ is the expected instantaneous rate of stock return, $\sigma_S \geq 0$ is the volatility of the stock price, and $\{W_{S,t}\colon t \geq 0\}$ is a standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, \mathbb{P})$.

If we further assume that there are no jumps, the Paulsen et al.'s work [15] suggested that the return on investment process R_t will be given by the Black-Scholes option pricing formula of the form

$$R_t = rt + \sigma_R W_{R,t} \quad t \geq 0,\ R_0 = 0, \tag{5}$$

where r is the risk-free part of the investment. Hence, $R_t = rt$ means that one unit invested at time zero will be worth e^{rt} at time t.

2.2. Risk Process with Refinancing and Investment. This section obtains the insurance process compounded with refinancing and investment. It is a careful combination of equations (2) and (5). In the case of reinsurance and investment, the process has been extensively studied for ultimate ruin probability in studies such as Kasozi et al. [16], Paulsen et al. [15], Paulsen and Gjessing [17], and Paulsen and Rasmussen [18] among others. This paper follows a similar approach as in Kasozi et al. [16]. The process $P^M = \{P_t^M\}_{t\in[0,\infty)}$, which represents the insurance portfolio, is given by

$$P_t^M = X_t^M + \int_0^t P^M(s^-)\mathrm{d}R(s), \tag{6}$$

which is the solution of the stochastic differential equation

$$dP_t^M = dX_t^M + P^M(t^-)dR(t), \tag{7}$$

where $p = P_0^M > 0$ is the initial capital of the insurance company same as in equation (2), X_t^M is the primary insurance process given in equation (2), $R(t)$ is the return on investment process in equation (5), and $P^M(t^-)$ stands for the insurer's surplus just before time t.

2.3. Maximizing Survival Function or Minimizing Probability of Ruin. Let us consider equation (6) to maximize survival function or minimize the probability of ruin for the insurance company. Since both X and R have stationary independent increments, P is a homogeneous robust Markov process. By using Itô's formula, the infinitesimal generator for P can be given by

$$\begin{aligned}\mathcal{L}g(p) =& \frac{1}{2}\left(\sigma_R^2 p^2 + \sigma_X^2\right)g''(p) + (rp + c_M)g'(p)\\ &+ \lambda_X \int_0^\infty (g(p - (y\wedge M)) - g(p))\mathrm{d}F_X(y).\end{aligned} \tag{8}$$

The integro-differential operator presented in equation (8) is quite complicated, and explicit analytical computations are hard to perform. However, Paulsen and Gjessing [17] have introduced and proved the following beneficial results.

Let $\tau_p = \inf\{t\colon P_t < 0\}$ be the time of ruin where $\tau_p = \infty$ means ruin never occurs, and then, let $\psi(p) = P(\tau_p < \infty)$ be the probability of eventual ruin to occur.

Assuming that $\psi(p)$ is bounded and twice continuously differentiable on $p \geq 0$ with a bounded first derivative there, where at $p = 0$ is meant the right-hand derivative, and that ψ solves $\mathcal{L}\psi(p) = 0$ on $p > 0$ together with the boundary conditions

$$\begin{aligned}\psi(p) &= 1 \quad \text{on } p < 0,\\ \psi(0) &= 1 \quad \text{if } \sigma_X^2 > 0,\\ \lim_{p\longrightarrow\infty}\psi(p) &= 0.\end{aligned} \tag{9}$$

Paulsen and Gjessing [17] have shown that

$$\psi(p) = P\left(\tau_p < \infty\right). \tag{10}$$

Assuming that $q_\alpha(p)$ is bounded and twice continuously differentiable on $p \geq 0$ with a bounded first derivative there, where at $p = 0$ is meant the right-hand derivative, and Paulsen and Gjessing [17] have also shown that if q_α solves $\mathcal{L}q_\alpha(p) = 0$ on $p > 0$ together with the boundary conditions

$$\begin{aligned}q_\alpha(p) &= 1 \quad \text{on } p < 0,\\ q_\alpha(0) &= 1 \quad \text{if } \sigma_X^2 > 0,\\ \lim_{p\longrightarrow\infty}q_\alpha(p) &= 0,\end{aligned} \tag{11}$$

then

$$q_\alpha(p) = \mathbb{E}\left[e^{\alpha\tau_p}\right]. \tag{12}$$

Now, using Paulsen et al. [15] idea, we replace the first part of the theorem with the survival function $\phi(p) = 1 - \psi(p)$ with boundary conditions given as follows:

$$\begin{aligned}\phi(p) &= 0 \quad \text{on } p < 0,\\ \phi(0) &= 0 \quad \text{if } \sigma_X^2 > 0,\\ \lim_{p\longrightarrow\infty}\phi(p) &= 1.\end{aligned} \tag{13}$$

Because maximizing the survival function influences the minimization of the probability of ruin which increases the probability of survival for the insurance company, the goal is now to maximize the survival function $\phi(p)$. Therefore, the value function, in this case, is defined by

$$V(p) = \sup_{M\geq 0}\phi^M(p), \tag{14}$$

and if it exists, we determine the corresponding refinancing strategy $M_t \in [0,\infty)$ that will satisfy the objective function. Therefore, we are interested in finding the optimal refinancing strategy in the presence of investments in risky and risk-free assets. We refer to this strategy as optimal because it maximizes survival function, which is the same as minimizing the probability of ultimate ruin. In other words, the survival function is the objective function, and the refinancing strategy M_t is the control variable to be adjusted such that the objective function is maximized.

2.4. Hamilton–Jacobi–Bellman Equation and Integro-Differential Equation. In this section, the HJB equation for the value function given by equation (14) is derived and solved. Then, the integro-differential equation for the survival function is formulated and solved too. In the literature, several HJB equations of a similar kind have been used, and for example, the reader may refer to Schmidli [10], Paulsen et al. [15], Kasozi et al. [11], and Kasumo et al. [12] for more details.

2.4.1. Hamilton–Jacobi–Bellman Equation. To derive the HJB equation for the value function given by equation (14), we follow a similar approach as in Kasozi et al. [11]. Let $(0, h]$ be a small interval and suppose that for each surplus $p(h) > 0$ at time h we have a refinancing strategy M^{ε} such that $\delta M^{\varepsilon}(p(h)) > \delta(p(h)) - \varepsilon$. Let also that $M_t = M \in [0, \infty)$ for $t \leq h$. Then, as in Kasozi et al. [11], by Markov property, one has the following equation:

$$\begin{aligned}\phi(p) \geq \phi^{M}(p) &= \mathbb{E}\left[\left(\phi^{M^{\varepsilon}}\left(P^{M}(h)\right)\right); \tau_p > h\right]\\ &= \mathbb{E}\left[\left(\phi^{M^{\varepsilon}}\left(P^{M}\left(\tau_p \wedge h\right)\right)\right)\right]\\ &\geq \mathbb{E}\left[\left(\phi^{M^{\varepsilon}}\left(P^{M}\left(\tau_p \wedge h\right)\right)\right)\right] - \varepsilon,\end{aligned} \tag{15}$$

where $\varepsilon \in (-\infty, \infty)$ one can choose $\varepsilon = 0$ to get

$$\phi(p) \geq \mathbb{E}\left[\left(\phi^{M^{\varepsilon}}\left(P^{M}\left(\tau_p \wedge h\right)\right)\right)\right]. \tag{16}$$

Let us assume that $\phi(p)$ is twice continuously differentiable; by using Itô's formula, we obtain

$$\begin{aligned}\phi\left(P^{M}\left(\tau_p \wedge h\right)\right) = \phi(p) + \int_0^{\tau_p \wedge h} &\left\{(rp + c_M)\phi'\left(P^{M}(s)\right) + \frac{1}{2}\left(\sigma_R^2 p^2 + \sigma_X^2\right)\phi''\left(P^{M}(s)\right)\right.\\ &\left.+ \lambda_X\left[\int_0^p \phi\left(P^{M}(s) - (y \wedge M)\right)\mathrm{d}F_X(y) - \phi\left(P^{M}(s)\right)\right]\right\}\mathrm{d}s,\end{aligned} \tag{17}$$

where $y \wedge M = \max(M, Y_i)$ denote the retained amount to the insurance company.

Now, put (17) into (16) to get

$$\begin{aligned}\mathbb{E}\Bigg[\int_0^{\tau_p \wedge h} &\left\{(rp + c_M)\phi'\left(P^{M}(s)\right) + \frac{1}{2}\left(\sigma_R^2 p^2 + \sigma_X^2\right)\phi''\left(P^{M}(s)\right)\right.\\ &\left.+ \lambda_X\left[\int_0^p \phi\left(P^{M}(s) - \max(M, Y_i)\right)\mathrm{d}F_X(y) - \phi\left(P^{M}(s)\right)\right]\right\}\mathrm{d}s\Bigg] \leq 0.\end{aligned} \tag{18}$$

Provided the limit and expectation can be interchanged, then dividing the later equation by h and letting $h \longrightarrow 0$ gives the following equation:

$$(rp + c_M)\phi'(p) + \frac{1}{2}\left(\sigma_R^2 p^2 + \sigma_X^2\right)\phi''(p) + \lambda_X\left[\int_0^p \phi(p - \max(M, Y_i))\mathrm{d}F_X(y) - \phi(p)\right] \leq 0. \tag{19}$$

This equation (19) must hold for all $M > 0$, that is, to write

$$\sup_{M>0}\left[(rp + c_M)\phi'(p) + \frac{1}{2}\left(\sigma_R^2 p^2 + \sigma_X^2\right)\phi''(p) + \lambda_X\left[\int_0^p \phi(p - \max(M, Y_i))\mathrm{d}F_X(y) - \phi(p)\right]\right] \leq 0. \tag{20}$$

Suppose that there is an optimal strategy $M \in [0, \infty)$ such that $\lim_{t \downarrow 0} M(t) = M(0)$. Then, using a similar approach, we have

Table 1: Model parameters and their values.

Symbol	Definition	Value (s)	Source
σ_X	Premium volatility	0.25	Zou et al. [22]
σ_R	Return volatility	0.1	Mtunya et al. [23]
p	Initial capital	10000	Assumed
λ_X	Intensity of the counting process	2, 5, 10	Kasumo et al. [12]
c_M	Premium rate left to the company	2	Kasumo [24]
M	Capital refinanced	600	Assumed
r	Instantaneous rate of stock return	0.05, 0.5	Kasozi et al. [11]
β	Mean of the exponential distribution	0.5	Kasumo [24]
Y	Capital risked	500	Assumed

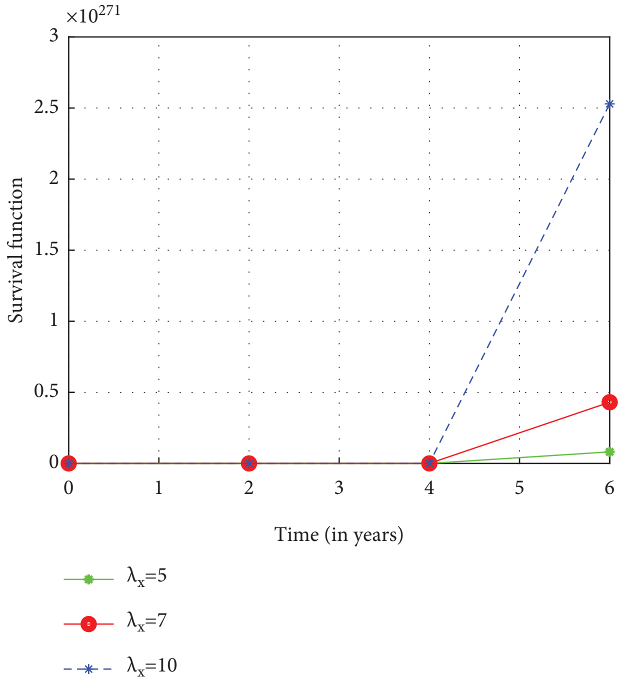

Figure 1: Dependence of the survival function on the intensity of the counting process.

$$\left(rp + c_M\right)\phi'(p) + \frac{1}{2}\left(\sigma_R^2 p^2 + \sigma_X^2\right)\phi''(p) + \lambda_X\left[\int_0^p \phi\left(p - \max\left(M, Y_i\right)\right)\mathrm{d}F_X(y) - \phi(p)\right] = 0. \tag{21}$$

Finally, this gives the HJB equation

$$\sup_{M>0}\left[\left(rp + c_M\right)\phi'(p) + \frac{1}{2}\left(\sigma_R^2 p^2 + \sigma_X^2\right)\phi''(p) + \lambda_X\left[\int_0^p \phi\left(p - \max\left(M, Y_i\right)\right)\mathrm{d}F_X(y) - \phi(p)\right]\right] = 0, \tag{22}$$

whose boundary conditions are $\phi(p) = 0$ on $p < 0$ and $\lim_{p\longrightarrow\infty}\phi(p) = 1$.

An optimal strategy is obtained from the solution set $(\phi(p), M^*(p))$ of equation (22), in which $M^*(p)$ is a point at which the supremum in (22) is obtained. The insurance company has a nonnegative net premium income if $c > (1+\eta)\lambda_X\mathbb{E}[(M - Y_i)^+]$.

Let $\underline{M}$ be the value where the equality holds, that is, $c = (1+\eta)\lambda_X\mathbb{E}[(M - Y_i)^+]$, but the aim is to find a nondecreasing solution to equation (22), and let us write it as follows:

$$\sup_{M>\underline{M}}\left[(rp+c_M)\phi'(p)+\frac{1}{2}\left(\sigma_R^2p^2+\sigma_X^2\right)\phi''(p)+\lambda_X\left[\int_0^p\phi(p-\max(M,Y_i))\mathrm{d}F_X(y)-\phi(p)\right]\right]=0, \tag{23}$$

whose boundary conditions are $\phi(p)=0$ on $p<0$ and $\lim_{p\longrightarrow\infty}\phi(p)=1$.

According to Hipp and Plum [19], the function $\phi(p)$ will satisfy equation (23), only if $\phi(p)$ is strictly increasing, strictly concave, twice continuously differentiable, and it satisfies the second condition; that is, $\lim_{p\longrightarrow\infty}\phi(p)=1$.

Now, let us assume that $\phi(p)$ solves the HJB equation (22), according to Hipp and Vogt [20] if $\phi(p)$ is a smooth solution of the HJB equation (22), then the supremum over $M>\underline{M}$ is either attained at $M=0$ when there is no refinancing for small claims or at $M=p$ or $\underline{M}<M<p$.

2.4.2. Integro-Differential Equation. From the HJB equation (22), the integro-differential equation for the survival function $\phi(p)$ takes the following form:

$$\mathscr{L}\phi(p)=0,\quad p\geq 0, \tag{24}$$

where $\mathscr{L}$ is the infinitesimal generator defined by equation (8) for the underlying risk process with refinancing and investment given by equation (6). Thus, from the HJB equation (22), the integro-differential equation for the survival function is given by

$$\begin{aligned}&(rp+c_M)\phi'(p)+\frac{1}{2}\left(\sigma_R^2p^2+\sigma_X^2\right)\phi''(p)\\&+\lambda_X\int_0^p\phi(p-\max(M,Y_i))\mathrm{d}F_X(y)-\lambda_X\phi(p)=0,\end{aligned} \tag{25}$$

for $0<p\leq\infty$.

Equation (25) is a second-order integro-differential equation of Volterra type (VIDE). In this paper, the VIDE given by equation (25) is converted into an ordinary differential equation (ODE) that can be solved numerically to determine the optimal strategies.

2.4.3. Converting VIDE into ODEs. In this section, we begin the process of solving VIDE given by equation (25). The equation is firstly converted into an ODE, and later, it will be solved numerically. The equation will be solved, assuming that the claims are exponentially distributed. If $\sigma_R=0$ and $r=0$, then there is no investment. For this case, the analytical solution to a similar problem is given by Belhaj [21], and if $\lambda_X=0$, a similar case was solved analytically by Paulsen and Gjessing [17]; however, when $\lambda_X\neq 0$, $\sigma_R\neq 0$, and $r\neq 0$, equation (25) has no analytical solution.

Consider exponential distribution given by

$$\begin{aligned}f_X(y)&=\beta e^{-\beta y},\\F_X(y)&=1-e^{-\beta y},\\dF_X(y)&=\beta e^{-\beta y}dy\ .\end{aligned} \tag{26}$$

Then, equation (25) become

$$\begin{aligned}&(rp+c_M)\phi'(p)+\frac{1}{2}\left(\sigma_R^2p^2+\sigma_X^2\right)\phi''(p)\\&+\lambda_X\int_0^p\phi(p-\max(M,Y_i))\beta e^{-\beta y}\mathrm{d}y-\lambda_X\phi(p)=0,\end{aligned} \tag{27}$$

for $0<p\leq\infty$.

Differentiating the above equation with respect to p and simplifications give

$$\begin{aligned}&\frac{1}{2}\left(\sigma_R^2p^2+\sigma_X^2\right)\phi'''(p)+\left(rp+c_M+\sigma_R^2p\right)\phi''(p)\\&\quad+(r-\lambda_X)\phi'(p)-\lambda_X\beta e^{-\beta p}\phi(p-\max(M,Y_i))=0,\end{aligned} \tag{28}$$

for $0<p\leq\infty$.

Equation (28) is an ODE that will be solved numerically.

3. Numerical Results and Discussion

We transform the third-order ODE given by equation (28) into a system of first-order ODEs that will be solved numerically by using the Runge–Kutta method. Letting $Z_1(x)=\phi(p)$, $Z_2(x)=\phi'(p)=Z_1'(x)$, and $Z_3(x)=\phi''(p)=Z_2'(x)$, then by using equation (28), the following system of first-order ODE is obtained.

$$\begin{cases}Z_1'=Z_2,\\Z_2'=Z_3,\\Z_3'=\dfrac{2}{\left(\sigma_R^2p^2+\sigma_X^2\right)}\left[\lambda_X\beta e^{-\beta p}Z_1(p-\max(M,Y_i))-(r-\lambda_X)Z_2-\left(rp+c_M+\sigma_R^2p\right)Z_3\right].\end{cases} \tag{29}$$

In this section, system (29) of first-order ODEs is solved numerically using the fourth-order Runge–Kutta method, implemented using MATLAB codes, and results are discussed. Simulations and graphics were performed using MATLAB R2020a. Values of the parameters used are presented in Table 1.

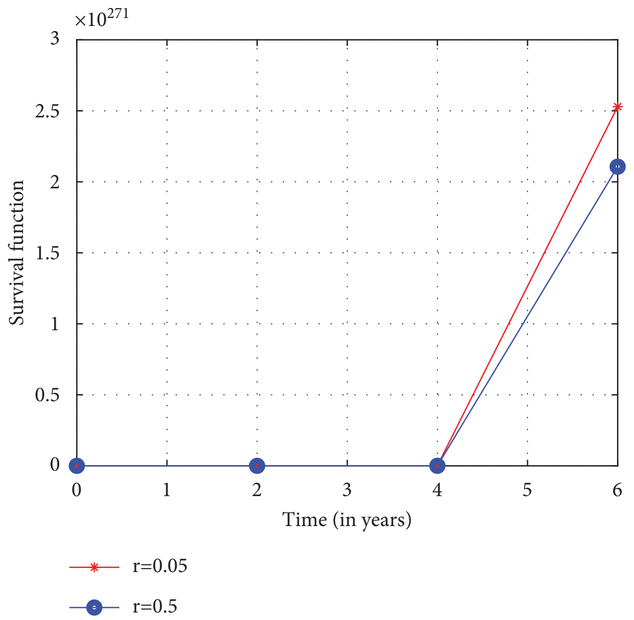

Figure 2: Dependence of the survival function on the instantaneous rate of stock return.

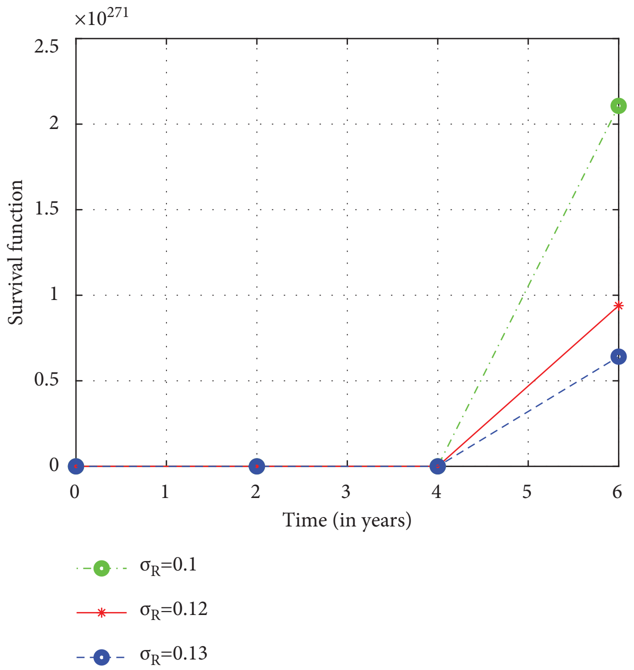

Figure 3: Dependence of the survival function on the return volatility.

In Figure 1, we observe an increase in survival function in the fourth year. The survival function is observed to increase as the intensity of the counting process increases. This is because as the counting process increases, the insurance company services its clients much faster. As a result, it becomes healthier, and hence, its probability of ruin is reduced.

In Figure 2, we observe that an insurance company has an increasing survival function due to capital injection. Still, we further note that when the instantaneous rate of stock return increases, the survival function decreases since the risk is much higher. These results are comparable to those of Gajek and Zagrodny [25], who studied reinsurance arrangements to maximize insurers' survival probability, similar to minimizing ruin.

Since volatility measures the dispersion of returns for a given security, we observe in Figure 3 that an insurance company has an inverse relationship with the return volatility. We note that when returning volatility increases just by 0.02, the survival function decreases very quickly because the higher the volatility, the riskier the security. These results show that the survival function is extremely sensitive to the return volatility and support the earlier observations on the instantaneous rate of stock return. As a result, this paper suggests that more investments should be made in risk-free assets rather than in risky assets when this situation occurs.

4. Conclusion

In this paper, we have used the basic Cramér-Lundberg model to derive the Volterra integro-differential equation (VIDE) for the survival function of the insurance company. After some conversions, the VIDE was solved numerically by the Runge-Kutta method of order four. The results indicate that it is possible to maximize the survival function of the insurance company's portfolio which helps the company reduce the possibility of ruin. The study further established that increasing the claim intensity has a positive effect in terms of increasing the survival function and reducing the probability of ruin. Therefore, it is recommended that insurance companies increase their claim intensities. This paper has also concluded that insurance companies should invest more in risk-free assets when the instantaneous rate of stock return and return volatility is much higher or increased.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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POSTER ON MAXIMIZING THE SURVIVAL FUNCTION

Reducing the Possibility of Ruin by Maximizing the Survival Function for the Insurance Company's Portfolio

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Introduction

The insurance industry is currently undergoing a fundamental transformation in terms of operations and competitiveness. Several disruptive factors in business have given rise to new players in the market with disruptive business models to outperform their competitors. Investment and refinancing can be used as survival approaches when insurance players consider how they should react to this major shift. With investing and refinancing, an insurance company can manage to operate much better even if it had suffered from ruin provided the investments are done properly and refinancing is done adequately and timely.

Shareholders of insurance businesses are interested in optimizing the returns from the insurance portfolio as well as ensuring that the business remains afloat over a long-time horizon. To achieve this, the company managers have to optimally run the business to maximize returns and reduce ruin probability. Even with extra care, many times, ruin is inevitable.

Model Formulation

MODEL EQUATION WITHOUT REFINANCING

$$X_t = p + ct + \sigma_X W_{X,t} - \sum_{i=1}^{N_{X,t}} Y_{X,i} \quad , \quad t \geq 0.$$

MODEL EQUATION WITH REFINANCING

$$X_t^M = p + c_M t + \sigma_X W_{X,t} - \sum_{i=1}^{N_{X,t}} Y_{X,i} + M_t \quad , \quad t \geq 0.$$

RISK-FREE (BOND)

$$dB_t = r_0 B_t dt$$

RISKY-FREE (STOCK)

$$dS_t = \sigma_S S_t dW_{S,t} + rS_t dt$$

RETURN ON INVESTMENTS

$$R_t = rt + \sigma_R W_{R,t} \quad t \geq 0, \quad R_0 = 0$$

Risk Process with Refinancing and Investment

Carefully combining model equation for refinancing and an equation of return on investment we obtain the following model equation for refinancing and investments

$$P_t^M = X_t^M + \int_0^t P^M(s^-)dR(s)$$

Which is the solution of the stochastic differential equation

$$dP_t^M = dX_t^M + P^M(t^-)dR(t)$$

Where $p = P_0^M > 0$p is the initial capital of the insurance company

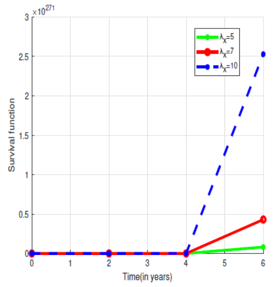
Maximizing Survival Function

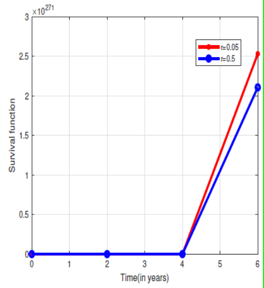
We are interested in finding the optimal refinancing strategy in the presence of investments in risky and risk-free assets. We refer to this strategy as optimal because it maximizes survival function, which is the same as minimizing the probability of ultimate ruin.

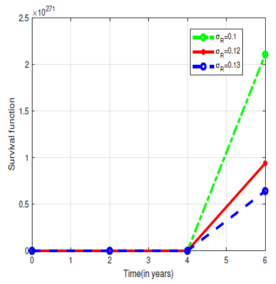
Numerical results and discussion

Model parameters and their values.

Symbol	Definition	Value(s)
σ_X	Premium volatility	0.25
σ_R	Return Volatility	0.1
p	Initial capital	10000
λ_X	Intensity of the counting process	2, 5, 10
c_M	Premium rate left to the company	2
M	Capital refinanced	600
r	Instantaneous rate of stock return	0.05, 0.5
β	Mean of the exponential distribution	0.5
Y	Capital risked	500







- The survival function is observed to increase as the intensity of the counting process increases.
- An insurance company has an increasing survival function due to capital injection. Still, we further note that when the instantaneous rate of stock return increases, the survival function decreases since the risk is much higher.
- Finally, we observe that an insurance company has inverse relationship with the return volatility.

Conclusion

We have used the basic Cramer-Lundberg model to derive the Volterra Integro Differential Equation (VIDE) for the survival function of the insurance company. After some conversions, the VIDE was solved numerically by the Runge-Kutta method of order four. The results indicate that it is possible to maximize the survival function of the insurance company's portfolio which helps the company reduce the possibility of ruin. The study further established that increasing the claim intensity has a positive effect in terms of increasing the survival function and reducing the probability of ruin. Therefore, it is recommended that insurance companies increase their claim intensities. We also concluded that insurance companies should invest more in risk-free assets when the instantaneous rate of stock return and return volatility are much higher or increased.

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